Supplemental material to “Multi-stage Kondo effects in a multi-terminal geometry: a quantum lego interferometer”

D. B. Karki

Division of Quantum State of Matter, Beijing Academy of Quantum Information Sciences, Beijing 100193, China

Andrei I. Pavlov and Mikhail N. Kiselev

The Abdus Salam International Centre for Theoretical Physics (ICTP), Strada Costiera 11, I-34151 Trieste, Italy

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In this Supplemental Materials we present additional details for the derivation of key equations. All used notations are in accordance with the main text.

S1. THREE TERMINALS: THE MINIMAL MODEL OF MULTI-STAGE KONDO SCREENING

(a). Three-terminal setup

We consider a three level ($i = 1, 2, 3$) quantum impurity tunnel-coupled to three non-interacting reservoirs ($\alpha = L, M, R$ - Left, Middle and Right). The impurity is described by three-orbital Anderson model (as presented in the main text). The Schrieffer-Wolff transformation of Anderson model eliminates the charge fluctuations among the orbitals resulting in the effective Hamiltonian

$$H_{eff} = \sum_{k} \sum_{\alpha=L,M,R} \sum_{\sigma=\uparrow,\downarrow} \epsilon_k C^{\dagger}_{\alpha k\sigma} C_{\alpha k\sigma} + \sum_{\alpha,\alpha'=L,M,R} \mathcal{J}_{\alpha\alpha'} \mathbf{s}_{\alpha\alpha'} \mathbf{S}, \tag{S1.1}$$

where the matrix of exchange coupling is represented by $\mathcal{J}_{\alpha,\alpha'}$ and the spin density of conduction electron writes

$$\mathbf{s}_{\alpha\alpha'} = \frac{1}{2} \sum_{\sigma,\sigma'} C^{\dagger}_{\alpha k\sigma} \tau_{\sigma\sigma'} C_{\alpha k\sigma'}. \tag{S1.2}$$

Denoting the tunneling elements from the lead α to the i-th orbital by $t_{\alpha i}$, we obtain the exchange matrix

$$J_{\alpha\alpha'} = \frac{2}{SE_C} \begin{pmatrix} |t_{L1}|^2 + |t_{L2}|^2 + |t_{L3}|^2 & t^*_{L1} t_{M1} + t^*_{L2} t_{M2} + t^*_{L3} t_{M3} & t^*_{L1} t_{R1} + t^*_{L2} t_{R2} + t^*_{L3} t_{R3} \\ t^*_{M1} t_{L1} + t^*_{M2} t_{L2} + t^*_{M3} t_{L3} & |t_{M1}|^2 + |t_{M2}|^2 + |t_{M3}|^2 & t^*_{M1} t_{R1} + t^*_{M2} t_{R2} + t^*_{M3} t_{R3} \\ t^*_{R1} t_{L1} + t^*_{R2} t_{L2} + t^*_{R3} t_{L3} & t^*_{R1} t_{M1} + t^*_{R2} t_{M2} + t^*_{R3} t_{M3} & |t_{R1}|^2 + |t_{R2}|^2 + |t_{R3}|^2 \end{pmatrix}. \tag{S1.3}$$

Let us denote the eigenvalues of this matrix as $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$.
Any 3×3 square matrix has three invariants, they are

$$Tr(\mathbb{J}) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 = \frac{2}{\mathcal{S}E_c} \sum_{\alpha=L,M,R} \sum_{i=1}^{3} |t_{\alpha i}|^2, \tag{S1.4}$$

$$\frac{1}{2}\left[(Tr(\mathbb{J}))^2 - Tr(\mathbb{J}^2)\right] = \mathcal{J}_1\mathcal{J}_2 + \mathcal{J}_2\mathcal{J}_3 + \mathcal{J}_1\mathcal{J}_3 = \left(\frac{2}{\mathcal{S}E_c}\right)^2 \sum_{\alpha\beta} \sum_{i\neq j} |t_{\alpha i} t_{\beta j} - t_{\alpha j} t_{\beta i}|^2, \tag{S1.5}$$

$$Det(\mathbb{J}) = \mathcal{J}_1\mathcal{J}_2\mathcal{J}_3 = \left(\frac{2}{\mathcal{S}E_c}\right)^3 \left| \sum_{ijk=1}^{3} \varepsilon^{ijk} t_{Li} t_{Mj} t_{Rk} \right|^2, \tag{S1.6}$$

