Supplemental Material to "Charge Kondo circuit as a detector for electron-electron interactions in a Luttinger Liquid"

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S1. GREEN'S FUNCTION

In the Supplemental Material we present the derivation of the spatially inhomogeneous Green's function (Eq. (7) in the main text). We follow the method developed in [S1]. Using saddle point for the Euclidean action of the spatially inhomogeneous Luttinger liquid

$$S_0 = \frac{1}{2\pi}\int dx \int_0^\beta dt \frac{1}{g(x)}\left[\frac{[\partial_t\phi(x,t)]^2}{u(x)} + u(x)[\partial_x\phi(x,t)]^2\right], \tag{S1.1}$$

we obtain the differential equation for the Green's function in Matsubara representation:

$$\left\{\frac{\omega_n^2}{u(x)g(x)} - \partial_x\frac{u(x)}{g(x)}\partial_x\right\}G_{\omega_n}(x,x') = \delta(x-x'). \tag{S1.2}$$

We assume the following protocols for the interaction's strength outside and inside the interacting area: $g(x) = 1$, $u(x) = v_F$ for $|x| > L/2$, and $g(x) = g$, $u(x) = v \approx v_F/g$ for $|x| < L/2$.

Equation for the Green's function $G_{\omega_n}(x,x')$ Eq. (S1.2) is completed by the boundary conditions: i) the propagator $G_{\omega_n}(x,x')$ is continuous function at $x = \pm L/2$ and $x = x'$; ii) $[u(x)/g(x)]\partial_x G_{\omega_n}(x,x')$ is continuous at $x = \pm L/2$. In addition, there is a discontinuity (jump) of $G_{\omega_n}(x,x')$ at $x = x'$, i.e.,

$$-\frac{u(x)}{g(x)}\partial_x G_{\omega_n}(x,x')|_{x=x'-0}^{x=x'+0} = 1. \tag{S1.3}$$

Besides, we assume the adiabatic switching off the interaction at $x = \pm\infty$: $G_{\omega_n}(\pm\infty, x') = 0$.

Thus, using the boundary conditions (S1.3) allows to obtain the solution of Eq. (S1.2) in four different regions as follows:

$$G_{\omega_n}(x,x') = \begin{cases} Ae^{\frac{|\omega_n|x}{v_F}}, & x < -L/2, \\ Be^{\frac{|\omega_n|x}{v}} + Ce^{-\frac{|\omega_n|x}{v}}, & -L/2 < x < x', \\ De^{\frac{|\omega_n|x}{v}} + Ee^{-\frac{|\omega_n|x}{v}}, & x' < x < L/2, \\ Fe^{-\frac{|\omega_n|x}{v_F}}, & x > L/2. \end{cases} \tag{S1.4}$$

By matching Eqs. (S1.4) at $x = \pm L/2$, and $x = x'$ we obtain:

$$A(x') = -\frac{g}{|\omega_n|}\frac{\left\{\kappa_- e^{\frac{|\omega_n|x'}{v}} e^{\frac{-|\omega_n|L}{v}} e^{\frac{|\omega_n|L}{2v_F}} + \kappa_+ e^{\frac{-|\omega_n|x'}{v}} e^{\frac{|\omega_n|L}{v}} e^{\frac{|\omega_n|L}{2v_F}}\right\}}{\left\{\kappa_-^2 e^{-\frac{|\omega_n|L}{v}} - \kappa_+^2 e^{\frac{|\omega_n|L}{v}}\right\}}, \tag{S1.5}$$

$$B(x') = \frac{\kappa_+}{2}e^{\frac{|\omega_n|L}{2v}} e^{\frac{-|\omega_n|L}{2v_F}} A(x') \quad , \quad C(x') = \frac{\kappa_-}{2}e^{-\frac{|\omega_n|L}{2v}} e^{\frac{-|\omega_n|L}{2v_F}} A(x'), \tag{S1.6}$$

$$F(x') = -\frac{g}{|\omega_n|}\frac{\left\{\kappa_+ e^{\frac{|\omega_n|x'}{v}} e^{\frac{|\omega_n|L}{v}} e^{\frac{|\omega_n|L}{2v_F}} + \kappa_- e^{\frac{-|\omega_n|x'}{v}} e^{-\frac{|\omega_n|L}{v}} e^{\frac{|\omega_n|L}{2v_F}}\right\}}{\left\{\kappa_-^2 e^{-\frac{|\omega_n|L}{v}} - \kappa_+^2 e^{\frac{|\omega_n|L}{v}}\right\}}, \tag{S1.7}$$

$$E(x') = \frac{\kappa_+}{2}e^{\frac{|\omega_n|L}{2v}} e^{\frac{-|\omega_n|L}{2v_F}} F(x') \quad , \quad D(x') = \frac{\kappa_-}{2}e^{-\frac{|\omega_n|L}{2v}} e^{\frac{-|\omega_n|L}{2v_F}} F(x'), \tag{S1.8}$$

