

# Superconductivity in systems with excitonic instability

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Received 2 October 1992

**Abstract.** This paper discusses the nonphonon superconductivity mechanism related to the high polarizability of the medium, with the polarizability being of an electronic nature. As an example of a system with such properties the authors consider the model of an excitonic insulator with doping, which is a two-band crystal with a narrow band gap. Assuming that the system is close to the transition to the state of an excitonic insulator in the excitonic and non-excitonic phases, they show that at low frequencies the dielectric constant exhibits a strong dispersion and that the highly nonideal Fermi liquid of the doping electrons possesses superconductive properties.

## 1. Introduction

The superconductivity of semiconducting compounds is a constant object of both theoretical and experimental studies, as can be seen by the recent discovery of superconductivity with  $T_c \sim 10$  K in heavily doped GaAs [1]. This value of  $T_c$  is, apparently, too high to be explained by the traditional electron–phonon mechanism. The present paper considers a possible variant of the nonphonon superconductivity mechanism, which in principle can be realized in systems of the semiconductor III–V group, whose representative GaAs is. All these systems, apparently, can be assumed to be close to the transition to the state of an excitonic insulator [2, 3], and a large number of papers are devoted to a theoretical study of the properties of such an insulator [2–8]. If dopants are added to a system that is close to the transition to the excitonic-insulator state, the free electrons emerging as a result of doping begin to strongly interact with the easily polarizable medium consisting of real or virtual excitons, which may lead to effective attraction between these electrons, that is, superconductivity.

Study of the superconductivity mechanism in systems close to an excitonic transition is also of interest for the problem of high- $T_c$  superconductivity. Several researchers (see [9–14]) have attempted to explain the normal and superconducting properties of copper-oxide superconductors by the effect of the excitonic subsystem.

This paper considers the case where there is no excitonic phase in the system but the polarizability is high and the case when a Bose gas of excitons interacting with free charge carriers is formed.

## 2. Effective action in the two-band model

We discuss the model of a crystal with two non-intersecting bands whose extrema are at a single point of the Brillouin band (figure 1). Let us assume that the energy

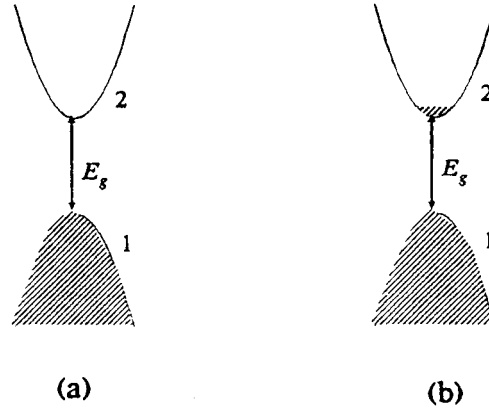


Figure 1. The two-band model: (a) in the absence of doping, and (b) with doping of the upper band by electrons.

spectrum of the initial system is known: in the lower band it is  $\varepsilon_1(\mathbf{p})$  and in the upper  $\varepsilon_2(\mathbf{p})$ . For the sake of simplicity we assume the functions  $\varepsilon_i$  to be isotropic, that is, depending on  $p = |\mathbf{p}|$ :  $\varepsilon_1(\mathbf{p}) = \max \varepsilon_1 - p^2/2m_1$  and  $\varepsilon_2(\mathbf{p}) = \min \varepsilon_2 + p^2/2m_2$ , with  $E_g = |\max \varepsilon_1 - \min \varepsilon_2|$  the energy gap. In what follows we assume for simplicity of calculations that  $m_1 = m_2 = m_e$ , but the final formulae will be given for the case of arbitrary mass ratios, e.g. when one of the bands is narrow ( $m_1 \sim m_e$  and  $m_2 \rightarrow \infty$ ). The Hamiltonian of the given system is

$$H = \sum_i \int d^3x \bar{\Psi}_i(\mathbf{x}, \tau) [\varepsilon_i - \mu] \Psi_i(\mathbf{x}, \tau) + \frac{1}{2} \sum_i \sum_j \int d^3x d^3y \bar{\Psi}_i(\mathbf{x}, \tau) \Psi_i(\mathbf{x}, \tau) V_{\mathbf{x}-\mathbf{y}} \bar{\Psi}_j(\mathbf{y}, \tau) \Psi_j(\mathbf{y}, \tau) \quad (2.1)$$

where  $i$  and  $j$  are the numbers of the bands,  $\varepsilon_{1,2} = \mp \frac{1}{2} E_g \pm \frac{1}{2} m^{-1} \nabla^2$  is the dispersion law for the electrons in the respective band, and  $V_{\mathbf{r}} = e^2/r$  the Coulomb interaction. Below we consider the case of a fully occupied band 1 and an empty band 2, which means that the chemical potential  $\mu$  is zero. The case where band 2 is doped with electrons or band 1 with holes occupies a special place in this paper.

Within this system let us consider  $Z = \int \exp(S_0) D\Psi$ , where we have introduced the notation  $D\Psi = \prod_{i=1,2} d\bar{\Psi}_i d\Psi_i$ . Quantization of Fermi systems is done by integrating over the space of anti-commuting functions (elements of an infinite-dimensional Grassmann algebra). To obtain correct statistics, anti-periodicity conditions in  $\tau$  are imposed on  $\bar{\Psi}_i$  and  $\Psi_i$  [15]. The quantity

$$S_0 = - \int_0^\beta d\tau \int d^3x \sum_{i=1,2} \bar{\Psi}_i \partial_\tau \Psi_i - \int_0^\beta d\tau H(\tau)$$

is a functional of  $\bar{\Psi}_1, \Psi_1, \bar{\Psi}_2,$  and  $\Psi_2$  having the meaning of action, with  $\beta = 1/T$  and  $H(\tau)$  the Hamiltonian of the system (2.1).

A new complex-valued field  $\Delta(\mathbf{x}, \mathbf{y}, \tau) = \Delta_1 + i\Delta_2$  is introduced at this point. We put  $x_1 = \text{Re}(\bar{\Psi}_1(\mathbf{x}, \tau)\Psi_2(\mathbf{y}, \tau))$  and  $x_2 = \text{Im}(\bar{\Psi}_1(\mathbf{x}, \tau)\Psi_2(\mathbf{y}, \tau))$ . Then

$Z = \int \exp(S_0) \delta(\Delta_1 - x_1) \delta(\Delta_2 - x_2) D\Psi D\Delta_1 D\Delta_2$ . Substituting the functional delta function in the form  $\delta(x) = \int d\phi \exp(i\phi x)$  and performing fairly simple transformations, we find that  $Z = \int \exp(S) D\Psi D\Delta D\phi$ , with  $D\Delta = d\Delta d\Delta^*$ ,  $D\phi = d\phi d\phi^*$ , and

$$S = S_0 + i \left\{ \int_0^\beta d\tau \int d^3x d^3y \phi^*(\mathbf{x}, \mathbf{y}, \tau) [\Delta(\mathbf{x}, \mathbf{y}, \tau) - \bar{\Psi}_1(\mathbf{x}, \tau) \Psi_2(\mathbf{y}, \tau)] + \text{CC} \right\}. \quad (2.2)$$

Next we go over to the momentum representation:

$$\begin{aligned} \Psi_i(\mathbf{x}, \tau) &= \frac{1}{\sqrt{\beta}} \sum_{\mathbf{p}, \varepsilon_n} a_i(\mathbf{p}, \varepsilon_n) \exp(i\mathbf{p} \cdot \mathbf{x} - i\varepsilon_n \tau) \\ \phi(\mathbf{x}, \mathbf{y}, \tau) &= \frac{1}{\sqrt{\beta}} \sum_{\mathbf{p}_1, \mathbf{p}_2, \omega_n} \phi(\mathbf{p}_1, \mathbf{p}_2, \omega_n) \exp \left[ -i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y}) + \frac{i}{2} \mathbf{p}_2 \cdot (\mathbf{x} + \mathbf{y}) - i\omega_n \tau \right] \\ \Delta(\mathbf{x}, \mathbf{y}, \tau) &= \frac{1}{\sqrt{\beta}} \sum_{\mathbf{p}_1, \mathbf{p}_2, \omega_n} \Delta(\mathbf{p}_1, \mathbf{p}_2; \omega_n) \exp \left[ -i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y}) + \frac{i}{2} \mathbf{p}_2 \cdot (\mathbf{x} + \mathbf{y}) - i\omega_n \tau \right]. \end{aligned} \quad (2.3)$$

Below we denote the Fermi frequencies by  $\varepsilon_n \equiv (2n+1)\pi T$  and the Bose frequencies by  $\omega_n \equiv 2\pi nT$ . By introducing the bilocal variables  $\phi$  and  $\Delta$  we go over to the centre-of-mass reference frame. The momentum  $\mathbf{p}_1$  corresponds to the relative motion of an electron and hole, and  $\mathbf{p}_2$  represents the motion of the two particles as a whole. After this transformation the action acquires the form

$$\begin{aligned} S &= \sum_i \sum_p \bar{a}_i(p) [i\varepsilon_n - \varepsilon_i(p) + \mu] a_i(p) \\ &\quad - \frac{T}{2} \sum_i \sum_j \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4} \bar{a}_i(\mathbf{p}_1) a_i(\mathbf{p}_2) V_{\mathbf{p}_1 - \mathbf{p}_2} \bar{a}_j(\mathbf{p}_3) a_j(\mathbf{p}_4) \delta_{\mathbf{p}_1 + \mathbf{p}_3 = \mathbf{p}_2 + \mathbf{p}_4} + S_{\phi\Delta} \\ S_{\phi\Delta} &= -\frac{i}{\sqrt{\beta}} \left[ \sum_{\mathbf{p}, \omega} \bar{a}_1(\mathbf{p}_1) a_2(\mathbf{p}_2) \phi \left( \frac{\mathbf{p}_1 + \mathbf{p}_2}{2}, \mathbf{p}_1 - \mathbf{p}_2, \omega_{n_1} - \omega_{n_2} \right) + \text{CC} \right] \\ &\quad + i \left[ \sum_{\mathbf{p}, \omega} \phi^*(\mathbf{p}_1, \mathbf{p}_2, \omega_n) \Delta(\mathbf{p}_1, \mathbf{p}_2, \omega_n) + \text{CC} \right] \end{aligned} \quad (2.4)$$

where  $p = (\mathbf{p}, \nu_n)$ , with  $\nu_n$  the Fermi or Bose frequency.

We introduce the density operator

$$\begin{aligned} \rho(\mathbf{x}, \tau) &= \sum_i \bar{\Psi}_i(\mathbf{x}, \tau) \Psi_i(\mathbf{x}, \tau) \\ &= \sum_i \rho_i(\mathbf{x}, \tau). \end{aligned} \quad (2.5)$$

In the term in  $S$  of (2.4) that represents interaction we isolate two channels:

1.  $|\mathbf{p}_1 - \mathbf{p}_2| \ll |\mathbf{p}_0|$
2.  $|\mathbf{p}_1 - \mathbf{p}_4| \ll |\mathbf{p}_0|$

with  $\mathbf{p}_0$  a characteristic momentum of the system,  $|\mathbf{p}_0| \sim a_B^{-1}$ , where  $a_B = (me^2)^{-1}$  is the Bohr radius of the electron ( $\hbar = 1$ ).

