Quantum thermal transport in the charged Sachdev-Ye-Kitaev model: thermoelectric Coulomb blockade

Supplemental Material

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We adopt the notations and definitions of the main text and use the numeration of equations and references of the *Letter*.

DERIVATION OF G, G_T AND κ

In accordance with Eq.(1), the transport coefficients can be found by taking derivatives of Eq.(4) by voltage at uniform temperature (coefficient G) and by temperature at zero voltage (coefficient G_T). Here we perform this procedure explicitly.

Let us start with the conductance G. It is defined as

$$G = \left.\frac{I}{\Delta V}\right|_{\Delta V=0} = 2\pi \int_{-\infty}^{\infty} d\varepsilon \rho_a(\epsilon)\rho_c(\epsilon)\partial_{\Delta V}\left(f(\epsilon+\Delta V, T+\Delta T) - f(\epsilon, T)\right)\Big|_{\Delta V=0}. \tag{S1.1}$$

$\rho_a(\epsilon) = (2\pi v_F)^{-1}$ (see the main text), so this expression is further simplified as

$$G = -\frac{1}{4 v_F T}\int_{-\infty}^{\infty} d\epsilon \frac{\rho_c(\epsilon)}{\cosh^2\left(\frac{\epsilon}{2T}\right)}. \tag{S1.2}$$

It was shown in [28] that the density of states $\rho_c(\epsilon)$ can be expressed in a general form via the T-matrix $\mathcal{T}$ in the real time representation

$$\rho_c(\epsilon) = -\frac{1}{\pi}\cosh\left(\frac{\epsilon}{2T}\right)\int_{-\infty}^{\infty} dt \mathcal{T}\left(\frac{1}{2T}+it\right)e^{i\epsilon t}. \tag{S1.3}$$

Combining Eqs.(S1.2) and (S1.3), we have

$$G = \frac{1}{4\pi v_F T}\int_{-\infty}^{\infty} dt \mathcal{T}\left(\frac{1}{2T}+it\right)\int_{-\infty}^{\infty} d\epsilon \frac{e^{i\epsilon t}}{\cosh\left(\frac{\epsilon}{2T}\right)} = \frac{1}{2v_F}\int_{-\infty}^{\infty}\frac{dt}{\cosh(\pi T t)}\mathcal{T}\left(\frac{1}{2T}+it\right). \tag{S1.4}$$

G_T is obtained in the same way.

$$G_T = \left.\frac{I}{\Delta T}\right|_{\Delta T=0} = 2\pi \int_{-\infty}^{\infty} d\varepsilon \rho_a(\epsilon)\rho_c(\epsilon)\partial_{\Delta T}\left(f(\epsilon, T+\Delta T) - f(\epsilon, T)\right)\Big|_{\Delta T=0}. \tag{S1.5}$$

Plugging DOS Eq.(S1.3) in Eq.(S1.5), we obtain

$$G_T = \frac{i}{4 v_F \pi T^2}\int_{-\infty}^{\infty} dt \mathcal{T}(\frac{1}{2T}+it)\frac{\partial}{\partial t}\frac{2\pi T}{\cosh(\pi T t)} = -\frac{i\pi}{2v_F}\int_{-\infty}^{\infty} dt \frac{\sinh(\pi T t)}{\cosh^2(\pi T t)}\mathcal{T}\left(\frac{1}{2T}+it\right). \tag{S1.6}$$

By treating the thermal current in the similar way, we get the coefficient K and the thermal conductance κ.

$$K = -\frac{1}{4 v_F T}\int_{-\infty}^{\infty} dt \mathcal{T}\left(\frac{1}{2T}+it\right)\frac{\partial^2}{\partial t^2}\frac{1}{\cosh(\pi T t)} = -\frac{\pi^2 T}{2 v_F}\int_{-\infty}^{\infty} dt \mathcal{T}\left(\frac{1}{2T}+it\right)\left\{\frac{1}{\cosh(\pi T t)} - \frac{2}{\cosh^3(\pi T t)}\right\}, \tag{S1.7}$$

here we used the identity $\sinh^2 x \equiv \cosh^2 x - 1$.

$$\kappa = K - \frac{1}{T}\frac{G_T^2}{G} = \frac{\pi^2 T}{v_F}\int_{-\infty}^{\infty} dt \mathcal{T}\left(\frac{1}{2T}+it\right)\frac{1}{\cosh^3(\pi T t)} - T\pi^2 G - \frac{1}{T}\frac{G_T^2}{G} \tag{S1.8}$$

These expressions are Eqs. (5-7) from the main text.