ε^{ijk} is the antisymmetric Levi-Civita tensor. Eqs. (S1.4-S1.6) are related to the Vieta's formulas for a cubic polynomial. These formulas state that roots r_1, r_2, r_3 of a cubic polynomial $P(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$ satisfy

$$\begin{aligned} r_1 + r_2 + r_3 &= -a, \\ r_1 r_2 + r_1 r_3 + r_2 r_3 &= b, \\ r_1 r_2 r_3 &= -c. \end{aligned}$$

The polynomial $P(\lambda)$ corresponds to the equation for eigenvalues of Eq. (S1.3),

$$\lambda^3 - \lambda^2 \frac{2}{\mathcal{S}E_c}\sum_{\alpha,i}|t_{\alpha i}|^2 + \lambda\left(\frac{2}{\mathcal{S}E_c}\right)^2 \sum_{\alpha\beta}\sum_{i\neq j}|t_{\alpha i}t_{\beta j} - t_{\alpha_j}t_{\beta i}|^2 - \left(\frac{2}{\mathcal{S}E_c}\right)^3 \left|\sum_{ijk}\varepsilon^{ijk}t_{Li}t_{Mj}t_{Rk}\right|^2 = 0.$$

Let us find conditions allowing Eq.(S1.6) to be positive. It allows the matrix Eq.(S1.3) to have three non-zero eigenvalues ($\mathcal{J}_1\mathcal{J}_2\mathcal{J}_3 \neq 0$).
We choose the tunneling coefficients $t_{\alpha i}$ as

$$\begin{aligned} t_{L1} = |t|e^{i\varphi_1},\, t_{L2} = |t|e^{i\varphi_2},\, t_{L3} = |t|e^{i\varphi_3}, \\ t_{M1} = |t|e^{i\varphi_4},\, t_{M2} = |t|e^{i\varphi_5},\, t_{M3} = |t|e^{i\varphi_6}, \\ t_{R1} = t_{R2} = t_{R3} = |t|. \end{aligned} \tag{S1.7}$$

Eq.(S1.6) is now equivalent to

$$Det(\mathbb{J}) \sim |t|^2 \left|e^{i(\varphi_1+\varphi_5)} - e^{i(\varphi_1+\varphi_6)} + e^{i(\varphi_3+\varphi_4)} - e^{i(\varphi_3+\varphi_5)} + e^{i(\varphi_2+\varphi_6)} - e^{i(\varphi_2+\varphi_4)}\right|^2. \tag{S1.8}$$

To satisfy $Det(\mathbb{J}) > 0$, it is enough to choose two phases from different terminals with different i, j indexes as non-zero (e.g. $\varphi_1 \neq 0$ and $\varphi_5 \neq 0$, or other combinations corresponding to sums in the exponents above) with all other phases being zero. Naturally, this condition guaranties that (S1.4) and (S1.5) are also positive. It is easy to see that if all the phases related to either L or M terminal are equal (e.g. $\varphi_1 = \varphi_2 = \varphi_3$), the determinant (S1.8) is zero. If we choose $\varphi_1 \neq 0$, $\varphi_4 \neq 0$ and all other $\varphi_i = 0$, we again obtain $Det(\mathbb{J}) = 0$.