Under the sign of the functional integral over the Fermi fields  $\Psi_i$  we put the functional integral over an auxiliary Bose field  $\Phi(\mathbf{x}, \tau)$  in such a way so as to cancel the term representing the Coulomb interaction in the initial action (2.4) [15]. Such decoupling can be done for channel 1. It is local since the fields decouple at one point. The part  $S_\phi$  of decoupling assumes the form

$$S_\phi = -i \int_0^\beta d\tau \int d^3x \rho(\mathbf{x}, \tau) \Phi(\mathbf{x}, \tau) - \frac{1}{2} \int_0^\beta d\tau \int d^3x \frac{(\nabla \Phi(\mathbf{x}, \tau))^2}{4\pi e^2}. \quad (2.6)$$

Note that the decoupling is done by the same field for both bands.

However, in channel 2 in  $S_0$  the decoupling of Fermi fields is not local. Using the commutation relations for elements of a Grassmann algebra, we can write the term with the Coulomb interaction in such a way that the Fermi fields are placed together in combinations like  $\Psi_i(\mathbf{x}, \tau) \Psi_i(\mathbf{y}, \tau)$ . Introducing auxiliary Bose fields  $\chi_i$  for each band in a similar manner, with the fields already depending on two coordinates ( $\mathbf{x}$  and  $\mathbf{y}$ ), we exclude the term with the Coulomb interaction. We then use  $\Delta$  and  $\Delta^*$  to express the terms which after allowing for the commutation relations contain Fermi fields from different bands. Here  $|\mathbf{p}_1 - \mathbf{p}_2|$  is already not small compared to  $|\mathbf{p}_0|$ . The result of the transformations assumes the form in the momentum representation:

$$\begin{aligned} S = & \sum_{\mathbf{p}} \bar{a}_i(\mathbf{p}_1) K_{\mathbf{p}_1 \mathbf{p}_2}^{ij} a_j(\mathbf{p}_2) + \sum_{\mathbf{k}, \mathbf{p}, \omega} V_{\mathbf{k}} \Delta(\mathbf{p}_1, \mathbf{p}_2, \omega_n) \Delta^*(\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2, \omega_n) \\ & + i \sum_{\mathbf{p}, \omega} [\phi^*(\mathbf{p}_1, \mathbf{p}_2, \omega_n) \Delta(\mathbf{p}_1, \mathbf{p}_2, \omega_n) + \text{c.c.}] \\ & - \frac{1}{2} \sum_{\mathbf{k}} V_{\mathbf{k}}^{-1} \left[ \Phi(\mathbf{k}) \Phi(-\mathbf{k}) + \sum_{\mathbf{p}, \omega} \chi_i(\mathbf{p}_1, \mathbf{p}_2, \omega_n) \chi_i(\mathbf{p}_1 - \mathbf{k}, -\mathbf{p}_2, -\omega_n) \right] \end{aligned} \quad (2.7)$$

where the  $K_{\mathbf{p}_1 \mathbf{p}_2}^{ij}$  are defined as

$$\hat{K}_{\mathbf{p}_1 \mathbf{p}_2} = \hat{G}_{\mathbf{p}_1 \mathbf{p}_2}^{-1} - i \hat{\Phi}_{\mathbf{p}_1 \mathbf{p}_2} \quad \hat{G}_{\mathbf{p}_1 \mathbf{p}_2}^{-1} = \hat{G}_0^{-1}(\mathbf{p}_1) \delta_{\mathbf{p}_1 \mathbf{p}_2} - \hat{\Lambda}_{\mathbf{p}_1 \mathbf{p}_2}. \quad (2.8)$$

Here we have introduced the following notation:

$$\begin{aligned} \hat{G}_0^{-1}(\mathbf{p}) &= \begin{pmatrix} i\varepsilon_n - \varepsilon_1(\mathbf{p}) + \mu & 0 \\ 0 & i\varepsilon_n - \varepsilon_2(\mathbf{p}) + \mu \end{pmatrix} \\ \hat{\Lambda}_{\mathbf{p}_1 \mathbf{p}_2} &= \frac{1}{\sqrt{\beta}} \begin{pmatrix} \chi_1(\mathbf{p}_1, \mathbf{p}_2) & i\phi^*(\mathbf{p}_1, \mathbf{p}_2) \\ i\phi(\mathbf{p}_1, \mathbf{p}_2) & \chi_2(\mathbf{p}_1, \mathbf{p}_2) \end{pmatrix} \\ \hat{\Phi}_{\mathbf{p}_1 \mathbf{p}_2} &= \frac{1}{\sqrt{\beta}} \begin{pmatrix} \Phi(\mathbf{p}_1, \mathbf{p}_2) & 0 \\ 0 & \Phi(\mathbf{p}_1, \mathbf{p}_2) \end{pmatrix}. \end{aligned} \quad (2.9)$$

Now let us employ the fact that band 1 is completely filled and band 2 is vacant. We integrate over all the Fermi fields in both bands:

$$\int \exp[S] \prod_i D\bar{a}_i D a_i = \exp[\text{Tr} \ln(K_{\mathbf{p}_1 \mathbf{p}_2}^{ij})]. \quad (2.10)$$

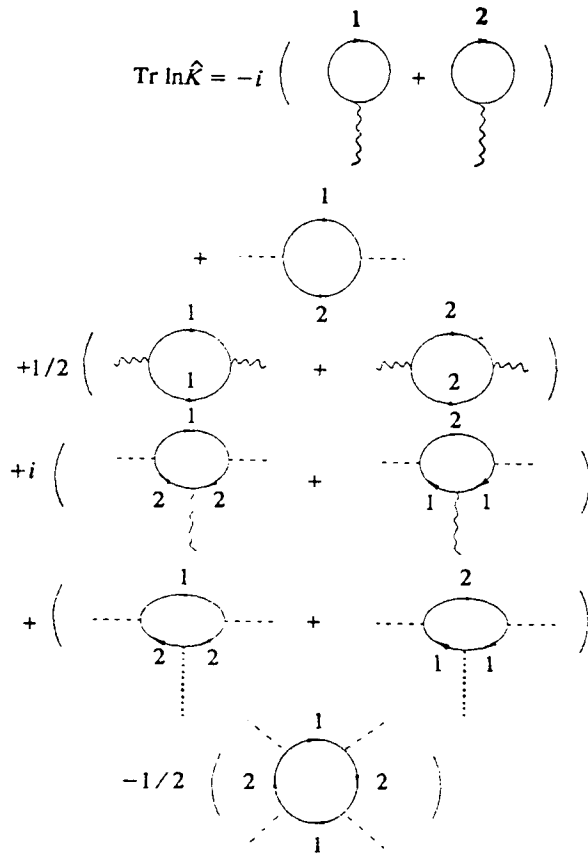


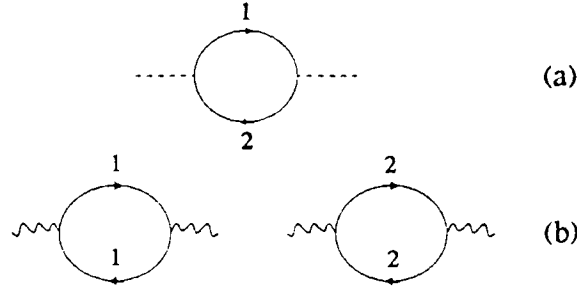
Figure 2. Diagrammatic structure of the expansion of  $\text{Tr} \ln \hat{K}$ . A solid line corresponds to the Green function of a free electron,  $G_{0i}$ , a wavy line to the local field  $\Phi$ , a dashed line to fields  $\phi$  and  $\phi^*$ , and a dotted line to the  $\chi_i$  fields.

Assuming that exciton number density is low, we expand  $\text{Tr} \ln \hat{K}$  in a series in  $\phi$ ,  $\phi^*$ , and  $\Phi$ :

$$\begin{aligned} \text{Tr} \ln \hat{K} &= \text{Tr} \ln [\hat{G}^{-1} - i\hat{\Phi}] \\ &= \text{Tr} \ln \hat{G}_0^{-1} - \text{Tr} \sum_{n=1}^{\infty} \frac{i^n}{n} [\hat{G}_0(\hat{\Phi} - i\hat{\Lambda})]^n. \end{aligned} \tag{2.11}$$

The result of expansion (2.1) is depicted graphically in figure 2. The expansion is done in parameter  $\phi$  up to  $\phi^4$ , and only terms linear in  $\Phi$  and  $\chi$  are retained, with the exception of two terms quadratic in  $\Phi$  that screen the Coulomb interaction in nonzero doping. Here  $\text{Tr} \ln \hat{G}_0^{-1}$  is independent of the fields and can be discarded. Calculations and analysis of diagrams constitute the main topic of this section. We also note that for coinciding momenta the fields  $\chi_{i,p_1,p_2}$  represent the respective self-energy parts  $\Sigma_i(p)$ . It can be assumed, however, that the untransformed spectrum  $\epsilon(p)$  has already been renormalized in the necessary manner.