CORRELATORS

Two-point Coulomb correlator $D(\tau_1,\tau_2)$

Let us reproduce here the derivation on then so-called Coulomb boson correlator arising due to averaging of the $U(1)$ gauge field:

$$D(\tau_1,\tau_2)=\left\langle e^{-i\phi(\tau_1)}e^{i\phi(\tau_2)}\right\rangle_{\phi}. \tag{S2.1}$$

The Green's function antiperiodicity condition $G(\frac{\beta}{2})=-G(-\frac{\beta}{2})$ imposes that $\phi(\frac{\beta}{2})=\phi(-\frac{\beta}{2})+-2\pi i\mathcal{E}+2\pi W$, where $\mathcal{E}$ is the spectral asymmetry, W in the winding number. We decompose the ϕ field such that $\phi(\tau)=\eta(\tau)+2\pi(W-i\mathcal{E})T\tau$ introducing a periodic function $\eta(\tau)$: $\eta(\frac{\beta}{2})=\eta(-\frac{\beta}{2})$. The correlator (S2.1) now reads

$$D(\tau_1,\tau_2)=\frac{1}{Z_C}\sum_{W\in Z}\int D[\eta]e^{-i\eta(\tau_1)}e^{i\eta(\tau_2)}e^{-2\pi i(W-\mathbf{i}\mathcal{E})T(\tau_1-\tau_2)}e^{-\int d\tau\eta'(\tau)\frac{1}{4E_C}\eta'(\tau)-\pi^2T\frac{(W-i\mathcal{E})^2}{E_C}} \tag{S2.2}$$

$$=\left\langle e^{-i\eta(\tau_1)}e^{i\eta(\tau_2)}\right\rangle_{\eta}\left\langle e^{-2\pi i(W-i\mathcal{E})T(\tau_1-\tau_2)}\right\rangle_{W}, \tag{S2.3}$$

Z_C is the partition sum which normalizes the correlator

We start with averaging over the η fields (we use here the Fourier image $\eta_m=\frac{1}{\beta}\int_0^\beta d\tau\eta_\tau e^{i\omega_m\tau}$, $\omega_m=2\pi Tm$):

$$\langle ...\rangle_\eta=\int D[\eta]e^{i\int_{\tau_1}^{\tau_2}d\tau\eta'(\tau)}e^{-\int d\tau\eta'(\tau)\frac{1}{4E_C}\eta'(\tau)}=\int D[\eta]e^{\beta\sum_{m\neq0}\left(\frac{\omega_m^2\eta_{-m}\eta_m}{4E_C}-i\eta_m J_{-m}^{\tau_1;\tau_2}\right)}, \tag{S2.4}$$

where $J_m^{\tau_1,\tau_2}=e^{i\omega_m\tau_1}-e^{i\omega_m\tau_2}$ is the Fourier image of $\delta(\tau-\tau_1)-\delta(\tau-\tau_2)$. The resulting integral over η is Gaussian, it gives

$$\langle ...\rangle_\eta=e^{-\beta\sum_{m\neq0}\frac{E_C}{\omega_m^2}J_{-m}^{\tau_1,\tau_2}J_m^{\tau_1,\tau_2}},\qquad qJ_{-m}^{\tau_1,\tau_2}J_m^{\tau_1,\tau_2}=2\left(1-e^{i\omega_m(\tau_2-\tau_1)}\right) \tag{S2.5}$$

This $\langle ...\rangle_\eta$ correlator was discussed in [72–76]. Using the Sommerfeld-Watson transformation for Eq.(S2.4), one obtains

$$\langle ...\rangle_\eta=e^{-E_C(|\tau_2-\tau_1|-\frac{(\tau_2-\tau_1)^2}{\beta})}. \tag{S2.6}$$

