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Semi-Fermionic Approach for Quantum Spin Systems*

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Abstract. We present a general derivation of semi-fermionic representation for generators of $SU(N)$ group as a bilinear combination of Fermi operators. The constraints are fulfilled by means of imaginary Lagrange multipliers. The important case of $SU(2)$ group is discussed. We demonstrate how the idea of semi-fermionic representation might be extended to the groups possessing dynamic symmetries. As an example, $SO(4)$ group is considered. We illustrate the application of semi-fermionic representations for various problems of strongly correlated physics.

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Introduction

It is known that spin operators satisfy neither Fermi nor Bose commutation relations. For example, the Pauli matrices for $S = 1/2$ operator commute on different sites and anticommute on the same site. The commutation relations for spins are determined by $SU(2)$ algebra, leading to the absence of a Wick theorem for the generators. To avoid this difficulty and construct a diagrammatic technique and path integral representation for spin systems various approaches have been used. The first class of approaches is based on representation of spins as bilinear combination of Fermi or Bose operators [1]-[6], whereas the representations belonging to the second class deal with more complex objects like, e.g. the Hubbard [7] and supersymmetric [8] operators, the nonlinear sigma model [9] etc. However, in all cases the fundamental problem which is at the heart of the difficulty is the local constraint problem. To illustrate it, let's consider e.g., first class of representations. Introducing the auxiliary Fermi or Bose fields makes the dimensionality of the Hilbert space, where these operators act, greater than the dimensionality of the Hilbert space for the spin operators. As a result, the spurious unphysical states should be excluded from the consideration which leads in turn to some restrictions (constraints) on bilinear combinations of Fermi/Bose operators, resulting in substantial complication of corresponding rules of the diagrammatic technique. The representations from the second class suffer from the same kind of problem, transformed either into a high nonlinearity of resulting model (non-linear sigma model) or hierarchical structure of perturbation series in the absence of Wick theorem (Hubbard operators). The exclusion of double occupied and empty states

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for a $S = 1/2$ impurity interacting with conduction electron bath (single impurity Kondo model), is controlled by fictitious chemical potential (Lagrange multiplier) of Abrikosov pseudofermions [4]. At the end of calculations this "chemical potential" λ should be put $\lambda \to -\infty$ to "freeze out" all unphysical states. In other words, there exists an additional $U(1)$ gauge field which freezes the charge fluctuations associated with this representation. The method works for dilute systems where all the spins can be considered independently. Unfortunately, attempts to generalize this technique to the lattice of spins results in the replacement of the local constraint (the number of particles on each site is fixed) by the so-called global constraint where the number of particles is fixed only on an average for the whole crystal. There is no reason to believe that such an approximation is a good starting point for the description of the strongly correlated systems. Another possibility to treat the local constraint rigorously is based on Majorana fermion representation. In this case fermions are "real" and corresponding gauge symmetry is Z_2. The difficulty with this representation is mostly related to the physical regularization of the fluctuations associated with the discrete symmetry group.

An alternative approach for spin Hamiltonians, free from local constraint problem, has been proposed in the pioneering paper of Popov and Fedotov [10]. Based on the exact fermionic representation for $S = 1/2$ and $S = 1$ operators, where the constraint is controlled by purely imaginary Lagrange multipliers, these authors demonstrated the power and simplification of the corresponding Matsubara diagram technique. The semi-fermionic representation (we discuss the meaning of this definition in the course of our paper) used by Popov and Fedotov is neither fermionic, nor bosonic, but reflects the fundamental Pauli nature of spins. The goal of this paper is to give a brief introduction to a semi-fermionic (SF) approach. A reader can find many useful technical details, discussion of mathematical aspects of semi-fermionic representation and its application to various problems in the original papers [10]-[21]. However, we reproduce the key steps of important derivations contained in [18],[19] in order to make the reader's job easier.