Since the perturbation theory is built with the gas parameter ( $na_B^d \ll 1$ , where  $d$  is the dimensionality of the system) and the expansion of  $\text{Tr} \ln \hat{K}$  lacks terms linear in  $\phi$ , we start calculations with the diagram of figure 3(a). The analytical expression

Figure 3. Diagrams quadratic in  $\phi$  and  $\Phi$ .

corresponding to the diagram has the form

$$\begin{aligned}
 T \sum_{\mathbf{p}, \mathbf{k}, \omega} \phi^*(\mathbf{p}, \mathbf{k}, \omega_m) \phi(\mathbf{p}, \mathbf{k}, \omega_m) \sum_n G_{01}\left(\mathbf{p} - \frac{\mathbf{k}}{2}, \varepsilon_n - \frac{\omega_m}{2}\right) G_{02}\left(\mathbf{p} + \frac{\mathbf{k}}{2}, \varepsilon_n + \frac{\omega_m}{2}\right) \\
 = \sum_{\mathbf{p}, \mathbf{k}, \omega} \phi^*(\mathbf{p}, \mathbf{k}, \omega_m) \phi(\mathbf{p}, \mathbf{k}, \omega_m) \Pi_{12}(\mathbf{p}, \mathbf{k}, \omega_m)
 \end{aligned} \quad (2.12)$$

where  $\Pi_{12}$  is the sum over the frequency of the two Green functions of electrons from different bands. Calculating the sum according to [16], we obtain (see formulae (A6)-(A8) in the appendix)

$$\begin{aligned}
 \Pi_{12}(\mathbf{p}, \mathbf{k}, \omega_m) &= -\frac{n_2(\mathbf{p} + \mathbf{k}/2) - n_1(\mathbf{p} - \mathbf{k}/2)}{i\omega_m - E_{\mathbf{p}\mathbf{k}}} \\
 E_{\mathbf{p}\mathbf{k}} &= \varepsilon_2\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) - \varepsilon_1\left(\mathbf{p} - \frac{\mathbf{k}}{2}\right) = E_g + \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{k}^2}{2M}.
 \end{aligned} \quad (2.13)$$

Here we have introduced the following notation:  $m = m_1 m_2 / (m_1 + m_2)$  is the reduced mass,  $M = m_1 + m_2$  the exciton mass, and  $n_i(\mathbf{p}) = \{\exp[(\varepsilon_i(\mathbf{p}) - \mu)/T] + 1\}^{-1}$  are the Fermi distribution functions for electrons and holes from bands 2 and 1, respectively. Assuming that  $m_1 = m_2 = m_e$ , we find that  $m = m_e/2$ ,  $M = 2m_e$ , and

$$n_2(\mathbf{p}) = \frac{1}{\exp[(\varepsilon(\mathbf{p}) - \mu)/T] + 1} \quad n_1(\mathbf{p}) = 1 - \frac{1}{\exp[(\varepsilon(\mathbf{p}) + \mu)/T] + 1}$$

with  $n_2 = 0$  and  $n_1 = 1$  at  $T = 0$  and  $\mu = 0$ . Note that the diagrams in figure 3 are calculated in a similar manner and expressed in terms of  $\Pi_{11}$  and  $\Pi_{22}$ . Obviously, at  $T = 0$  and  $\mu = 0$  these diagrams vanish. Allowing for the diagrams when  $\mu \neq 0$  leads to screening of the Coulomb interaction of electrons in the bands. Thus, substituting the calculated term into the action, we obtain

$$\begin{aligned}
 S &= \sum_{\mathbf{p}, \mathbf{k}, \omega} \frac{\phi^*(\mathbf{p}, \mathbf{k}, \omega_n) \phi(\mathbf{p}, \mathbf{k}, \omega_n)}{i\omega_n - E_{\mathbf{p}\mathbf{k}}} + \sum_{\mathbf{k}, \mathbf{p}, \omega} V_{\mathbf{k}} \Delta^*(\mathbf{p}_1 - \mathbf{k}, \mathbf{p}_2, \omega_n) \Delta(\mathbf{p}_1, \mathbf{p}_2, \omega_n) \\
 &+ i \sum_{\mathbf{p}, \mathbf{k}, \omega} [\phi^*(\mathbf{p}, \mathbf{k}, \omega_n) \Delta(\mathbf{p}, \mathbf{k}, \omega_n) + \text{c.c.}] + F_1(\phi \phi^*, \Phi) + F_2(\phi \phi^*, \chi) + F_3(\phi^4) + \dots
 \end{aligned} \quad (2.14)$$

We calculate the integral  $\int \exp(S) d\phi^* d\phi$  by the saddle-point method of perturbation theory, ignoring the terms  $F_j$ . As we shortly show, allowing for these terms leads to higher-order corrections. Varying in  $\phi^*$  and  $\phi$ , nullifying the variation, and expressing  $\Delta$  in terms of  $\phi$  at the saddle point, we get

$$\phi(\mathbf{p}, \mathbf{k}, \omega_n) = i[E_{\mathbf{p}\mathbf{k}} - i\omega_n]\Delta(\mathbf{p}, \mathbf{k}, \omega_n) \quad \phi^*(\mathbf{p}, \mathbf{k}, \omega_n) = i[E_{\mathbf{p}\mathbf{k}} - i\omega_n]\Delta^*(\mathbf{p}, \mathbf{k}, \omega_n) \quad (2.15)$$

$$S = \sum_{\mathbf{p}, \mathbf{k}, \omega} \Delta^*(\mathbf{p}, \mathbf{k}, \omega_n)[i\omega_n - E_{\mathbf{p}\mathbf{k}}]\Delta(\mathbf{p}, \mathbf{k}, \omega_n) + \sum_{\mathbf{p}, \mathbf{k}, \mathbf{l}, \omega} \Delta^*(\mathbf{p}, \mathbf{k}, \omega_n)V_{\mathbf{l}}\Delta(\mathbf{p} + \mathbf{l}, \mathbf{k}, \omega_n) + F_1(\Delta^* \Delta, \Phi) + F_2(\Delta^* \Delta, \chi) + F_3(\Delta^4) + \dots \quad (2.16)$$

Varying in  $\Delta$  or  $\Delta^*$  and nullifying the variation, we obtain the equation for  $\Delta$  (or  $\Delta^*$ ) at the saddle point. This is the Bethe–Salpeter equation:

$$\left( \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{k}^2}{2M} + E_g - i\omega_n \right) \Delta(\mathbf{p}, \mathbf{k}, \omega_n) = \sum_{\mathbf{l}} V_{\mathbf{l}} \Delta(\mathbf{p} + \mathbf{l}, \mathbf{k}, \omega_n). \quad (2.17)$$

To find the solution to this equation, we expand  $\Delta$  in the complete set of the eigenfunctions  $\psi_i(\mathbf{p})$  of the discrete spectrum of the hydrogen atom with eigenvalues  $-E_c^i = -me^4/2(i+1)^2$  ( $i = 0, 1, 2, \dots$ ):

$$\Delta(\mathbf{p}, \mathbf{k}, \omega_n) = \sum_{\mathbf{i}} \psi_{\mathbf{i}}(\mathbf{p}) C_{\mathbf{i}}(\mathbf{k}, \omega_n) \quad (2.18)$$

where the  $\psi_{\mathbf{i}}$  are determined from the Schrödinger equation

$$\left( \frac{\mathbf{p}^2}{2m} + E_c^i \right) \psi_{\mathbf{i}}(\mathbf{p}) = \sum_{\mathbf{l}} V_{\mathbf{l}} \psi_{\mathbf{i}}(\mathbf{p} + \mathbf{l}). \quad (2.19)$$

Factorizing the function  $\Delta$  in this way and replacing the sums over the momenta  $\mathbf{p}$  with integrals in the usual manner, we arrive at an expression for the action in terms of the functions  $C$  and  $C^*$ :

$$S = \sum_{\mathbf{k}} C_{\mathbf{i}}^*(\mathbf{k}, \omega_n) \left[ i\omega_n - \frac{\mathbf{k}^2}{2M} + (E_c^i - E_g) \right] C_{\mathbf{i}}(\mathbf{k}, \omega_n) + \dots \quad (2.20)$$

In what follows we keep in the sum only the ground states  $i = 0$  of an exciton.

After the action is represented in the form (2.20), the physical meaning of the introduced notation becomes clear. For instance,  $C$  is the excitonic field,  $\mathbf{k}$  is the exciton momentum,  $M$  the exciton mass, and  $\lambda_0 = E_c^0 - E_g$ , where  $E_c^0$  the exciton binding energy. The possibility of Bose condensation of the excitons is determined by the sign of  $\lambda_0$ . In the present paper the non-ideal Bose gas of excitons is considered both in the condensate range, where  $\lambda_0 > 0$ , and in the range  $\lambda_0 < 0$ , where there can be only virtual excitons.

Let us examine the diagrams corresponding to the local field  $\Phi$ . There are two classes of diagrams here. The diagrams of figure 4(a) contribute nothing to the action since the system as a whole is electroneutral [15]. The diagrams representing the

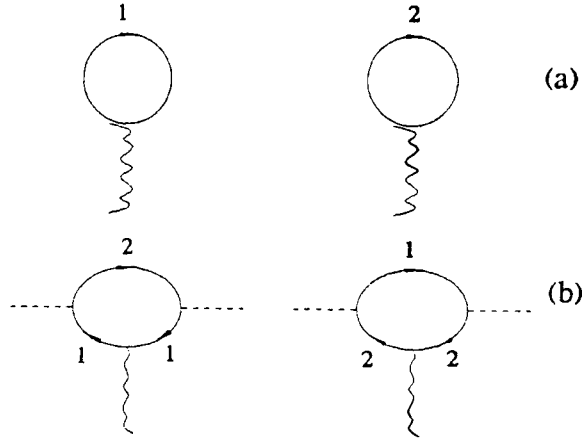


Figure 4. Diagrams corresponding to local decoupling.

combination of fields  $\phi$  and  $\Phi$  are depicted in figure 4(b). The analytical expression for the first diagram in figure 4(b) is

$$i\sqrt{T} \sum_{\mathbf{q}, \mathbf{p}, \omega} \phi^* \left( \mathbf{q} + \frac{1}{2} \mathbf{p}_1, \mathbf{p}_2, \omega_m \right) \phi \left( \mathbf{q} + \frac{1}{2} \mathbf{p}_2, \mathbf{p}_1, \omega_n \right) \Phi \left( \mathbf{p}_2 - \mathbf{p}_1, \omega_m - \omega_n \right) \Pi_{112} \left( \mathbf{q}, \mathbf{p}_1, \mathbf{p}_2, \omega_m, \omega_n \right). \quad (2.21)$$

Similarly, for the second diagram in figure 4(b) we have

$$i\sqrt{T} \sum_{\mathbf{q}, \mathbf{p}, \omega} \phi^* \left( \mathbf{q} + \frac{1}{2} \mathbf{p}_2, \mathbf{p}_1, \omega_n \right) \phi \left( \mathbf{q} + \frac{1}{2} \mathbf{p}_1, \mathbf{p}_2, \omega_m \right) \Phi \left( \mathbf{p}_2 - \mathbf{p}_1, \omega_m - \omega_n \right) \Pi_{221} \left( \mathbf{q}, \mathbf{p}_1, \mathbf{p}_2, \omega_n, \omega_m \right) \quad (2.22)$$

where  $\Pi_{112}$  and  $\Pi_{221}$  are sums over the ring frequency of three Green functions (see formulae (A8)–(A13) in the appendix). Here is the expression for  $\Pi_{112}$ :

$$\begin{aligned} \Pi_{112} \left( \mathbf{q}, \mathbf{p}_1, \mathbf{p}_2, \omega_n, \omega_m \right) = T \sum_{\epsilon} G_{01} \left( \mathbf{q} - \frac{\mathbf{p}_2 - \mathbf{p}_1}{2}, \epsilon_q - \frac{\omega_m - \omega_n}{2} \right) \\ \times G_{01} \left( \mathbf{q} + \frac{\mathbf{p}_2 - \mathbf{p}_1}{2}, \epsilon_q + \frac{\omega_m - \omega_n}{2} \right) \\ \times G_{02} \left( \mathbf{q} + \frac{\mathbf{p}_2 + \mathbf{p}_1}{2}, \epsilon_q + \frac{\omega_m + \omega_n}{2} \right). \end{aligned} \quad (2.23)$$