where $\kappa_\pm = 1 \pm g$. Combining Eqs. (S1.5)-(S1.8) and Eq. (S1.4), we obtain the equation for the Green's function (Eq. (7) in the main part of the manuscript):

$$G_{\omega_n}(x,x') = \frac{g}{2|\omega_n|}\frac{2\kappa_+\kappa_-\cosh\left[\frac{|\omega_n|(x+x')}{v}\right] + \kappa_-^2 e^{\pm\frac{|\omega_n|(x-x')}{v}} e^{-\frac{|\omega_n|L}{v}} + \kappa_+^2 e^{\frac{\mp|\omega_n|(x-x')}{v}} e^{\frac{|\omega_n|L}{v}}}{\kappa_+^2 e^{\frac{|\omega_n|L}{v}} - \kappa_-^2 e^{\frac{-|\omega_n|L}{v}}}. \tag{S1.9}$$

Here the upper sign in exponents corresponds to the case $x > x'$, while the lower sign corresponds to $x < x'$. The Green's function Eq. (S1.9) has different asymptotic values at two regimes: i) $G_{\omega_n}(0,0) \approx g/2|\omega_n|$ at "high" frequencies (temperatures), $\omega_n \gg v/L$, and ii) $G_{\omega_n}(0,0) \approx 1/2|\omega_n|$ at the "low" frequency (temperature) regime, $\omega_n \ll v/L$. It indicates that the long-range properties become more relevant as the frequency (temperature) decreases. The crossover between two regimes occurs at the so-called critical temperature $T^{cr} \sim v/L$.

S2. DIFFERENTIAL CONDUCTANCE: A LINEAR RESPONSE

As it is mention in the main part of the text, using the Green's function Eq. (S1.9) and integrating out bosonic fields $\phi_{c,s}(x,t)$, we obtain the following effective action:

$$S = \frac{1}{2\pi\beta}\sum_n \sum_{i=c,s} G^{-1}_{\omega_n}(0,0)\phi_i(i\omega_n)\phi_i(-i\omega_n) + \frac{E_C}{\pi^2}\int_0^\beta dt\left[\sqrt{2}\phi_c(t) - \pi N\right]^2$$
$$+\frac{D}{\pi}\int_0^\beta dt\left\{|r_1|\cos[\sqrt{2}(\phi_s(t)-\phi_c(t))] + |r_2|\cos[\sqrt{2}(\phi_s(t)+\phi_c(t))]\right\}. \tag{S2.1}$$

We use the perturbation theory assuming the electron backscattering amplitude $|r_{1,2}| \ll 1$ being a small parameter in order to obtain the correlator $\langle\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle\rangle$. For this, we expand the partition function over S' [third term in Eq. (S2.1)] up to the second order:

$$e^{-S'} = 1 + \frac{1}{2!}\frac{D^2}{\pi^2}\int dt\int dt'\frac{1}{2}\left\{|r_1|^2\left[\cos[\sqrt{2}(\phi_s-\phi_s'-\phi_c+\phi_c')] + \cos[\sqrt{2}(\phi_s+\phi_s'-\phi_c-\phi_c')]\right]\right. \tag{S2.2}$$
$$+|r_2|^2\left[\cos[\sqrt{2}(\phi_s-\phi_s'+\phi_c-\phi_c')] + \cos[\sqrt{2}(\phi_s+\phi_s'+\phi_c+\phi_c')]\right] \tag{S2.3}$$
$$+|r_1||r_2|\left[\cos[\sqrt{2}(\phi_s-\phi_s'-\phi_c-\phi_c')] + \cos[\sqrt{2}(\phi_s+\phi_s'-\phi_c+\phi_c')]\right] \tag{S2.4}$$
$$\left.+|r_1||r_2|\left[\cos[\sqrt{2}(\phi_s-\phi_s'+\phi_c+\phi_c')] + \cos[\sqrt{2}(\phi_s+\phi_s'+\phi_c-\phi_c')]\right]\right\}. \tag{S2.5}$$

Here, we use shorthand notations $\phi_{c,s} = \phi_{c,s}(t)$ and $\phi'_{c,s} = \phi_{c,s}(t')$.