The other part of the Coulomb correlator, namely contribution from the winding numbers was evaluated in [72]. Using the Poisson formula

$$\sum_{k=-\infty}^{\infty}e^{-\frac{a}{2}k^2+ixk}=\sqrt{\frac{2\pi}{a}}\sum_{n=-\infty}^{\infty}e^{-\frac{1}{2a}(x-2\pi n)^2}, \tag{S2.7}$$

one can evaluate the partition sum Z_C and the two-point propagator $\langle e^{2\pi iT(W-i\mathcal{E})(\tau_2-\tau_1)}\rangle_W$.

$$Z_C=\sum_W e^{-\frac{4\pi^2T(W-i\mathcal{E})^2}{4E_C}}=\sum_m e^{-\frac{E_C}{T}m^2+2\pi m\mathcal{E}}, \tag{S2.8}$$

$$\langle ...\rangle_W=\frac{1}{Z_C}\sum_m e^{-\frac{\pi^2T(W-i\mathcal{E})^2}{E_C}+2\pi iT(W-i\mathcal{E})(\tau_2-\tau_2)}=\frac{1}{Z_C}\sum_m e^{-\beta E_C(m-(\tau_2-\tau_1)T)^2+2\pi m\mathcal{E}}. \tag{S2.9}$$

The correlators above are normalized by corresponding partition sums. Quadratic terms in the exponents of both correlators (S2.6) and (S2.9) cancel each other, so the result is

$$D(\tau_1,\tau_2)=\frac{e^{-E_C|\tau_2-\tau_1|}}{Z_C}\sum_{m=-\infty}^{\infty}e^{-2E_Cm(\tau_2-\tau_1)-\beta E_Cm^2+2\pi m\mathcal{E}}. \tag{S2.10}$$

Four-point Coulomb correlator

Now we consider the same procedure applied to the four point Coulomb correlator

$$F(\tau_1,\tau_2,\tau_3,\tau_4)=\left\langle e^{-i\phi(\tau_1)}e^{i\phi(\tau_2)}e^{-i\phi(\tau_3)}e^{i\phi(\tau_4)}\right\rangle_\phi. \tag{S2.11}$$

The approach is analogous to the former (two-point) case. Decomposing $\phi(\tau)$ field into the periodic field $\eta(\tau)$ and the winding number contribution, we get factorization of two propagators.

$$\left\langle e^{\beta\sum_{m\neq 0}\omega_m\eta_m\left(\int_{\tau_1}^{\tau_2}d\tau e^{i\omega_m\tau}+\int_{\tau_3}^{\tau_4}d\tilde{\tau}e^{i\omega_m\tilde{\tau}}\right)}\right\rangle_\eta=e^{-\beta\sum_{m\neq 0}\frac{E_C}{\omega_m^2}\left(J_{-m}^{\tau_1;\tau_2}+J_{-m}^{\tau_3;\tau_4}\right)\left(J_m^{\tau_1;\tau_2}+J_m^{\tau_3;\tau_4}\right)}. \tag{S2.12}$$

$$\left(J_m^{\tau_1;\tau_2}+J_m^{\tau_3;\tau_4}\right)=2\left(1-e^{i\omega_m\tau_{41}}\right)+2\left(1-e^{i\omega_m\tau_{32}}\right)+2e^{i\omega_m\tau_{31}}-2e^{i\omega_m\tau_{21}}+2e^{i\omega_m\tau_{42}}-2e^{i\omega_m\tau_{43}}, \tag{S2.13}$$

where we denoted $\tau_{ij}\equiv\tau_i-\tau_j$.

$$\langle\ldots\rangle_\eta=e^{-E_C(|\tau_{41}|+|\tau_{32}|-|\tau_{31}|-|\tau_{42}|+|\tau_{21}|+|\tau_{43}|)+\frac{E_C}{\beta}\left(\tau_{41}^2+\tau_{32}^2-\tau_{31}^2-\tau_{42}^2+\tau_{21}^2+\tau_{43}^2\right)} \tag{S2.14}$$

Averaging over winding numbers, we get

$$\left\langle e^{2\pi i(W-i\mathcal{E})T(\tau_{21}+\tau_{43})}\right\rangle_W=\frac{1}{Z_C}\sum_m e^{-\beta E_C(m-(\tau_{21}+\tau_{43})T)^2+2\pi m\mathcal{E}}. \tag{S2.15}$$