Calculating the sum over frequencies and employing the relation (2.15) between  $\phi$  and  $\Delta$  at the saddle point, we arrive at an expression for the diagrams in figure 4(b) in terms of  $\Delta$  and  $\Delta^*$ :

$$i\sqrt{T} \sum_{\mathbf{k}} \Phi(\mathbf{k}, \omega_k) \sum_{\mathbf{q}, \mathbf{p}, \omega} \left[ \Delta^* \left( \mathbf{q}, \mathbf{p} + \mathbf{k}, \omega_n + \omega_k \right) \Delta \left( \mathbf{q} + \frac{1}{2} \mathbf{k}, \mathbf{p}, \omega_n \right) \right. \\ \left. - \Delta^* \left( \mathbf{q} + \frac{1}{2} \mathbf{k}, \mathbf{p}, \omega_n \right) \Delta \left( \mathbf{q}, \mathbf{p} - \mathbf{k}, \omega_n - \omega_m \right) \right]. \quad (2.24)$$



Let us now use the expansion (2.18) for the order parameter  $\Delta$  and go over to the coordinate representation for functions  $\psi_i$ :

$$\Delta(\mathbf{p}, \mathbf{k}, \omega_n) = \sum_i \int d^3r \psi_i(\mathbf{r}) \exp(-i\mathbf{p} \cdot \mathbf{r}) C_i(\mathbf{k}, \omega_n). \quad (2.25)$$

We restrict our discussion to 0-1 transitions and introduce the following notation:

$$\begin{aligned} D_{01}(\mathbf{k}) &= 2 \int d^3r \psi_1(\mathbf{r}) \sin(\tfrac{1}{2}\mathbf{k} \cdot \mathbf{r}) \psi_0(\mathbf{r}) = \mathbf{k} \cdot \mathbf{d}_{10} + \dots \\ \pi(\mathbf{k}, \omega_k) &= \sum_p [C_1^*(p+k)C_0(p) + C_0^*(p)C_1(p-k)] \end{aligned} \quad (2.26)$$

where  $\mathbf{d}_{10}$  is the dipole moment of the transition between the ground state and the first excited state of the hydrogen atom divided by the electron charge. Integrating over the 'fast' momenta, we write (2.24) in terms of excitonic fields:

$$\sqrt{T} \sum_{\mathbf{k}} \Phi(\mathbf{k}, \omega_k) D_{01}(\mathbf{k}) \pi(\mathbf{k}, \omega_k). \quad (2.27)$$

Note that after integration over  $\Phi$  the diagrams depicted in figure 4 contribute to the interaction of excitons over great distances, which, as we will shortly see, resembles the van der Waals attractive forces. They also contribute to the interaction of the doping electrons of band 2 with excitons, which resembles the dipole-charge interaction.

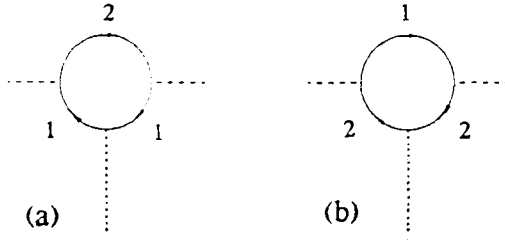


Figure 5. Diagrams corresponding to bilocal decoupling.

The difference between the diagrams in figure 4 for local decoupling and those in figure 5 for bilocal decoupling is that in the former the field  $\Phi$  depends only on the difference of the momenta  $\mathbf{p}_1$  and  $\mathbf{p}_2$  and is independent of the momentum transferred in the ring, while the  $\chi_i$  fields depend on this momentum. Using equation (2.23) for  $\Pi_{112}$  and  $\Pi_{221}$  and the equation (2.15) that links  $\phi$  and  $\Delta$  at the saddle point, we can immediately write the sum of the two diagrams:

$$\begin{aligned} T \left[ \sum_{\mathbf{q}, \mathbf{p}, \omega} \chi_1(\mathbf{q}, \mathbf{p}_2 - \mathbf{p}_1, \omega_m - \omega_n) \Delta^*(\mathbf{q} + \tfrac{1}{2}\mathbf{p}_1, \mathbf{p}_2, \omega_m) \Delta(\mathbf{q} + \tfrac{1}{2}\mathbf{p}_2, \mathbf{p}_1, \omega_n) \right. \\ \left. - \chi_2(\mathbf{q}, \mathbf{p}_1 - \mathbf{p}_2, \omega_n - \omega_m) \Delta^*(\mathbf{q} - \tfrac{1}{2}\mathbf{p}_2, \mathbf{p}_1, \omega_n) \Delta(\mathbf{q} - \tfrac{1}{2}\mathbf{p}_1, \mathbf{p}_2, \omega_m) \right]. \end{aligned} \quad (2.28)$$

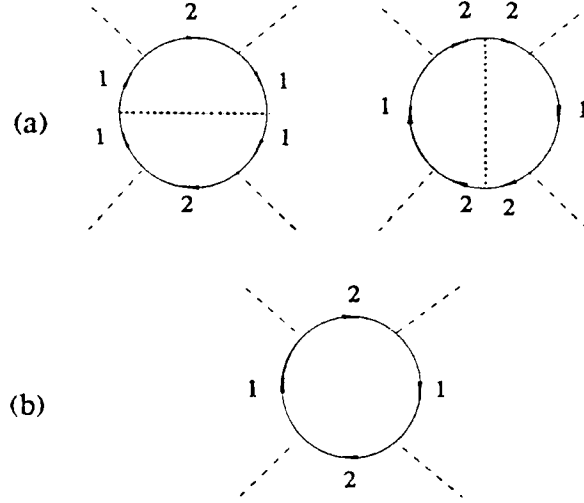


Figure 6. Diagrams determining the excitonic scattering amplitude. The dotted lines depict the Coulomb interaction.

Now we can integrate over the  $\chi_i$  fields. The diagrams after integration over the  $\chi_i$  are shown in figure 6(a). Note that in addition to the diagrams of figure 6 there are diagrams that contribute to the electron–exciton scattering amplitude. The analysis of such diagrams, however, lies outside the scope of the present paper and require special consideration. The contribution  $S_2 = \int \exp(F_2(\Delta^* \Delta, \chi)) d\chi_1 d\chi_2$  of these diagrams in equation (2.16) has the form

$$S_2 = T \sum_{l, p, k} V_l \Delta^*(p, k_1) \Delta\left(p + \frac{k_2 - k_1}{2}, k_2\right) \times \Delta^*\left(p + l + \frac{k_2 - k_1}{2}, k_3 - k_1 + k_2\right) \Delta\left(p + l + \frac{k_1 - k_3}{2}, k_3\right). \quad (2.29)$$

Since the exciton's centre-of-mass momentum is much smaller than the characteristic momentum of relative motion, it can be ignored in the first argument of the  $\Delta$  functions. Using the expansion (2.18), restricting our discussion to the ground excitonic states, and integrating over the relative-motion momenta, we arrive at

$$S_2 = T \sum_{k_i} B C_0^*(k_1) C_0(k_2) C_0^*(k_3) C_0(k_4) \delta_{k_1 + k_3 = k_2 + k_4} \quad (2.30)$$

$$B = \frac{1}{(2\pi)^6} \int d^3 k d^3 p \psi^2(p) V_k \psi^2(p + k) = \frac{50\pi}{6}$$

where we have employed the system of units in which  $\hbar = e = m = 1$ . Here  $B$  contributes to the exciton–exciton scattering amplitude, just as  $F_3(\Delta^4)$  in (2.16) does. The respective diagram is shown in figure 6(b), and the analytical expression corresponding to this diagram has the form

$$F_3(\phi^4) = -\frac{T}{2} \sum_{p, k, \omega} \phi^*\left(p - \frac{k_1 - k_3}{2}, k_2, \omega_{n_2}\right) \phi\left(p - \frac{k_1 - k_2}{2}, k_3, \omega_{n_3}\right) \Pi_{1212}(p, k, \omega) \times \phi^*(p, k_1 - k_2 + k_3, \omega_{n_1} - \omega_{n_2} + \omega_{n_3}) \phi\left(p - \frac{k_2 - k_3}{2}, k_1, \omega_{n_1}\right) \quad (2.31)$$

$$\begin{aligned} \Pi_{1212}(\mathbf{p}, \mathbf{k}, \omega) = T \sum_{\epsilon} G_{01} \left( p - \frac{\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3}{2} \right) G_{02} \left( p - \frac{\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3}{2} \right) \\ \times G_{01} \left( p - \frac{\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3}{2} \right) G_{02} \left( p + \frac{\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3}{2} \right). \end{aligned} \quad (2.32)$$

The reader will recall that  $p = (\mathbf{p}, \epsilon_n)$  and  $k_i = (\mathbf{k}_i, \omega_{n_i})$  and that summation is performed over the ring frequency  $\epsilon_n$ . Performing fairly simple but cumbersome transformations that involve using the relationship (2.15) between  $\phi$  and  $\Delta$  at the saddle point,

$$\begin{aligned} S_3 = -T \sum_{l, \mathbf{p}, \mathbf{k}} V_l \Delta^*(\mathbf{p}, \mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3) \Delta^* \left( p - \frac{\mathbf{k}_1 - \mathbf{k}_3}{2}, k_2 \right) \\ \times \Delta \left( p - l - \frac{\mathbf{k}_1 - \mathbf{k}_2}{2}, k_3 \right) \Delta \left( p - \frac{\mathbf{k}_2 - \mathbf{k}_3}{2}, k_1 \right). \end{aligned} \quad (2.33)$$

Ignoring the exciton's centre-of-mass momentum in the first argument of  $\Delta$ , as we did in the case of bilocal decoupling, employing the expansion (2.18) of the order parameter, and keeping only the contribution from the ground exciton state, we get

$$\begin{aligned} S_3 = -T \sum_{\mathbf{k}_i} A C_0^*(k_1) C_0(k_2) C_0^*(k_3) C_0(k_4) \delta_{\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4} \\ A = \frac{1}{(2\pi)^6} \int d^3p d^3k \psi^3(\mathbf{p}) V_{\mathbf{k}} \psi(\mathbf{p} + \mathbf{k}) = \frac{21\pi}{2}. \end{aligned} \quad (2.34)$$

Combining the terms  $S_2$  and  $S_3$  and introducing the exciton-exciton scattering length  $f$ , we can finally write

$$\begin{aligned} S_2 + S_3 = -\frac{Tf}{2} \sum_{\mathbf{k}_i} C_0^*(k_1) C_0(k_2) C_0^*(k_3) C_0(k_4) \delta_{\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4} \\ f = 2(A - B) = \frac{13\pi}{3}. \end{aligned} \quad (2.35)$$

For the case of Coulomb interaction in a three-dimensional electron-hole system the amplitude  $f$  was calculated by Keldysh and Kozlov [4]. Note, however, that  $f$  is not the amplitude of scattering of isolated excitons on each other because many-particle effects are taken into account in view of the fact that an exciton is a compound particle. The sign of  $f$  corresponds to the Pauli principle.