We redefine $\phi_c \to \pi N/\sqrt{2} + \phi_c$ to proceed with the calculations. Second terms which contain $\phi_s + \phi_s'$ in the Eqs. (S2.2)-(S2.5) vanish after averaging procedure. Thus, for instance, the correlation function $\langle\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle\rangle$ accounted from Eq. (S2.2) is written as follows:

$$\langle\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle\rangle = \langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle + \frac{1}{4}\frac{D^2}{\pi^2}|r_1|^2\mathrm{Re}\int dt\int dt'\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)e^{i\sqrt{2}[\phi_s-\phi_s']}e^{-i\sqrt{2}[\phi_c-\phi_c']}\rangle -$$
$$-\frac{1}{4}\frac{D^2}{\pi^2}|r_1|^2\mathrm{Re}\int dt\int dt'\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle\langle e^{i\sqrt{2}[\phi_s-\phi_s']}e^{-i\sqrt{2}[\phi_c-\phi_c']}\rangle. \tag{S2.6}$$

In the absence of backscattering ($|r_{1,2}| = 0$), the correlation function $\langle\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle\rangle$ at the dc limit ($\omega \to 0$) is determined by the low-frequency limit of Eq. (S1.9), $\langle\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle\rangle = \pi/2|\omega_n|T$. However, the situation is completely different for finite backscattering. Applying the Wick theorem in Eq. (S2.6), one find that a small correction due to backscattering is characterized by the function $\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle^2\exp\left[-\langle(\phi_s-\phi_s')^2\rangle\right]\exp\left[-\langle(\phi_c-\phi_c')^2\rangle\right]$. After performing the Fourier transformation, correlations functions in exponents can be obtained by summing the series over Matsubara frequencies. It turns that $\omega_n = 0$ term does not contribute to the series, see [S2, S3]. Therefore, at $T \gg v/L$, the high-temperature limit of Eq. (S1.9) should be considered for further calculations. Using the following relations:

$$\pi T g \sum_n \frac{1}{|\omega_n|}(1-\cos[\omega_n\tau])e^{-\frac{|\omega_n|}{D}} = g\log\left(\frac{D}{\pi T}\sin[\pi T\tau]\right), \tag{S2.7}$$

$$\pi T g \sum_n \frac{1}{|\omega_n|+\frac{2gE_C}{\pi}}e^{-\frac{|\omega_n|}{D}} = g\log\left(\frac{\pi D}{2\gamma g E_C}\right), \tag{S2.8}$$

$$\pi T g \sum_n \frac{\cos[\omega_n\tau]}{|\omega_n|+\frac{2gE_C}{\pi}}e^{-\frac{|\omega_n|}{D}} = g\frac{\pi^4 T^2}{4g^2E_C^2}\frac{1}{\sin^2[\pi T\tau]}, \tag{S2.9}$$

we compute the Gaussian integrals and rewrite Eq. (S2.6) as follows:

$$\langle\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle\rangle = \frac{\pi}{2|\omega_n|T} - \frac{D^2}{4}\frac{|r_1|^2}{\omega_n^2}\int_0^\beta dt\int_0^\beta dt'(1-\cos[\omega_n(t-t')])$$
$$\times\left(\frac{\pi T}{D}\frac{1}{|\sin[\pi T(t-t')]|}\right)^g\left(\frac{2\gamma g E_C}{\pi D}\right)^g\left\{1-g\frac{\pi^4T^2}{4g^2E_C^2}\frac{1}{\sin^2[\pi T(t-t')]}\right\}. \tag{S2.10}$$

Using the Kubo formula [note, the current operator reads as $\hat{I}=(e/\sqrt{2}\pi)\dot{\phi}_s(0,t)$]

$$G = \frac{e^2T}{2\pi^2 i}\lim_{\omega\to 0}\omega\lim_{i\omega_n\to\omega+i0}\langle\langle\phi_s(i\omega_n)\phi_s(-i\omega_n)\rangle\rangle \tag{S2.11}$$

and generalizing the above perturbative expansion for all terms in Eqs. (S2.2)-(S2.5), we get

$$G-\frac{e^2}{4\pi\hbar} = -\frac{e^2}{16\pi^3\hbar}(2\gamma)^g\frac{\sqrt{\pi}\Gamma[g/2]}{\Gamma\left[1/2+g/2\right]}|r_+|^2\left(\frac{gE_C}{D}\right)^{2g-2}\left(\frac{T}{gE_C}\right)^{g-2}$$
$$-\frac{e^2}{4\pi\hbar}g\frac{\pi^2(2\gamma)^g}{16}\frac{\sqrt{\pi}\Gamma[1+g/2]}{\Gamma\left[3/2+g/2\right]}|r_-|^2\left(\frac{gE_C}{D}\right)^{2g-2}\left(\frac{T}{gE_C}\right)^{g}, \tag{S2.12}$$

where we denote $|r_\pm|^2=\left[|r_1|^2+|r_2|^2\pm 2|r_1||r_2|\cos(2\pi N)\right]$.

In the low-temperature limit $T\ll v/L$, the interaction-induced renormalization is cut-off by finite interaction length L [S2, S3]. One can roughly estimate the differential conductance in the low-temperature regime by replacing $T\to v/L$ in Eq. (S2.12)[S2]. Therefore, the conductance does not depend on the temperature.

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[S2] D. Maslov, Phys. Rev. B **52**, R14368 (1995).
[S3] A. Furusaki, and N. Nagaosa, Phys. Rev. B **54**, R5239 (1996).