The quadratic terms in these two correlators cancel each other, so the resulting four-point function is

$$F(\tau_1,\tau_2,\tau_3,\tau_4)=\frac{1}{Z_C}\sum_m e^{-\frac{E_C}{T}m^2-E_C(|\tau_{41}|+|\tau_{32}|-|\tau_{31}|-|\tau_{42}|+|\tau_{21}|+|\tau_{43}|)+E_Cm(\tau_{21}+\tau_{43})+2\pi m\mathcal{E}}. \tag{S2.16}$$

Let us choose some particular time ordering, for instance, $\tau_1>\tau_4>\tau_2>\tau_3$. In this case, the correlator becomes

$$F(\tau_1,\tau_2,\tau_3,\tau_4)=\frac{1}{Z_C}\sum_m e^{-\frac{E_C}{T}m^2+E_C(\tau_{14}+\tau_{23})(2m-1)+2\pi m\mathcal{E}}. \tag{S2.17}$$

The main contribution to this sum comes from $m=0$. This term was discussed in [62] and reads

$$e^{-E_C(\tau_{14}+\tau_{23})}. \tag{S2.18}$$

Note that only certain time orderings are relevant for the inelastic co-tunneling process. Namely, $\tau_1,\tau_4>\tau_2,\tau_3$ or $\tau_2,\tau_3>\tau_1,\tau_4$ are relevant, while other orderings (e.g. $\tau_1,\tau_2>\tau_3,\tau_4$) correspond to two sequential direct tunnelings ([79]). For all 8 possible relevant time orderings, (S2.16) simplifies to

$$F(\tau_1,\tau_2,\tau_3,\tau_4)=\frac{1}{Z_C}\sum_m e^{-\frac{E_C}{T}m^2-E_C(|\tau_{41}|+|\tau_{32}|)+E_Cm(\tau_{41}+\tau_{23})+2\pi m\mathcal{E}}. \tag{S2.19}$$

G and G_T in the small tunneling approximation are proportional to

$$\int d^4\tau e^{-E_C(\tau_{14}+\tau_{23})}\left\langle G_{\tau_1,\tau_2}[h]G_{\tau_3,\tau_4}[h]\right\rangle_h, \tag{S2.20}$$

so the contributions from times $\tau_4\neq\tau_1$ and $\tau_3\neq\tau_2$ are exponentially suppressed. This allows to approximate

$$\left\langle G_{\tau_1,\tau_2}[h]G_{\tau_3,\tau_4}[h]\right\rangle_h\simeq\left\langle G_{\tau_1,\tau_2}[h]G_{\tau_2,\tau_1}[h]\right\rangle_h \tag{S2.21}$$

in the integral above.

We are interested in terms with non-zero m, which are exponentially small at low temperatures ($T\ll E_C$) comparing to (S2.18), so we can consider only $m=\pm 1$ terms. Integrating Eq.(S2.20) for various time orderings, one obtains a time-independent constant in the leading order. This term is dominant for conductivity, but first non-vanishing contribution to thermal conductivity comes from next terms, proportional to $e^{-E_C\tau_{12}+2\pi\mathcal{E}}$ and $e^{E_C\tau_{12}-2\pi\mathcal{E}}$.