Let us check whether it's possible to have three identical non-zero eigenvalues (i.e. three identical finite Kondo temperatures). This condition is satisfied if all non-diagonal elements of matrix (S1.3) are zero. Choosing the tunneling elements in the form of Eq.(S1.7), we get the following conditions

$$\begin{aligned} e^{i(\varphi_4-\varphi_1)} + e^{i(\varphi_5-\varphi_2)} + e^{i(\varphi_6-\varphi_3)} = 0, \\ e^{-i\varphi_1} + e^{-i\varphi_2} + e^{-i\varphi_3} = 0, \\ e^{-i\varphi_4} + e^{-i\varphi_5} + e^{-i\varphi_6} = 0. \end{aligned}$$

These equations are satisfied for $\varphi_2 = \varphi_5 = 0$, $\varphi_1 = \varphi_6 = -\varphi_3 = -\varphi_4 = \{\frac{2\pi}{3}, \frac{4\pi}{3}\}$, and we get $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}_3 = 3\mathcal{J}_0$ with $\mathcal{J}_0 = \frac{2|t|^2}{\mathcal{S}E_C}$, so it's possible to have three identical Kondo temperatures.
Let us choose

$$\begin{aligned} t_{L1} = |t|e^{i\varphi},\, t_{L2} = |t|,\, t_{L3} = |t|e^{-i\varphi}, \\ t_{M1} = |t|e^{-i\varphi},\, t_{M2} = |t|,\, t_{M3} = |t|e^{i\varphi}, \\ t_{R1} = t_{R2} = t_{R3}, \end{aligned} \tag{S1.9}$$

with arbitrary phase φ. Manipulating this phase, one can have any possible relation between Kondo temperatures. The matrix $\mathbb{J}$ takes form

$$\mathbb{J} = \mathcal{J}_0 \begin{pmatrix} 3 & 1+2\cos 2\phi & 1+2\cos\varphi \\ 1+2\cos 2\varphi & 3 & 1+2\cos\varphi \\ 1+2\cos\varphi & 1+2\cos\varphi & 3 \end{pmatrix}, \tag{S1.10}$$

here we chose $t_{R1} = t_{R2} = t_{R3} = |t|$ (other choices that do not affect eigenvalues of this matrix are possible, see discussion below).

The eigenvalues of this matrix are

$$\mathcal{J}_{1,3} = \frac{\mathcal{J}_0}{2}\left(7 + 2\cos 2\varphi \pm \sqrt{32\cos\varphi + (5+2\cos 2\varphi)^2}\right),\ \ \mathcal{J}_2 = 4\mathcal{J}_0 \sin^2\varphi. \tag{S1.11}$$

In general, these eigenvalues give us three different corresponding non-zero Kondo temperatures, but for specific values of φ we can have $\mathcal{J}_2 = \mathcal{J}_3 = 0$, $\mathcal{J}_1 > 0$ ($\varphi = 0$); $\mathcal{J}_3 = 0$, $\mathcal{J}_{1,2} > 0$ ($\varphi = \pi$); $\mathcal{J}_1 = \mathcal{J}_3 > \mathcal{J}_2 > 0$ ($\varphi = \frac{\pi}{2}$); $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}_3 > 0$ ($\varphi = \frac{2\pi}{3}, \frac{4\pi}{3}$). Eigenvalues $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ as functions of φ are given in Fig.2 of the main text.

Redefining eigenvalue indexes $\{1,2,3\} \to \{e, o1, o2\}$ to have a hierarchy $\mathcal{J}_e > \mathcal{J}_{o1} > \mathcal{J}_{o2}$, we have the corresponding Kondo temperatures $T_K^a \sim D \exp\left(-\frac{1}{2\nu_F \mathcal{J}_a(\varphi)}\right)$, $a = \{e, o1, o2\}$, D is a bandwidth of conduction electrons band, ν_F is the density of states. Note that the choice of the phase for the tunneling constants associated with the third terminal (R) in Eq.(S1.9) does not affect the eigenvalues as long that these tunneling constants have the same phase, since that does not change Eq.(S1.6) (we can choose $t_{R1} = t_{R2} = t_{R3} = |t|$ for simplicity).

S2. ENERGY GAP

Now we write down the wave functions of the electrons in the dot. The dot under consideration has spin $S = 3/2$. The three electrons form 8 states ($2 \times 2 \times 2$): one quartet with total spin 3/2 and two doublets with total state 1/2 each.

