Supplemental Material.
Derivation of Friedel sum rule for moving shuttle

Original Friedel sum rule [1] formulated for the stationary quantum mechanical states establishes connection between the scattering phase shifts and the number of occupied states inserted by an impurity potential and screened by the free electrons in the Fermi sea. This universal rule is valid also for strongly correlated systems involving many-particle effects like orthogonality catastrophe (Fermi-edge singularity, Kondo effect). In the latter case the Friedel-Langreth rule fixes the Kondo screening effect [2], i.e. counts the number of free electron states involved in formation of the Kondo singlet ground state and thus excluded from the Fermi bath with N electrons. In case of single-channel Kondo effect for impurity spin 1/2 the s-partial wave is scattered with the phase shift $\delta_\sigma(\varepsilon_F) = \sigma\pi/2$. This phase shift fixes the number of electronic degrees of freedom spent for complete dynamical screening of impurity spin.

Specific feature of our case is that we consider dynamical screening of slowly moving spin by the electrons in the left lead in a strong coupling regime near the unitarity limit fixed point following the Nozieres' phenomenological Fermi liquid approach [3] under conditions of finite $(T, eV_{\rm bias}, \omega) \ll T_K$ (we adopt notations k_B=$\hbar$=1). Kondo temperature plays part of effective energy scale ("bandwidth") for fermionic excitations above the ground state (Kondo singlet plus Fermi sphere with $N-1$ electrons). In adiabatic regime the hierarchy of energy levels is conserved, but T_K and therefore δ_σ become slow functions of time.

The phase shifts enter the spin-dependent $\mathcal{T}$ - matix defined as

$$-\pi\rho\mathcal{T}_\sigma(\omega) = \frac{1}{2i}\left[e^{2i\delta_\sigma(\omega)} - 1\right] + e^{2i\delta_\sigma(\omega)}\left[-\pi\rho\mathcal{T}_{in}(\omega)\right] \quad (1)$$

where $\mathcal{T}_{in}$ accounts for inelastic processes and ρ is a total density of states at the Fermi level. To derive Eq. (10) of the main text we start with definition

$$N_1 - N_2 \equiv \langle \hat{N}_1 - \hat{N}_2 \rangle = \quad (2)$$
$$T\sum_n \sum_\sigma \int \frac{d\vec{p}}{(2\pi)^3}\left[\mathcal{G}_{1\sigma}(\vec{p},\omega_n) - \mathcal{G}_{2\sigma}(\vec{p},\omega_n)\right] e^{i\omega_n\tau}|_{+0}.$$

Here $\mathcal{G}_{i\sigma}$ are the Matsubara Green functions and the channels 1 (2) are symmetric (antisymmetric) combinations of left $\pm$ right leads. Only the channel 1 is involved in the Kondo interaction with moving island. Averaging is performed with Nozieres Kondo fixed-point Fermi liquid Hamiltonian. The Green functions in the channels 1,2 are given by

$$\mathcal{G}_{1\sigma} = \mathcal{G}_{0\sigma} + \mathcal{G}_{0\sigma}\mathcal{T}_\sigma\mathcal{G}_{0\sigma}, \quad \mathcal{G}_{2\sigma} = \mathcal{G}_{0\sigma} \quad (3)$$

($\mathcal{G}_{0\sigma}(\vec{p},\omega_n) = (i\omega_n - \xi_{p,\sigma})^{-1}$ is the bare propagator in channels 1 and 2 and $\xi_{p,\sigma} = \varepsilon_{p,\sigma} - \mu$ is the dispersion of Fermi excitations with the bandwidth $2T_K$ in the Nozieres' Fermi liquid model).

Near the fixed point one may use the conformal field theory approach for $\mathcal{T}$-matrix [4], which gives

$$-\pi\rho\mathcal{T}_{in}(\omega) = i\frac{\omega^2 + \pi^2T^2}{2T_K^2}. \quad (4)$$

As usual, we introduce the density of states for given spin projection:

$$\varrho_\sigma(\varepsilon) = \int \frac{d\vec{p}}{(2\pi)^3}\delta(\varepsilon - \xi_{p,\sigma})$$

Then

$$N_1 - N_2 = \quad (5)$$
$$T\sum_n\sum_\sigma \int_{-T_K}^{T_K} d\varepsilon \mathcal{G}_{0\sigma}(\varepsilon,\omega_n)\varrho_\sigma(\varepsilon)\mathcal{T}_\sigma(\omega_n)\mathcal{G}_{0\sigma}(\varepsilon,\omega_n).$$

Substituting (4) and (1) in (5) and integrating over ε, we get

$$N_1 - N_2 = \frac{2}{\pi}\mathcal{R}e\left\{T\sum_n\sum_\sigma \frac{T_K}{\omega_n^2 + T_K^2}\left[-\pi\rho\mathcal{T}_\sigma\right]\right\} \quad (6)$$

where $\rho = \varrho_\sigma(\mu)$. At $T \ll T_K$ we replace Matsubara summation for integration $T\sum_n \to \int d\omega/2\pi$ [5] and perform Wick's rotation $i\omega_n \leftrightarrow \omega$. Combining Eqs. (4) and (1) we obtain after simple transformations:

$$N_1 - N_2 \to \frac{\delta_t}{\pi} + O(\tilde{\delta}_\uparrow^3 + \tilde{\delta}_\downarrow^3) \quad (7)$$

This equation is in fact Eq. (10) of the main text plus higher order corrections in the phase shifts. It should be stressed that we work near the unitarity limit $\delta_\sigma \to \sigma\pi/2$ (here $\tilde{\delta}_\sigma = \delta_\sigma - \sigma\pi/2$) and total phase $\delta_t = \delta_\uparrow + \delta_\downarrow$ is defined in the interval $0 \le \delta_t \le \pi$. Index t stands for parametric time dependence in the adiabatic regime.

Summarizing, presented derivation is nothing but generalization of Affleck - Ludwig - Pustilnik - Glazman procedure [4, 6] for time-dependent adiabatic motion of quantum impurity. Less general procedure (based on the Langreth perturbation theory [2] applied to Anderson's model of localized impurity states in metals) has been proposed for the Friedel sum rule in Ref. [7] devoted to the charge pumping in the Kondo regime.

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