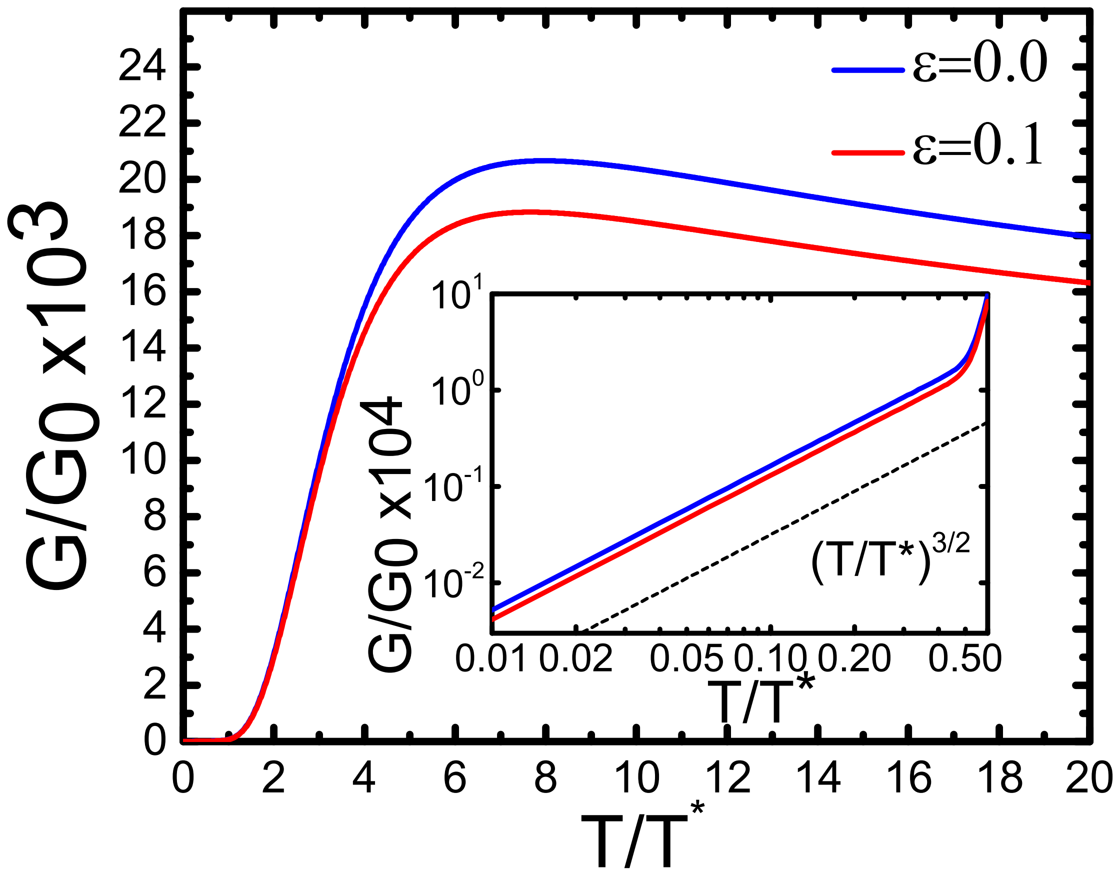

FIG. S1. Electric conductance G accounting for elastic and inelastic co-tunneling in the conformal regime, $E_C/T^* = 10$, $N = 50$, $G_0 = \frac{(T^*)^2}{v_F}$, $(\lambda/T^*)^2 = 0.03$. The lines correspond to $\mathcal{E} = 0$ (blue) and $\mathcal{E} = 0.1$ (red). Inset: $Ln-ln$ plot for electric conductance G in the Schwarzian regime for the same values of $\mathcal{E}$.

ELECTRIC CONDUCTANCE G

Here we provide our results for the electric conductance G obtained by evaluation of Eq. (5). Fig. S1 shows G accounting for both elastic and inelasic processes. In the conformal regime, electric conductance scales as $G \sim \frac{1}{\sqrt{T}}$ for $T \gg E_C$ (this scaling comes from direct tunneling), while $G \sim T$ at $T^* \ll T \ll E_C$ (it stems from inelastic co-tunneling). The direct tunneling contribution becomes dominant with increase of temperature, so the intermediate region with dominant direct tunneling $G \sim e^{-E_C/T}$ is seen there at $T \simeq E_C$. The inset demonstrates the electric conductance in the Schwarzian regime of the theory $T \ll T^* \ll E_C$, here G scales as $G \sim T^{3/2}$, this scaling stems from the inelastic co-tunneling. These results are in agreement with [60, 62]. They are further used for evaluation of the thermopower $\mathcal{S}$ Eq. (12) of the main text.