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = |\uparrow\uparrow\uparrow\rangle\,,$$

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\right),$$

$$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left(|\downarrow\downarrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle\right),$$

$$\left|\frac{3}{2}, -\frac{3}{2}\right\rangle = |\downarrow\downarrow\downarrow\rangle\,,$$

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle_1 = \frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle\right),$$

$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle_1 = \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle\right),$$

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle_2 = \frac{1}{\sqrt{6}}\left(|\downarrow\uparrow\uparrow\rangle - 2\,|\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\right),$$

$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle_2 = \frac{1}{\sqrt{6}}\left(2\,|\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle\right).$$

Let us find the ground state configuration of the dot. For that, we suppose that we have three spins S_1, S_2, S_3 on a ring interacting via the ferromagnetic Heisenberg Hamiltonian. This interaction allows us to reproduce the Hund's rules in the dot.

$$H = \mathcal{I}(\mathbf{S}_1\mathbf{S}_2 + \mathbf{S}_2\mathbf{S}_3 + \mathbf{S}_1\mathbf{S}_3), \ \ (\mathcal{I} < 0).$$

The scalar product in the brackets reads

$$(\mathbf{S}_1\mathbf{S}_2 + \mathbf{S}_2\mathbf{S}_3 + \mathbf{S}_1\mathbf{S}_3) = \frac{1}{2}\left(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3\right)^2 - \frac{1}{2}\left(\mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2\right).$$

The spin 3/2 (quartet) state and the 1/2 (doublet) states are separated by the gap

$$\Delta = E(S_{total} = 1/2) - E(S_{total} = 3/2) = -\frac{|\mathcal{I}|}{2}\left(\frac{1}{2}\cdot\frac{3}{2} - 3\cdot\frac{3}{4}\right) - \frac{|\mathcal{I}|}{2}\left(\frac{3}{2}\cdot\frac{5}{2} - 3\cdot\frac{3}{4}\right) = \frac{3}{2}|\mathcal{I}|, \tag{S2.1}$$

so the ground state is the quartet, the doublets are excited states separated from it by the gap Δ. This quartet, appearing as the ground state from Eq.(S2.1) corresponds to the ground state of dot considered in the main text.

S3. ROTATION OF ELECTRON STATES

Now we discuss a generalization of the Glazman-Raikh (GR) rotation on the three-terminal case. Let us construct an effective basis with one even and two odd states C_e, C_{o1}, C_{o2} out of three original states of the leads C_L, C_M, C_R. The original GR approach deals with the two-channel case, so the transformation is given by the real-valued matrix $\mathbb{U}_2$ of the $SU(2)$ representation (the rotation matrix in 2D) defined by one parameter α:

$$\begin{pmatrix} C_e \\ C_o \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} C_L \\ C_R \end{pmatrix}. \tag{S3.1}$$

Let us start with a two-terminal two-channels case and find α for this case. There we have 4 tunneling parameters chosen to be real for simplicity. $\mathbb{J}_{2\times 2}$ matrix composed of $\mathcal{J}_{\alpha\alpha'}$ tunneling elements for the two-terminal two-channel case is

$$\mathbb{J}_{2\times 2} = \frac{2}{\mathcal{S}E_c} \begin{pmatrix} t_{L1}^2 + t_{L2}^2 & t_{L1}t_{R1} + t_{L2}t_{R2} \\ t_{L1}t_{R1} + t_{L2}t_{R2} & t_{R1}^2 + t_{R2}^2 \end{pmatrix},$$

and we parametrize tunneling elements as

$$t_{L1} = t\cos\theta\cos\varphi_L; t_{L2} = t\cos\theta\sin\varphi_L; t_{R1} = t\sin\theta\cos\varphi_R; t_{R1} = t\sin\theta\cos\varphi_R.$$