Thus, we have calculated all the terms in the expansion up to  $\phi^4$  and have made the transition to the order parameter that depends only on the 'slow' variables. The hydrodynamic action functional has the following form:

$$\begin{aligned} S = \sum_{\mathbf{k}} C_0^*(\mathbf{k}, \omega_n) \left[ i\omega_n - \frac{\mathbf{k}^2}{2M} + \lambda_0 \right] C_0(\mathbf{k}, \omega_n) \\ - \frac{Tf}{2} \sum_{\mathbf{k}_i} C_0^*(k_1) C_0(k_2) C_0^*(k_3) C_0(k_4) \delta_{\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4} \\ + \sqrt{T} \sum_{\mathbf{k}} \Phi(\mathbf{k}, \omega_n) D_{01}(\mathbf{k}) \pi(\mathbf{k}, \omega_n) - \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{8\pi e^2} \Phi(\mathbf{k}) \Phi(-\mathbf{k}). \end{aligned} \quad (2.36)$$

### 3. Electron–exciton interaction

Let us examine a situation in which dopants are added to a system that is close to the transition to an excitonic insulator. We assume that the medium possesses strong polarizability (this fact is subsequently proved). The presence of an easily polarizable medium may lead to the formation of free charge carriers in the band even at low impurity concentrations, since the interaction of electrons with impurities is determined by the static dielectric constant, which has a large value owing to the excitonic contribution. The result of all this is that the radius of the electron–impurity bound state is considerably larger than the exciton’s size, and the doping electron gas may have a density for which the average electron separation is much larger than the exciton size. This makes it possible to study the system of doping electrons interacting with excitons.

To calculate the vertex representing the interaction of the ‘slow’ electrons ‘added’ to band 2 (figure 1(b)) with excitons, we introduce sources corresponding to the ‘slow’ electrons. Separation of particles into ‘fast’ and ‘slow’ takes place when we compare the particle momentum with  $p_0 = a_B^{-1}$ . Thus, introducing source functions, we get

$$S_\eta = S + \sum_p [\bar{a}(p)\eta(p) + \bar{\eta}(p)a(p)] \quad (3.1)$$

with  $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$  and  $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ . To simplify notation we drop the arguments of functions and summation signs. We wish to calculate  $Z = \int \exp(S_\eta) Da D\Delta D\phi$ . This is done by integrating over all the Fermi fields, as we did in the absence of doping, employing the relationship

$$\frac{\int Dx \exp(-\bar{x}Ax - \bar{x}\eta - \bar{\eta}x)}{\int Dx \exp(-\bar{x}Ax)} = \exp(\bar{\eta}A^{-1}\eta).$$

Let us write the part of the action depending on the source functions:

$$S = S(\phi, \Delta, \Phi, \chi) + \text{Tr} \ln \hat{K} + \bar{\eta} \hat{K}^{-1} \eta \quad (3.2)$$

with  $\hat{K}$  defined in equations (2.8) and (2.9). Let us perform the following unitary transformation:  $\bar{\eta} \hat{K}^{-1} \eta = \bar{\eta} \hat{U}^\dagger \hat{U} \hat{K}^{-1} \hat{U}^\dagger \hat{U} = \bar{\gamma} \hat{\Theta} \gamma$ , where  $\hat{\Theta} = \hat{U} \hat{K}^{-1} \hat{U}^\dagger$  is a diagonal matrix,  $\gamma = \hat{U} \eta$  the source functions in the new basis, and  $U$  a unitary matrix. Solving the secular equation

$$\det \|\hat{K}^{-1} - \zeta \hat{1}\| = 0 \quad (3.3)$$

we find that

$$\zeta_{1,2} = \frac{1}{i\varepsilon_n + \mu - i\Phi \mp E(p)} \quad E(p) = [\varepsilon^2(p) + \delta^2(p)]^{1/2} \quad (3.4)$$

where  $\delta(p) = -i\phi(p, 0, 0)$  is the gap in the renormalized electron spectrum. Thus,

$$\hat{\Theta}(p) = \begin{pmatrix} [i\varepsilon_n - E(p) + \mu - i\Phi]^{-1} & 0 \\ 0 & [i\varepsilon_n + E(p) + \mu - i\Phi]^{-1} \end{pmatrix}. \quad (3.5)$$

Combining this with the unitarity condition for matrix  $U$ , we can represent  $\hat{U}$  as follows:

$$\begin{aligned} \hat{U}(\mathbf{p}) &= \begin{pmatrix} u(\mathbf{p}) & v(\mathbf{p}) \\ -v(\mathbf{p}) & u(\mathbf{p}) \end{pmatrix} \\ u^2(\mathbf{p}) &= \frac{1}{2} \left( 1 + \frac{\varepsilon(\mathbf{p})}{E(\mathbf{p})} \right) \quad v^2(\mathbf{p}) = \frac{1}{2} \left( 1 - \frac{\varepsilon(\mathbf{p})}{E(\mathbf{p})} \right). \end{aligned} \quad (3.6)$$

Now we require that after the unitary transformation the new source functions have the form  $\gamma = J \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , with  $J$  a  $c$ -number. Decoupling  $\bar{\eta} \hat{K}^{-1} \eta = \bar{\gamma} \mathcal{G}$  by introducing a new fermion field  $\alpha$ , we can write the action for this field as

$$S_\alpha = \sum_{\mathbf{p}} \bar{\alpha}(\mathbf{p}) [i\varepsilon_{\mathbf{p}} - E(\mathbf{p}) + \mu] \alpha(\mathbf{p}) - i\sqrt{T} \sum_{\mathbf{p}} \hat{\alpha}(\mathbf{p}_1) \alpha(\mathbf{p}_2) \Phi(\mathbf{p}_1 - \mathbf{p}_2). \quad (3.7)$$

The procedure of further integration over the auxiliary fields  $\phi$  and  $\chi$  is similar to that for no doping. Introducing the notation  $\rho_\alpha(\mathbf{x}, \tau) = \bar{\alpha}(\mathbf{x}, \tau) \alpha(\mathbf{x}, \tau)$  and combining (2.27), (2.36), and (3.7), we can write the total hydrodynamic action functional as

$$\begin{aligned} S &= \sum_{\mathbf{p}} \bar{\alpha}(\mathbf{p}) [i\varepsilon_{\mathbf{p}} - E(\mathbf{p}) + \mu] \alpha(\mathbf{p}) \\ &+ \sum_{\mathbf{k}} C_0^* \left[ i\omega_{\mathbf{k}} - \frac{\mathbf{k}^2}{2M} + \lambda \right] C_0(\mathbf{k}) \\ &- i\sqrt{T} \sum_{\mathbf{k}} \Phi(\mathbf{k}) [\rho_\alpha(\mathbf{k}) + i\mathbf{k} \cdot \mathbf{d}_{01} \pi(\mathbf{k})] - \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{8\pi e^2} \Phi(\mathbf{k}) \Phi(-\mathbf{k}) \\ &- \frac{T\tilde{f}}{2} \sum_{\mathbf{p}_i} C_0^*(\mathbf{p}_1) C_0(\mathbf{p}_2) C_0^*(\mathbf{p}_3) C_0(\mathbf{p}_4) \delta_{\mathbf{p}_1 + \mathbf{p}_3 = \mathbf{p}_2 + \mathbf{p}_4} \end{aligned} \quad (3.8)$$

where  $\tilde{f}$  is the exciton-exciton scattering amplitude renormalized by doping (we assume, however, that  $\tilde{f}$  is still positive), and  $\lambda = E_c^* - E_g$ , with  $-E_c^*$  the eigenvalue of the Schrödinger equation

$$\left( \frac{\mathbf{p}^2}{2m} + E_c^* \right) \psi(\mathbf{p}) = \sum_{\mathbf{l}} V_{\mathbf{l}} \psi(\mathbf{p} + \mathbf{l}) [1 - n(\mathbf{p} + \mathbf{l})] \quad (3.9)$$

with  $n(\mathbf{p}) = \{1 + \exp[(\mathbf{p}^2 - \mathbf{p}_F^2)/2mT]\}^{-1}$ . At  $\mathbf{p}_F = 0$  equation (3.9) transforms into equation (2.19). Generally,  $E_c^*$  depends on the amount of doping.

Integrating the action (3.8) over the  $\Phi$  field, we can easily obtain the vertices representing the electron-exciton interaction and the interaction of excitons over great distances. As expected, the electron-exciton interaction is of the dipole-charge interaction type and the exciton-exciton interaction of the dipole-dipole type. Since the system does not contain excitons in the excited state, the problem of exciton-exciton interaction over great distances is equivalent to that of the interaction of two atoms in the  $s$ -state. Solution of the latter problem according to perturbation theory leads, as is known (e.g. see [17]), to van der Waals attraction. After integration over

the  $\Phi$  field the action  $S$  assumes the form

$$\begin{aligned}
S = & \sum_p \bar{\alpha}(p)[i\varepsilon_n - E(p) + \mu]\alpha(p) + \sum_k C_0^*(k) \left[ i\omega_n - \frac{k^2}{2M} + \lambda \right] C_0(k) \\
& - T \sum_p V^{e-d}(\mathbf{p}_1 - \mathbf{p}_2) \bar{\alpha}(p_1) \alpha(p_2) [C_1^*(p_3) C_0(p_4) + C_0^*(p_3) C_1(p_4)] \delta_{p_1+p_3=p_2+p_4} \\
& - \frac{T}{2} \sum_p V^{d-d}(\mathbf{p}_1 - \mathbf{p}_3) [C_0^*(p_1) C_0^*(p_2) C_1(p_3) C_1(p_4) \\
& \quad + C_0^*(p_1) C_1^*(p_2) C_0(p_3) C_1(p_4) + c.c.] \delta_{p_1+p_2=p_3+p_4} \\
& - \frac{T}{2} \sum_p V(\mathbf{p}_1 - \mathbf{p}_2) \bar{\alpha}(p_1) \alpha(p_2) \bar{\alpha}(p_3) \alpha(p_4) \delta_{p_1+p_3=p_2+p_4} \\
& - \frac{T}{2} \sum_p \bar{f} C_0^*(p_1) C_0(p_2) C_0^*(p_3) C_0(p_4) \delta_{p_1+p_3=p_2+p_4}. \tag{3.10}
\end{aligned}$$

Here we have introduced the following notation:  $V_{\mathbf{k}} = 4\pi e^2/k^2 \tilde{\varepsilon}(\mathbf{k})$  is the potential of the Coulomb interaction of the upper-band electrons,  $\tilde{\varepsilon}(\mathbf{k})$  the medium's dielectric constant (we calculate it in section 4),  $V_{\mathbf{k}}^{e-d} = i\mathbf{k} \cdot \mathbf{d}_{01} V_{\mathbf{k}}$  the dipole-charge interaction potential,  $V_{\mathbf{k}}^{d-d} = (\mathbf{k} \cdot \mathbf{d}_{01})^2 V_{\mathbf{k}}$  the dipole-dipole interaction potential.