THERMOPOWER $\mathcal{S}$ AS A FUNCTION OF THE SPECTRAL ASYMMETRY PARAMETER $\mathcal{E}$

As discussed in the main text, the thermopower $\mathcal{S}$ is antisymmetric in the spectral asymmetry parameter $\mathcal{E}$. $\mathcal{S}$ is linear in the leading order of $\mathcal{E}$ close to the particle-hole symmetric point ($\mathcal{E} \ll 1$). This behavior of the thermopower (multiplied on the electric charge e to form dimensionless units) is plotted in Fig. S2. Note the scale of the inset showing the thermopower in the Schwarzian regime - $\mathcal{S}$ is exponentially suppressed by temperature in accordance with Eq. (12).

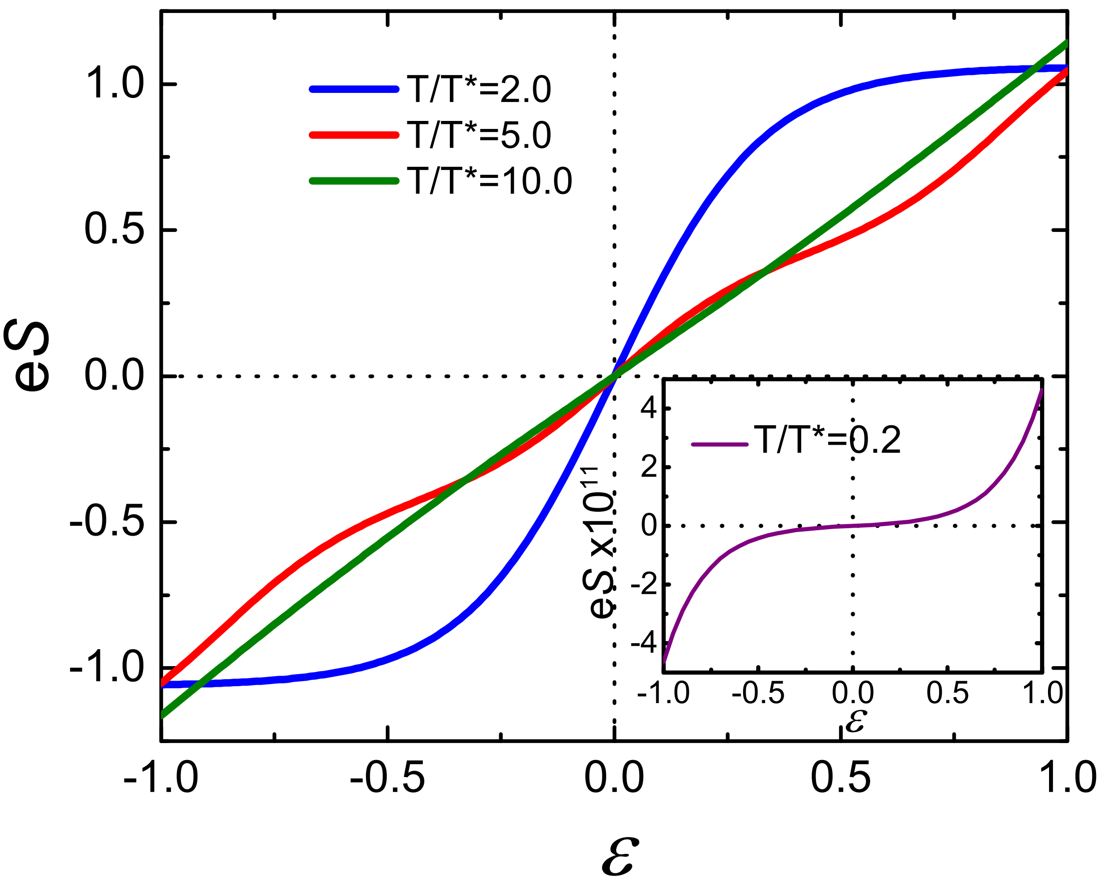

FIG. S2. Thermopower $\mathcal{S}$ accountable for elastic and inelastic co-tunneling in the conformal regime as a function of $\mathcal{E}$, $N = 50$, $E_C/T^* = 10$, $(\lambda/T^*)^2 = 0.03$. $T/T^* = 2$ (blue line), $T/T^* = 5$ (red line), $T/T^* = 10$ (green line). Inset: Thermopower $\mathcal{S}$ in the Schwarzian regime, $T/T^* = 0.2$.