The transformation $\mathbb{U}_2\mathbb{J}_{2\times 2}\mathbb{U}_2^{-1}$ ($\mathbb{U}_2$ is the rotation matrix from Eq.(S3.1)) diagonalizes the $\mathbb{J}_{2\times 2}$ matrix, this condition gives us the relation between angle α and angles θ, φ_L, φ_R:

$$\tan(2\alpha) = \tan(2\theta)\cos(\varphi_L - \varphi_R),$$

so $\alpha = \theta$ when all the tunneling amplitudes are equal.
In addition, we have the following relation for the eigenvalues $\tilde{\mathcal{J}}_1$ and $\tilde{\mathcal{J}}_2$ of the $\mathbb{J}_{2\times 2}$ matrix

$$\tilde{\mathcal{J}}_1 + \tilde{\mathcal{J}}_2 = \mathcal{J}_0,$$
$$\tilde{\mathcal{J}}_1\tilde{\mathcal{J}}_2 = \frac{\mathcal{J}_0^2}{4}\sin^2(2\theta)\sin^2(\varphi_L - \varphi_R).$$

where $\mathcal{J}_0 = 2t^2/E_c$ for $\mathcal{S} = 1$.

A general real-valued matrix of the $SU(3)$ representation is characterized by three angles α, β, γ, this matrix is composed of the eigenvector for the symmetric mode and modes orthogonal to it [S1–S3]:

$$\begin{pmatrix} C_e \\ C_{o1} \\ C_{o2} \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha & \cos\alpha\sin\beta \\ \sin\beta\sin\gamma - \sin\alpha\cos\beta\cos\gamma & \cos\alpha\cos\gamma & -\cos\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma \\ -\sin\alpha\cos\beta\sin\gamma - \sin\beta\cos\gamma & \cos\alpha\sin\gamma & \cos\beta\cos\gamma - \sin\alpha\sin\beta\sin\gamma \end{pmatrix} \begin{pmatrix} C_L \\ C_M \\ C_R \end{pmatrix}. \tag{S3.2}$$

γ defines a rotation of two mutually orthogonal "odd" states in a plane perpendicular to the even state (any two mutually orthogonal vectors belonging to the plane can be chosen as C_{o1} and C_{o2}). We choose $\gamma = \frac{\pi}{2}$ for simplicity (so the C_{o1} and C_{o2} states become antisymmetric), the generalized GR transformation then reads

$$\begin{pmatrix} C_e \\ C_{o1} \\ C_{o2} \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha & \cos\alpha\sin\beta \\ \sin\beta & 0 & -\cos\beta \\ -\sin\alpha\cos\beta & \cos\alpha & -\sin\alpha\sin\beta \end{pmatrix} \begin{pmatrix} C_L \\ C_M \\ C_R \end{pmatrix}. \tag{S3.3}$$

Other choices of γ are possible. For instance, putting $\gamma = 0$, we obtain the same matrix of Eq.(S3.3) with flipped second and third lines (the new third line additionally changes signs). In a general case of the arbitrary γ angle, we get a matrix with its lines being linear combinations of the lines (S3.3).
In general, the angles α, β, γ are expressed via the Euler angles and incorporate the asymmetries between the channels, the matrix in Eq.(S3.2) is a rotation matrix. It can be obtained by the rotation $R_z(\gamma)R_y(-\alpha)R_x(\beta)$, up to cyclic permutations, which mean simply relabelling of the axes. R_i are the rotation operators around the corresponding axis i on the angle given angle.

For $|t_L| = |t_M| = |t_R|$, we have $\alpha = \arcsin\frac{1}{\sqrt{3}}$, $\beta = \frac{\pi}{4}$, so the transformation becomes

$$\begin{pmatrix} C_e \\ C_{o1} \\ C_{o2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} c_L \\ c_M \\ c_R \end{pmatrix}.$$

The three terminals L, M, R are equivalent, but the parametrization (S3.2) breaks the symmetry between them, so the cyclic permutations of the leads are possible: $L \to M \to R$ [S4]. For instance, the permutation $\{R \to L,\, L \to M,\, M \to R\}$ gives

$$\begin{pmatrix} C_e \\ \tilde{C}_{o1} \\ \tilde{C}_{o2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} C_L \\ C_M \\ C_R \end{pmatrix}.$$

We will use this representation further throughout the paper. The observables do not depend on our choice of parametrization, so they must be averaged over the γ-angle.