However, a situation is possible in which there is no excitonic phase without doping, that is,  $E_g > E_c$ , but in the presence of doping excitons are formed owing to the formation of an exciton-electron bound state. The possibility of such a phenomenon in connection with problems of high- $T_c$  superconductivity was discussed in [18]. Below we consider the case where these bound states are disrupted by doping. We will devote a separate paper to studying the possibility of the formation of a bound electron-exciton state and the properties of such a strongly interacting system of electrons and excitons.

#### 4. Calculating the dielectric constant

To allow for the effect of band-to-band transitions in the model considered here on the nature of the phase transition, we introduce into the Hamiltonian terms corresponding to the transformation of electrons of one band into electrons of another, in addition to those present. Thus, instead of Hamiltonian (2.1) we consider the following:

$$\begin{aligned}
H(\tau) = & \sum_{\alpha} \int d^3x \bar{\Psi}_{\alpha}(\mathbf{x}, \tau) [\varepsilon_{\alpha} - \mu] \Psi_{\alpha}(\mathbf{x}, \tau) \\
& + \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \int d^3x d^3y \bar{\Psi}_{\alpha}(\mathbf{x}, \tau) \Psi_{\beta}(\mathbf{x}, \tau) V(\mathbf{x} - \mathbf{y}) \bar{\Psi}_{\gamma}(\mathbf{y}, \tau) \Psi_{\delta}(\mathbf{y}, \tau) \tag{4.1}
\end{aligned}$$

with  $V = e^2/r$  the Coulomb interaction. The difference between Hamiltonians (2.1) and (4.1) is that the latter allows for terms that lie beyond the scope of the density-density interaction type. Such terms were allowed for in the work of Keldysh and Guseinov [7, 8] by the canonical transformation method as small corrections in the case of low exciton densities in the ladder approximation. As a result the degeneracy in the phase of the order parameter is lifted. The semiconductor-excitonic-insulator

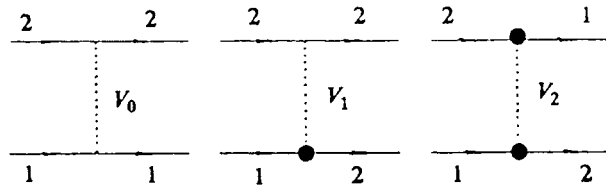


Figure 7. Diagrams describing the interaction (4.1).

phase transition becomes a first-order phase transition, and the collective excitation spectrum loses its acoustic nature. This disrupts the analogy between an excitonic insulator and the superfluid condensate of Bose particles.

Let us see how the introduction of additional terms in the Hamiltonian influences the Bethe-Salpeter equation in the functional integration formalism. The various diagrams corresponding to  $V$  are depicted in figure 7. We introduce the notation  $V_{1112} = V_{1121} = \dots = V_1$  and  $V_{1221} = V_{1212} = V_{2121} = V_{2112} = V_2$ . It can be demonstrated that  $V_1$  and  $V_2$  are linked by a certain relation that is a corollary of the fact that the wave functions belonging to different bands are not orthogonal to each other; hence,  $V_1$  and  $V_2$  are not independent parameters. We also assume that  $V_1$  and  $V_2$  are small and can be allowed for by perturbation theory techniques. Indeed, we are considering the case of a narrow band gap, with the emerging excitons being large-radius excitons. Since here all the characteristic momenta are small compared to the reciprocal lattice vector, the values of  $V_1$  and  $V_2$  prove to be small because of the orthogonality of the wave functions of different bands. And since  $V$  is a function of the difference of coordinates, it depends on one momentum in the momentum representation. We introduce the notation

$$V_1(\mathbf{k}) = g(\mathbf{k})V(\mathbf{k}) \quad V_2(\mathbf{k}) = g(\mathbf{k})g(-\mathbf{k})V(\mathbf{k}) \tag{4.2}$$

where  $g(\mathbf{k})$  is a form factor reflecting the fact that  $V_1$  and  $V_2$  are not independent. In what follows we employ the fact that generally  $g(\mathbf{k}) \sim \int d^3x [u_{1\mathbf{p}}(\mathbf{x})u_{2\mathbf{p}+\mathbf{k}}^*(\mathbf{x})] \sim i\mathbf{k} \cdot \mathbf{d}_{12}$ . Here  $u_{i\mathbf{p}}$  are the electron Bloch functions in band  $i$ , with the momentum  $\mathbf{p}$  being small in comparison to the reciprocal lattice vector (in view of which the dependence of  $g(\mathbf{k})$  on this momentum can be ignored), and  $\mathbf{d}_{12}$  is the dipole moment of the band-to-band transition divided by the electron charge (it is of the order of the interatomic distance). The additional terms in the local and bilocal decouplings must be taken into account in a self-consistent manner.

We start with local decoupling. By direct substitution we verify that the introduction of additional terms is equivalent to replacing  $\rho(\mathbf{k})$  in (2.5) with  $\tilde{\rho}(\mathbf{k})$  in the following manner:

$$\tilde{\rho}(\mathbf{k}) = \rho(\mathbf{k}) + \frac{g(\mathbf{k})}{\sqrt{T}} \sum_{\mathbf{p}} [\Delta(\mathbf{p}, \mathbf{k}, \omega_{\mathbf{k}}) + \Delta^*(\mathbf{p}, -\mathbf{k}, -\omega_{\mathbf{k}})].$$

In bilocal interaction the substitution is more complicated. Similar to the way we did in section 2, we regroup  $\Psi$  in the initial action in such a manner that adjoining it is  $\Psi$  in the combination  $P_i(\mathbf{x}, \mathbf{y}, \tau) = \tilde{\Psi}_i(\mathbf{x}, \tau)\Psi_i(\mathbf{y}, \tau)$ . After going over to the momentum representation, we arrive at a dependence of  $P$  on two momenta corresponding to

relative particle motion and to the motion of the particles as a whole. In bilocal decoupling  $P$  acts as  $\rho$  in local decoupling. Introducing additional terms into the Hamiltonian is equivalent to replacing  $P$  with  $\tilde{P}$ , which has the form

$$\tilde{P}_i(\mathbf{p}, \mathbf{P}, \omega) = P_i(\mathbf{p}, \mathbf{P}, \omega) + \frac{1}{\sqrt{T}} [g(\mathbf{k})\Delta(\mathbf{p}, \mathbf{P}, \omega) + \text{CC}] + \frac{1}{2}g(\mathbf{k})g(-\mathbf{k})P_{j \neq i}$$

$$\tilde{P}_i(\mathbf{p} + \mathbf{k}, \mathbf{P}, \omega) = P_i(\mathbf{p} + \mathbf{k}, \mathbf{P}, \omega) + \frac{1}{\sqrt{T}} [g(\mathbf{k})\Delta(\mathbf{p} + \mathbf{k}, \mathbf{P}, \omega) + \text{CC}] + \frac{1}{2}g(\mathbf{k})g(-\mathbf{k})P_{j \neq i}$$

where  $i, j = 1, 2$  are the band numbers. In expanding we must retain the terms up to  $|g|^2$  inclusive and discard the terms  $Vg^2(\Delta\Delta + \Delta^*\Delta^*)$ , which are taken into account twice. After performing all the transformations specified above and integrating over the  $\chi_i$  fields corresponding to bilocal decoupling, we arrive at the following expression for the action:

$$\begin{aligned} S = & \sum_{\mathbf{p}, \mathbf{k}, \omega} \Delta^*(\mathbf{p}, \mathbf{k}, \omega_n) \left[ i\omega_n - \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{k}^2}{2M} - E_g \right] \Delta(\mathbf{p}, \mathbf{k}, \omega_n) \\ & + \sum_{\mathbf{l}, \mathbf{p}, \mathbf{k}, \omega} V_l \Delta^*(\mathbf{p}, \mathbf{k}, \omega_n) \Delta(\mathbf{p} + \mathbf{l}, \mathbf{k}, \omega_n) \\ & + \frac{1}{2} \sum_{\mathbf{l}, \mathbf{p}, \mathbf{k}, \omega} V_l [g(\mathbf{l})g(-\mathbf{l})\Delta(\mathbf{p}, \mathbf{k}, \omega_n)\Delta(\mathbf{p} + \mathbf{l}, -\mathbf{k}, -\omega_n) + \text{CC}] \\ & - i \sum_{\mathbf{p}, \mathbf{k}} \Phi(\mathbf{k})g(\mathbf{k})[\Delta(\mathbf{p}, \mathbf{k}, \omega_n) + \Delta^*(\mathbf{p}, -\mathbf{k}, -\omega_n)] \\ & - \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{8\pi e^2} \Phi(\mathbf{k})\Phi(-\mathbf{k}) - F(\Delta^4). \end{aligned} \quad (4.3)$$

If we use the expansion of  $\Delta$  in the form (2.18), we can integrate over the ‘fast’ variables and arrive at the following expression for the effective action:

$$\begin{aligned} S_{\text{eff}} = & \sum_{\mathbf{k}} C_0^*(\mathbf{k}) \left[ i\omega_n - \frac{\mathbf{k}^2}{2M} + \lambda \right] C_0(\mathbf{k}) \\ & + \frac{\tilde{v}}{2} \sum_{\mathbf{k}} [C_0(\mathbf{k})C_0(-\mathbf{k}) + C_0^*(\mathbf{k})C_0^*(-\mathbf{k})] \\ & - i \sum_{\mathbf{k}} \Phi(\mathbf{k})\gamma(\mathbf{k}) [C_0(\mathbf{k}) + C_0^*(-\mathbf{k})] - \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{8\pi e^2} \Phi(\mathbf{k})\Phi(-\mathbf{k}) - F(C^4) \end{aligned} \quad (4.4)$$

with

$$\begin{aligned} \gamma(\mathbf{k}) &= \frac{g(\mathbf{k})}{(2\pi)^3} \int d^3p \psi_0(\mathbf{p}) \\ \tilde{v} &= \frac{1}{(2\pi)^6} \int d^3p_1 d^3p_2 \psi_0(\mathbf{p}_1) V_2(\mathbf{p}_1 - \mathbf{p}_2) \psi_0(\mathbf{p}_2). \end{aligned} \quad (4.5)$$

We can write the action in a more compact form by introducing the notation  $\mathbf{J}(\mathbf{k}) = i\gamma(\mathbf{k})\Phi(\mathbf{k})\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\mathbf{C}(\mathbf{k}) = \begin{pmatrix} C_0(\mathbf{k}) \\ C_0^*(-\mathbf{k}) \end{pmatrix}$ :

$$S_{\text{eff}} = S_0 + \frac{1}{2} \sum_{\mathbf{k}} \bar{\mathbf{C}}(\mathbf{k}) \Gamma(\mathbf{k}) \mathbf{C}(\mathbf{k}) - \frac{1}{2} \sum_{\mathbf{k}} [\bar{\mathbf{J}}(\mathbf{k}) \mathbf{C}(\mathbf{k}) + \bar{\mathbf{C}}(\mathbf{k}) \mathbf{J}(\mathbf{k})] - \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{8\pi e^2} \Phi(\mathbf{k})\Phi(-\mathbf{k}). \quad (4.6)$$