The manuscript is organized as follows: in Section I, the general concept of semi-fermions is introduced. We begin with the construction of the SF formalism for the fully antisymmetric representation of $SU(N)$ group and the fully symmetric SF representation of $SU(2)$ group using the imaginary-time (Matsubara) representation. We show a "bridge" between different representations using the simplest example of $S = 1$ in $SU(2)$ and discuss the SF approach for $SO(4)$ group. Finally, we show how to work with semi-fermions in real-time formalism and construct the Schwinger-Keldysh technique for SF. In this section, we will mostly follow original papers by the author [11], [18]. The reader acquainted with semi-fermionic technique can easily skip this section. In Section II, we illustrate the applications of SF formalism for various problems of condensed matter physics, such as ferromagnetic (FM), antiferromagnetic (AFM) and resonance valence bond (RVB) instabilities in the Heisenberg model, competition between local and non-local correlations in Kondo lattices in the vicinity of magnetic and spin glass critical points and the Kondo effect in quantum dots. In the Epilogue, we discuss some open questions

and perspectives.

1 Semi-fermionic representation

To begin with, we briefly reproduce the arguments contained in the original paper of Popov and Fedotov. Let's assume first $S = 1/2$. We denote as H_σ the Hamiltonian of spin system. The standard Pauli matrices can be represented as bilinear combination of Fermi operators as follows:

$$\sigma_j^z \to a_j^\dagger a_j - b_j^\dagger b_j, \qquad \sigma_j^+ \to 2a_j^\dagger b_j, \qquad \sigma_j^- \to 2b_j^\dagger a_j. \tag{1}$$

on each site i of the lattice. The partition function of the spin problem Z_σ is given by

$$Z_\sigma = Tr \exp(-\beta \hat{H}_\sigma) = i^N Tr \exp(-\beta(\hat{H}_F + i\pi \hat{N}_F/(2\beta)) \tag{2}$$

where $\hat{H}_F$ is the operator obtained from $\hat{H}_\sigma$ by the replacement (1) and

$$\hat{N} = \sum_{j=1}^{N} (a_j^\dagger a_j + b_j^\dagger b_j) \tag{3}$$

(N is the number of sites in the system and $\beta = 1/T$ is inverse temperature). To prove equation (2) we note that the trace over the nonphysical states of the i-th site vanishes

$$Tr_{unphys} \exp(-\beta(\hat{H}_F + i\pi \hat{N}_F/(2\beta)) = (-i)^0 + (-i)^2 = 0 \tag{4}$$

Thus, the identity (2) holds. The constraint of fixed number of fermions $\hat{N}_j = 1$, is achieved by means of the purely imaginary Lagrange multipliers $\mu = -i\pi/(2\beta)$ playing the role of imaginary chemical potentials of fermions. As a result, the Green's function

$$G = (i\omega_F - \epsilon)^{-1} \tag{5}$$

is expressed in terms of Matsubara frequencies $\omega_F = 2\pi T(n + 1/4)$ corresponding neither Fermi nor Bose statistics.

For $S = 1$ we adopt the representation of $\hat{H}_\sigma$ in terms of the 3-component Fermi field:

$$\sigma_j^z \to a_j^\dagger a - b_j^\dagger b, \quad \sigma_j^+ \to \sqrt{2}(a_j^\dagger c_j + c_j^\dagger b_j), \quad \sigma_j^- \to \sqrt{2}(c_j^\dagger a_j + b_j^\dagger c_j). \tag{6}$$

The partition function Z_σ is given by

$$Z_\sigma = Tr(-\beta \hat{H}_\sigma) = \left(\frac{i}{\sqrt{3}}\right)^N Tr \exp(-\beta(\hat{H}_F + i\pi \hat{N}_F/(3\beta)). \tag{7}$$

It is easy to note that the states with occupation numbers 0 and 3 cancel each other, whereas states with occupation 1 and 2 are equivalent due to the particle-hole symmetry and thus can be taken into account on an equal footing by proper

normalization of the partition function. As a result, the Green's function in the imaginary time representation is expressed in terms of $\omega_F = 2\pi T(n + 1/3)$ frequencies.