(a). Phase tunable regime

Now, let us consider a general case when the tunneling coefficients are complex and depend of the Aharonov-Bohm phases (S1.9). The eigenvectors corresponding to the three non-zero eigenvalues (S1.11) cannot be chosen to be the phase-independent in the three-terminal setup. The corresponding matrix $\mathbb{U}$ that diagonalizes Eq.(S1.3) is composed of the eigenvectors of $\mathbb{J}$ and reads

$$\mathbb{U}(\varphi) = \begin{pmatrix} v_{11}(\varphi) & v_{12}(\varphi) & v_{13}(\varphi) \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ v_{31}(\varphi) & v_{32}(\varphi) & v_{33}(\varphi) \end{pmatrix},$$

where

$$\begin{aligned} v_{11}(\varphi) = v_{12}(\varphi) &\equiv \frac{-1 + 2\cos\varphi + \sqrt{11 - 4\cos\varphi + 2\cos 2\varphi}}{4\sqrt{1 + \frac{1}{8}\left(-1 + 2\cos\varphi + \sqrt{11 - 4\cos\varphi + 2\cos 2\varphi}\right)^2}}, \\ v_{13}(\varphi) &\equiv \frac{1}{\sqrt{1 + \frac{1}{8}\left(-1 + 2\cos\varphi + \sqrt{11 - 4\cos\varphi + 2\cos 2\varphi}\right)^2}}, \\ v_{31}(\varphi) = v_{32}(\varphi) &\equiv \frac{-1 + 2\cos\varphi - \sqrt{11 - 4\cos\varphi + 2\cos 2\varphi}}{4\sqrt{1 + \frac{1}{8}\left(-1 + 2\cos\varphi - \sqrt{11 - 4\cos\varphi + 2\cos 2\varphi}\right)^2}}, \\ v_{33}(\varphi) &\equiv \frac{1}{\sqrt{1 + \frac{1}{8}\left(-1 + 2\cos\varphi - \sqrt{11 - 4\cos\varphi + 2\cos 2\varphi}\right)^2}}. \end{aligned}$$

This matrix is identical to Eq.(7) of the main text.

Note that due to presence of the phases in particular tunneling elements, the symmetry between the terminals is broken. In particular, choice (S1.9) makes terminal R different from terminals L and M.

(b). Tunneling amplitudes tunable regime

To ensure that Eq.(S1.6) is positive (i.e. there are three non-zero Kondo temperatures), one does not have to alter phases of the tunneling elements, since the same effect can be achieved by changing amplitudes of these elements. In general, tunneling processes of N-terminal M-channel Kondo problem are parametrized by $2 \times N \times M$ real parameters. The condition for N nonzero Kondo temperatures ($det(\mathbb{J}) \neq 0$) puts a constrain on a minimal number of parameters that must be tuned (i.e. a certain number of symmetries must be broken). On the other hand, if one wants to have a system where the Kondo effect splits into symmetric and antisymmetric channels, one has to impose a number of constrains that reduce the number of adjustable parameters. Effectively, there is a K-dimensional surface in the $2 \times N \times M$-dimensional parametric space ($0 < K < 2 \times N \times M$) which satisfies the necessary conditions.

Let us illustrate it for the 3-terminal 3-channel Kondo problem under consideration. There are 18 parameters which we reparametrize as

$$
\begin{aligned}
t_{L1} &= t\sin\theta_1\sin\theta_2\sin\phi_{La}\sin\phi_{Lb}e^{i\psi_{L1}},\\
t_{L2} &= t\sin\theta_1\sin\theta_2\cos\phi_{La}e^{i\psi_{L2}},\\
t_{L3} &= t\sin\theta_1\sin\theta_2\sin\phi_{La}\cos\phi_{Lb}e^{i\psi_{L3}},\\
t_{M1} &= t\cos\theta_1\sin\phi_{Ma}\sin\phi_{Mb}e^{i\psi_{M1}},\\
t_{M2} &= t\cos\theta_1\cos\phi_{Ma}e^{i\psi_{M2}},\\
t_{M3} &= t\cos\theta_1\sin\phi_{La}\cos\phi_{Mb}e^{i\psi_{M3}},\\
t_{R1} &= t\sin\theta_1\cos\theta_2\sin\phi_{Ra}\sin\phi_{Rb}e^{i\psi_{R1}},\\
t_{R2} &= t\sin\theta_1\cos\theta_2\cos\phi_{Ra}e^{i\psi_{R2}},\\
t_{R3} &= t\sin\theta_1\cos\theta_2\sin\phi_{Ra}\cos\phi_{Rb}e^{i\psi_{R3}},
\end{aligned}
\tag{S3.4}
$$