The matrix  $\Gamma$  and the functional  $S_0$  have different forms above and below the transformation point in the exciton spectrum. To find  $\Gamma$  and  $S_0$  in the excitonic phase, we must isolate the condensate part by representing the fields  $C$  and  $C^*$  as follows:

$$C_0(\mathbf{k}) = \nu_0 \delta_{\mathbf{k}0} + \tilde{C}(\mathbf{k}) \quad C_0^*(\mathbf{k}) = \nu_0^* \delta_{\mathbf{k}0} + \tilde{C}_0^*(\mathbf{k}) \quad (4.7)$$

where  $|\nu_0|^2$  has the meaning of the condensate's density and is determined from a balance equation obtained when the coefficient of  $\tilde{C}_0(0)$  vanishes, as in the case of a weakly nonideal Bose gas. Thus, for the functional  $S_0$  and the matrix  $\Gamma$  we have

$$S_0 = \begin{cases} 4 \frac{1}{2} \frac{(E_c - E_g + \tilde{\nu})^2}{f} \beta V & \text{if } E_g < E_c + \tilde{\nu} \\ 0 & \text{if } E_g > E_c + \tilde{\nu} \end{cases} \quad \Gamma = \begin{cases} \Gamma_{<} & \text{if } E_g < E_c + \tilde{\nu} \\ \Gamma_{>} & \text{if } E_g > E_c + \tilde{\nu} \end{cases} \quad (4.8)$$

$$\Gamma_{>} = \begin{pmatrix} i\omega_n - \mathbf{k}^2/2M - (E_g - E_c) & \tilde{\nu} \\ \tilde{\nu} & -i\omega_n - \mathbf{k}^2/2M - (E_g - E_c) \end{pmatrix}$$

$$\Gamma_{<} = \begin{pmatrix} i\omega_n - \mathbf{k}^2/2M - (E_c - E_g) - 2\tilde{\nu} & E_g - E_c \\ E_g - E_c & -i\omega_n - \mathbf{k}^2/2M - (E_c - E_g) - 2\tilde{\nu} \end{pmatrix}.$$

After the above transformations have been performed, it becomes possible to integrate the action over the excitonic fields, with the zeros of  $\det(\Gamma)$  determining the exciton spectrum  $\mathcal{E}(\mathbf{k})$  both in the instability region and above it and  $\text{Tr} \ln \Gamma$  contributes to the thermodynamics of the system:

$$\mathcal{E}(\mathbf{k}) = \begin{cases} [(E_g - E_c + \mathbf{k}^2/2M)^2 - |\tilde{\nu}|^2]^{1/2} & \text{if } E_g > E_c + \tilde{\nu} \\ [\delta^2 + s^2 \mathbf{k}^2 + (\mathbf{k}^2/2M)^2]^{1/2} & \text{if } E_g < E_c + \tilde{\nu} \end{cases} \quad (4.9)$$

where  $\delta$  is the gap in the exciton spectrum, and  $s$  is the speed of sound, with

$$\delta^2 = 4|\tilde{\nu}|(E_c - E_g + |\tilde{\nu}|) \quad (4.10)$$

$$Ms^2 = E_c - E_g + 2|\tilde{\nu}|. \quad (4.11)$$

After we integrate over the excitonic fields, we find that the coefficient of  $\Phi(\mathbf{k})\Phi(-\mathbf{k})$  determines the dielectric constant of the system,  $\bar{\epsilon}(\mathbf{k}, \omega_n)$ . For the sake of simplicity we give the expression for  $\bar{\epsilon}(\mathbf{k}, \omega = 0)$ :

$$\bar{\epsilon}(\mathbf{k}, \omega = 0) = \begin{cases} 1 + \frac{8\pi e^2 |\gamma(\mathbf{k})|^2 / \mathbf{k}^2}{E_g - E_c - |\tilde{\nu}| + \mathbf{k}^2/2M} & \text{if } E_g > E_c + |\tilde{\nu}| \\ 1 + \frac{4\pi e^2 |\gamma(\mathbf{k})|^2 / \mathbf{k}^2}{E_c - E_g + |\tilde{\nu}| + \mathbf{k}^2/4M} & \text{if } E_g < E_c + |\tilde{\nu}|. \end{cases} \quad (4.12)$$

As  $|\mathbf{k}| \rightarrow 0$ , the numerator of  $\bar{\epsilon}$  tends to a finite value  $v_0$  proportional to the dipole moment of a band-to-band transition.

Thus, allowing for additional terms in Hamiltonian (4.1) changes the nature of the phase transition, the exciton spectrum acquires a gap and the phase of the order parameter becomes fixed. Here the dielectric constant of the system near the spectrum transformation point tends to infinity.

Let us consider a system in which the upper band is doped. Integration over the excitonic fields yields the following expression for the effective action functional:

$$S_{\text{eff}} = \sum_p \bar{\alpha}(p)[i\varepsilon_n - E(p) + \mu]\alpha(p) - i\sqrt{T} \sum_{p,k} \bar{\alpha}(p)\alpha(p+k)\Phi(k) - \sum_k \frac{k^2}{8\pi e^2} \bar{\varepsilon}(\mathbf{k}, \omega) \Phi(k)\Phi(-k) \quad (4.13)$$

where  $E(p)$  is defined in (3.4), and  $\bar{\varepsilon}(\mathbf{k}, \omega)$  the medium's dielectric constant defined in (4.12), in which  $E_c$  generally depends on the doping degree. If we integrate over the  $\Phi$  field, we get an expression for the action depending only on the 'slow' doping electrons:

$$S_{\text{eff}} = \sum_p \bar{\alpha}(p)[i\varepsilon_n - E(p) + \mu]\alpha(p) - \frac{T}{2} \sum_{p_i, k} V_c(k) \bar{\alpha}_{p_1} \bar{\alpha}_{p_2} \alpha_{p_2+k} \alpha_{p_1-k} \quad (4.14)$$

$$V_c(k) = \frac{V(\mathbf{k})}{\bar{\varepsilon}(\mathbf{k}, \omega)}.$$

Thus, the strength of the effective interaction of the electrons belonging to the upper band decreases  $\bar{\varepsilon}$ -fold at frequencies that are low compared to the Fermi energy, with  $\bar{\varepsilon} \gg 1$ .

Let us now consider the case where one of the bands, say the upper, is much narrower than the other. We must write the Bethe-Salpeter equation in the electron-hole centre-of-mass system. After the fast variables are isolated, the equation reduces to the Schrödinger equation for the hydrogen atom. In this case the reduced mass  $m$  is the 'light' mass, and the exciton mass  $M = m_1 + m_2$  tends to infinity. Hence, in the exciton spectrum and the expression for  $\bar{\varepsilon}$  we can neglect terms of the order of  $k^2/2M$  not only when  $|\mathbf{k}| \rightarrow 0$  but also when  $|\mathbf{k}| \sim a^{-1}$ , with  $a$  the size of the crystal's unit cell. Equations (4.8)–(4.14) remain unchanged.

## 5. The superconductivity mechanism

Let us consider the Fermi liquid of electrons with the initial Coulomb interaction (4.14). The dielectric constant of the medium,  $\bar{\varepsilon}(\mathbf{k}, \omega)$ , caused by the closeness to the phase transition to the excitonic insulator state and given by formula (4.12), strongly depends on the frequency in the  $\omega \lesssim \omega_0 \ll \varepsilon_F$  range, with  $\omega_0 \sim |E_c - E_g|$ .

We wish to calculate the scattering amplitude for electrons with opposite momenta near the Fermi surface. The diagrams contributing to the vertex of interest to us can be divided into two classes: diagrams of class 1, which cannot be 'cut' along two parallel electron lines into two disconnected parts, and diagrams of class 2, which can be 'cut' into two parts along parallel electron lines but for which the electrons on the cut have an energy  $\Omega$  considerably higher than  $\omega_0$ , that is,  $\omega_0 \ll \Omega \ll \varepsilon_F$ .

Let us consider the diagrams of class 1 in greater detail. These diagrams can be divided into two subclasses (figure 8(c)):

(a) Diagrams irreducible in the Coulomb interaction, that is, those that cannot be cut into two disconnected parts along a single Coulomb-interaction line (figure 8(a)). We denote the vertex corresponding to the sum of all such diagrams by  $\Gamma_1$ .

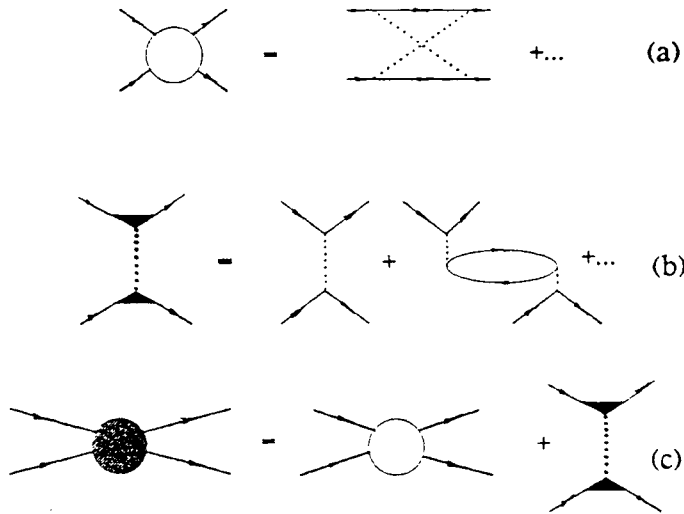


Figure 8. The sum of irreducible diagrams of class 1.

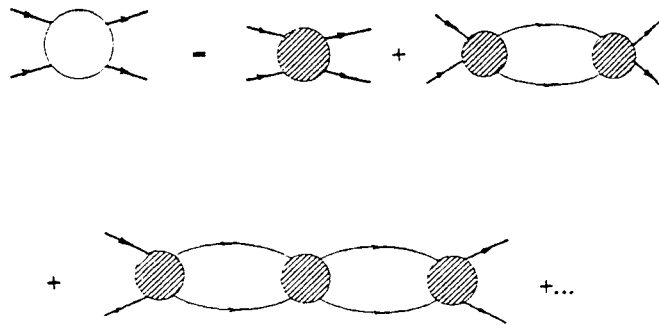


Figure 9. The diagrammatic equation for  $\Gamma_0$ .