$$
\begin{aligned}
\sin\phi_{Lb} = \frac{|t_{L1}|}{\sqrt{|t_{L1}^2|+|t_{L3}^2|}},\ \sin\phi_{Mb} = \frac{|t_{M1}|}{\sqrt{|t_{M1}^2|+|t_{M3}^2|}},\ \sin\phi_{Rb} = \frac{|t_{R1}|}{\sqrt{|t_{R1}^2|+|t_{R3}^2|}},\\
t_L = \sqrt{|t_{L1}^2|+|t_{L2}^2|+|t_{L3}^2|},\ t_M = \sqrt{|t_{M1}^2|+|t_{M2}^2|+|t_{M3}^2|},\ t_R = \sqrt{|t_{R1}^2|+|t_{R2}^2|+|t_{R3}^2|},\\
\sin\phi_{La} = \frac{\sqrt{|t_{L1}^2|+|t_{L3}^2|}}{t_L},\ \sin\phi_{Ma} = \frac{\sqrt{|t_{M1}^2|+|t_{M3}^2|}}{t_M},\ \sin\phi_{Ra} = \frac{\sqrt{|t_{R1}^2|+|t_{R3}^2|}}{t_R},\\
\sin\theta_2 = \frac{t_L}{\sqrt{t_L^2+t_R^2}},\ t = \sqrt{t_L^2+t_M^2+t_R^2},\ \sin\theta_1 = \frac{\sqrt{t_L^2+t_R^2}}{t},\\
\theta_1\,\theta_2,\ \phi_{La},\ \phi_{Lb},\ \phi_{Ma},\ \phi_{Mb},\ \phi_{Ra},\ \phi_{Rb}\in[0,\frac{\pi}{2}].
\end{aligned}
$$

Parameter t defines the absolute values of the Kondo temperatures but does not affect the rotation matrix $\mathbb{U}$. Let us choose all tunneling phases $\psi_{\alpha,i}$ to be zero. The most symmetric case (with all terminals and all channels being identical) corresponds to values $\theta_1 = \arccos\frac{1}{\sqrt{3}}$, $\theta_2 = \frac{\pi}{4}$, $\phi_{\alpha a} = \arccos\frac{1}{\sqrt{3}}$, $\phi_{\alpha b} = \frac{\pi}{4}$, $\alpha = \{L, M, R\}$. Now we break the symmetry between different channels in two terminals and introduce an asymmetry parameter φ so that

$$
\phi_{Lb} = \frac{\pi}{4}+\varphi,\ \phi_{Mb} = \frac{\pi}{4}-\varphi,\ \phi_{Rb} = \frac{\pi}{4}.
$$

Plugging tunneling elements (S3.4) into Eq.(S1.3), we exactly reproduce the $\mathbb{J}$ matrix (S1.10) and all the further calculations become identical to the case we considered above, where instead of asymmetry in tunneling amplitudes, asymmetry in tunneling phases was introduced.
We have here

$$
\begin{aligned}
\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3 &= 9\mathcal{J}_0;\\
\mathcal{J}_1\mathcal{J}_2+\mathcal{J}_1\mathcal{J}_3+\mathcal{J}_2\mathcal{J}_3 &= 2\mathcal{J}_0^2\left(9-4\cos\varphi-4\cos2\varphi-\cos4\varphi\right);\\
\mathcal{J}_1\mathcal{J}_2\mathcal{J}_3 &= 64\mathcal{J}_0^3\sin^4\left(\frac{\varphi}{2}\right)\sin^2\varphi.
\end{aligned}
$$

Note that we define here $\mathcal{J}_0 = 2|t|^2/(\mathcal{S}E_c) = 2t^2/(9\mathcal{S}E_c)$. Also angles α, β and γ acquire dependence on φ.

S4. STRONG-COUPLING FIXED POINT

The strong coupling fixed point Hamiltonian contains six leading irrelevant operators [S5]:

$$
H = -\sum_{i=1}^{3}\lambda_i :\vec{s}_i(0)\cdot\vec{s}_i(0): -\sum_{i=1}^{3}\sum_{j\neq i}^{3}\lambda_{ij} :\vec{s}_i(0)\cdot\vec{s}_j(0):
$$

There are two important limiting cases. The first case corresponds to the situation when all three eigenvalues of the exchange matrix (and therefore all three Kondo temperatures) are equal.

- $\lambda_1=\lambda_2=\lambda_3=\lambda_{12}=\lambda_{23}=\lambda_{13}=\Lambda$

$$H=-\Lambda:\left(\vec{s}_1(0)+\vec{s}_2(0)+\vec{s}_3(0)\right)^2:$$

This is the case of total destructive interference. The net current through the system is zero.

The second important limiting case corresponds to degeneracy of two eigenvalues corresponding to two orthogonal anti-symmetric modes.

- $\lambda_2=\lambda_3=\lambda_{23}=\Lambda_2$, $\lambda_{12}=\lambda_{13}=\Lambda_{12}$, $\lambda_1=\Lambda_1$

$$H=-\Lambda_1:\left(\vec{s}_1(0)\right)^2:-\Lambda_2:\left(\vec{s}_2(0)+\vec{s}_3(0)\right)^2:-2\Lambda_{12}:\vec{s}_1(0)\cdot\left(\vec{s}_2(0)+\vec{s}_3(0)\right):$$

This case corresponds to the two-stage Kondo effect. However, unlike conventional 2SK, the screening at the first stage is done by two orbital channels such a way that spin $S=1$ is screened first and $s=1/2$ is screened at the second stage.

The most general form of the low-energy FL Hamiltonian for the three-stage Kondo problem corresponding to the particle-hole symmetric limit of the three-orbital-level Anderson model is given by $H=H_0+H_\alpha+H_\phi+H_\Phi$ with $i,j=e,o1,o2$:

$$\begin{aligned}
H_0 &= \sum_{i\sigma}\int_\varepsilon \nu\left(\varepsilon+\varepsilon^Z_\sigma\right) b^\dagger_{i\varepsilon\sigma}b_{i\varepsilon\sigma}\\
H_\alpha &= -\sum_{i\sigma}\int_{\varepsilon_{1-2}} \frac{\alpha_i}{2\pi}\left(\varepsilon_1+\varepsilon_2\right) b^\dagger_{i\varepsilon_1\sigma}b_{i\varepsilon_2\sigma}\\
H_\phi &= \sum_{i}\int_{\varepsilon_{1-4}} \frac{\phi_i}{\pi\nu} :b^\dagger_{i\varepsilon_1\uparrow}b_{i\varepsilon_2\uparrow}b^\dagger_{i\varepsilon_3\downarrow}b_{i\varepsilon_4\downarrow}:\\
H_\Phi &= -\sum_{ij\sigma_{1-4}}\int_{\varepsilon_{1-4}} \frac{\Phi_{ij}}{2\pi\nu} :b^\dagger_{i\varepsilon_1\sigma_1}\boldsymbol{\tau}_{\sigma_{12}}b_{i\varepsilon_2\sigma_2}b^\dagger_{j\varepsilon_3\sigma_3}\boldsymbol{\tau}_{\sigma_{34}}b_{j\varepsilon_4\sigma_4}:,
\end{aligned}\tag{S4.1}$$

where $\alpha_i=\phi_i$ in accordance with Nozieres theory. The six-parametric strong coupling fixed point Hamiltonian (S4.1) accounts for both elastic and inelastic processes. Finite temperature conductance behaviour is controlled by three Kondo temperatures $T^i_K\propto\lambda_i^{-1}$ and three additional parameters $\mathcal{F}_{ij}\propto(\lambda_i-\lambda_{ij})(\lambda_j-\lambda_{ij})/(\lambda_i-\lambda_j)^2$ in full accordance with [S5].

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