(b) Diagrams reducible in the Coulomb interaction, that is, those that can be cut into two disconnected parts along a single Coulomb-interaction line (figure 8(b)). We denote the vertex corresponding to the sum of such diagrams by  $V^{\text{eff}}(0)$ . The expression for  $V^{\text{eff}}(0)$  has the form

$$\begin{aligned}
 V^{\text{eff}}(0) &= V^{\text{eff}}(\mathbf{k}, \omega \sim \omega_0) \\
 &= \Lambda \frac{V_C(\mathbf{k}, \omega_0)}{1 - \Pi(\mathbf{k}, \omega_0)V_C(\mathbf{k}, \omega_0)} \Lambda
 \end{aligned}
 \tag{5.1}$$

where  $\Pi$  is the total electron polarization operator,  $\Lambda$  the renormalized vertex (figure 8(b)) having two outer electron lines and one Coulomb line, and  $V_C(\mathbf{k}, \omega_0) = 4\pi e^2 / \mathbf{k}^2 \tilde{\epsilon}(\mathbf{k}, \omega_0)$  the Coulomb interaction that allows for the dielectric constant of the excitonic medium. We denote the sum of class-1 and class-2 diagrams by  $\Gamma_0$ . Diagrammatically the equation for  $\Gamma_0$  is depicted in figure 9. Solving it, we find that

$$\Gamma_0 = -V_\infty^{\text{eff}} + V^{\text{eff}}(0) + \frac{V_\infty^{\text{eff}} + \Gamma_1}{1 + N_0(V_\infty^{\text{eff}} + \Gamma_1) \ln(\epsilon_F/\Omega)}
 \tag{5.2}$$

where  $N_0$  is the density of states on the Fermi surface, and  $V_\infty^{\text{eff}} \sim 4\pi e^2/\mathbf{k}^2 \tilde{\epsilon}_\infty$ , with  $\tilde{\epsilon}_\infty$  the dielectric constant of the excitonic medium at frequencies  $\Omega$  satisfying the condition  $\omega_0 \ll \Omega \ll \epsilon_F$ , and  $\tilde{\epsilon}_\infty \sim 1$ . Since  $V^{\text{eff}}(0)$  is strongly weakened owing to  $\tilde{\epsilon}_0$ , the second term in (5.2) can be ignored. The third term is small compared to  $V_\infty^{\text{eff}}$  provided that  $\ln(\epsilon_F/\Omega) \gg 1$  (we assume that  $N_0 V_\infty^{\text{eff}} \sim 1$ ) and can also be discarded. This means that  $\Gamma_0 \approx -V_\infty^{\text{eff}}$  and, hence, corresponds to attraction. Summing the Cooper ladder for the scattering amplitude for electrons with opposite momenta, we find for the vertex  $\mathfrak{J}$  the following expression:

$$\mathfrak{J} = -\frac{V_\infty^{\text{eff}}}{1 - N_0 V_\infty^{\text{eff}} \ln(2\omega_0\gamma/\pi T_c)} \quad (5.3)$$

where  $T_c$  is the superconductive transition temperature, and  $\ln \gamma = C = 0.577$ . This expression gives the following estimate for the transition temperature:

$$T_c = \frac{2\omega_0\gamma}{\pi} \exp\left(-\frac{1}{N_0 V_\infty^{\text{eff}}}\right).$$

Since  $N_0 V_\infty^{\text{eff}} \sim 1$ , it can be assumed that the tight binding approximation is realized and  $T_c \sim \omega_0$ .

The first to discuss the mechanism of superconductivity in systems with strong dispersion of the dielectric constant caused by the interaction of electrons with optical phonons were Gurevich, Larkin, and Firsov [19]. The possibility of a strong reduction of the negative role of direct Coulomb interaction in high- $T_c$  systems by a similar mechanism was pointed out in [20]. The mechanism of superconductivity in high- $T_c$  systems related to a strong polarizability of the medium on the assumption that the system is close to an excitonic transition was studied in [21].

## 6. Conclusion

Study of the Fermi-liquid properties of the highly nonideal doping-electrons liquid in a medium with a strong dispersion of the low-frequency dielectric constant caused by the closeness to the transition to the state of an excitonic insulator has shown that a superconductive transition in such a system is possible. The mechanism of superconductivity here has a Coulomb nature. Note that in the system of interacting electrons and excitons an electron-exciton bound state can form, at least in principle, which may lead to the emergence of an excitonic transition induced by doping. We hope to return to this topic later.

## Acknowledgment

We would like to thank L A Maksimov and A N Kozlov for useful discussions and their constant interest in our work.

## Appendix

Here we present the final formulae for sums over the ring frequency in diagrams with two and three lines obtained from the expansion (2.11) for the case where the upper band has a nonzero occupation number. The expression for diagrams with four lines is so involved that, with the exception of zero doping, the volume of the present paper does not allow representing it. Here we use the following notation: the  $\varepsilon_n \equiv \pi(2n+1)T$  are the Fermi frequencies and the  $\omega_n \equiv 2\pi nT$  the Bose frequencies. Calculation of such sums is discussed in detail in [18].

$$\varepsilon(\mathbf{p}) = \frac{E_g}{2} + \frac{\mathbf{p}^2}{2m_e} \quad (\text{A1})$$

$$G_{01}(\mathbf{p}, \varepsilon_n) = \frac{1}{i\varepsilon_n + \varepsilon(\mathbf{p}) + \mu} \quad (\text{A2})$$

$$G_{02}(\mathbf{p}, \varepsilon_n) = \frac{1}{i\varepsilon_n - \varepsilon(\mathbf{p}) + \mu} \quad (\text{A3})$$

$$n_2(\mathbf{p}) = \frac{1}{\exp[(\varepsilon(\mathbf{p}) - \mu)/T] + 1} \quad (\text{A4})$$

$$n_1(\mathbf{p}) = 1 - \frac{1}{\exp[(\varepsilon(\mathbf{p}) + \mu)/T] + 1} \quad (\text{A5})$$

$$\Pi_{12}(\mathbf{p}, \mathbf{k}, \omega_m) = T \sum_n G_{01}(\mathbf{p} - \frac{1}{2}\mathbf{k}, \varepsilon_n - \frac{1}{2}\omega_m) G_{02}(\mathbf{p} + \frac{1}{2}\mathbf{k}, \varepsilon_n + \frac{1}{2}\omega_m) \quad (\text{A6})$$

$$\Pi_{12}(\mathbf{p}, \mathbf{k}, \omega_m) = -\frac{n_2(\mathbf{p} + \frac{1}{2}\mathbf{k}) - n_1(\mathbf{p} - \frac{1}{2}\mathbf{k})}{i\omega_m - E_{\mathbf{p}\mathbf{k}}} \quad (\text{A7})$$

$$E_{\mathbf{p}\mathbf{k}} = \varepsilon(\mathbf{p} + \frac{1}{2}\mathbf{k}) + \varepsilon(\mathbf{p} - \frac{1}{2}\mathbf{k}) = E_g + \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{k}^2}{2M} \quad (\text{A8})$$

$$\begin{aligned} \Pi_{112}(\mathbf{q}, \mathbf{p}_1, \mathbf{p}_2, \omega_n, \omega_m) = T \sum_{\varepsilon} & G_{01}\left(\mathbf{q} - \frac{\mathbf{p}_2 - \mathbf{p}_1}{2}, \varepsilon_q - \frac{\omega_m - \omega_n}{2}\right) \\ & \times G_{01}\left(\mathbf{q} + \frac{\mathbf{p}_2 - \mathbf{p}_1}{2}, \varepsilon_q + \frac{\omega_m - \omega_n}{2}\right) \\ & \times G_{02}\left(\mathbf{q} + \frac{\mathbf{p}_2 + \mathbf{p}_1}{2}, \varepsilon_q + \frac{\omega_m + \omega_n}{2}\right) \end{aligned} \quad (\text{A9})$$

$$\Pi_{112} = \frac{1}{i(\omega_m - \omega_n) + E_{212}} \left\{ \frac{n_1(\mathbf{q} + (\mathbf{p}_1 - \mathbf{p}_2)/2) - n_2(\mathbf{q} + (\mathbf{p}_1 + \mathbf{p}_2)/2)}{i\omega_m - [\varepsilon(\mathbf{q} + (\mathbf{p}_1 - \mathbf{p}_2)/2) + \varepsilon(\mathbf{q} + (\mathbf{p}_1 + \mathbf{p}_2)/2)]} \right. \\ \left. - \frac{n_1(\mathbf{q} + (\mathbf{p}_2 - \mathbf{p}_1)/2) - n_2(\mathbf{q} + (\mathbf{p}_1 + \mathbf{p}_2)/2)}{i\omega_n - [\varepsilon(\mathbf{q} + (\mathbf{p}_2 - \mathbf{p}_1)/2) + \varepsilon(\mathbf{q} + (\mathbf{p}_1 + \mathbf{p}_2)/2)]} \right\} \quad (\text{A10})$$

$$E_{212} = \varepsilon\left(\mathbf{q} + \frac{\mathbf{p}_2 - \mathbf{p}_1}{2}\right) - \varepsilon\left(\mathbf{q} + \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}\right) \quad (\text{A11})$$

$$\begin{aligned} \Pi_{221}(\mathbf{q}, \mathbf{p}_1, \mathbf{p}_2, \omega_n, \omega_m) = T \sum_{\varepsilon} & G_{01}\left(\mathbf{q} - \frac{\mathbf{p}_1 + \mathbf{p}_2}{2}, \varepsilon_q - \frac{\omega_m + \omega_n}{2}\right) \\ & \times G_{02}\left(\mathbf{q} + \frac{\mathbf{p}_2 - \mathbf{p}_1}{2}, \varepsilon_q + \frac{\omega_m - \omega_n}{2}\right) G_{02}\left(\mathbf{q} - \frac{\mathbf{p}_2 - \mathbf{p}_1}{2}, \varepsilon_q - \frac{\omega_m - \omega_n}{2}\right) \end{aligned} \quad (\text{A12})$$

$$\Pi_{221} = \frac{1}{i(\omega_n - \omega_m) - E_{212}} \left\{ \frac{n_1(q + (p_1 + p_2)/2) - n_2(q + (p_1 - p_2)/2)}{i\omega_m - [\varepsilon(q + (p_1 - p_2)/2) + \varepsilon(q + (p_1 + p_2)/2)]} - \frac{n_1(q + (p_1 + p_2)/2) - n_2(q + (p_2 - p_1)/2)}{i\omega_n - [\varepsilon(q + (p_2 - p_1)/2) + \varepsilon(q + (p_1 + p_2)/2)]} \right\} \quad (\text{A13})$$

$$\Pi_{1212}(p, k, \xi) = T \sum_{\omega} G_{01} \left( p - \frac{k_1 + k_2 - k_3}{2} \right) G_{02} \left( p - \frac{k_1 - k_2 - k_3}{2} \right) \times G_{01} \left( p - \frac{k_1 - k_2 + k_3}{2} \right) G_{02} \left( p + \frac{k_1 - k_2 + k_3}{2} \right). \quad (\text{A14})$$

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