In this section, we show how semi-fermionic (Popov-Fedotov) representation can be derived using the mapping of partition function of the spin problem onto the corresponding partition function of the fermionic problem. The cases of arbitrary N (even) for SU(N) groups and arbitrary S for SU(2) group are discussed.

1.1 SU(N) group

We begin with the derivation of SF representation for SU(N) group. The SU(N) algebra is determined by the generators obeying the following commutation relations:

$$[\hat{S}^{\beta}_{\alpha,i}\hat{S}^{\rho}_{\sigma j}] = \delta_{ij}(\delta^{\rho}_{\alpha}\hat{S}^{\beta}_{\sigma i} - \delta^{\beta}_{\sigma}\hat{S}^{\rho}_{\alpha i}), \tag{8}$$

where $\alpha, \beta = 1, ..., N$. We adopt the definition of the Cartan algebra [22] of the SU(N) group $\{H_\alpha\} = S^{\alpha}_{\alpha}$ similar to the one used in [23], noting that the diagonal generators S^{α}_{α} are not traceless. To ensure a vanishing trace, the diagonal generators should only appear in combinations

$$\sum_{\alpha=1}^{N} s_\alpha S^{\alpha}_{\alpha} \quad \text{with} \quad \sum_{\alpha=1}^{N} s_\alpha = 0, \tag{9}$$

which effectively reduce the number of independent diagonal generators to $N - 1$ and the total number of SU(N) generators to $N^2 - 1$.

In this paper we discuss the representations of SU(N) group determined by rectangular Young Tableau (YT) (see [23] and [18] for details) and mostly concentrate on two important cases of the fully asymmetric (one column) YT and the fully symmetric (one row) YT.

The generator $\hat{S}^{\alpha}_{\beta}$ may be written as biquadratic form in terms of the Fermi-operators

$$\hat{S}^{\alpha}_{\beta} = \sum_{\gamma} a^{\dagger}_{\alpha\gamma} a^{\beta\gamma} \tag{10}$$

where the "color" index $\gamma = 1, ..., n_c$ and the $n_c(n_c + 1)/2$ constraints

$$\sum_{\alpha=1}^{N} a^{\dagger}_{\alpha\gamma_1} a^{\alpha\gamma_2} = \delta^{\gamma_2}_{\gamma_1} m \tag{11}$$

restrict the Hilbert space to the states with $m * n_c$ particles and ensure the characteristic symmetry in the color index a. Here m corresponds to the number of rows in rectangular Young Tableau whereas n_c stands for the number of columns. The antisymmetric behavior with respect to α is a direct consequence of the fermionic representation.

Let us consider the partition function for the Hamiltonian, expressed in terms of SU(N) generators

$$Z_S = Tr \exp(-\beta H_S) = Tr' \exp(-\beta H_F) \tag{12}$$

where Tr' denotes the trace taken with constraints (11). As it is shown in [18], the partition function of $SU(N)$ model is related to partition function of corresponding fermion model through the following equation:

$$Z_S = \int \prod_j d\mu(j) P(\mu(j)) Tr \exp\left(-\beta(H_F - \mu(j) n_F)\right) = \int \prod_j d\mu(j) P(\mu(j)) Z_F(\mu(j)) \tag{13}$$

here $P(\mu_j)$ is a distribution function of imaginary Lagrange multipliers. We calculate $P(\mu_j)$ explicitely using constraints (11).

We use the path integral representation of the partition function

$$Z_S / Z_S^0 = \int \prod_j d\mu(j) P(\mu(j)) \exp(\mathcal{A}) / \int \prod_j d\mu(j) P(\mu(j)) \exp(\mathcal{A}_0) \tag{14}$$

where the actions $\mathcal{A}$ and $\mathcal{A}_0$ are determined by

$$\mathcal{A} = \mathcal{A}_0 - \int_0^\beta d\tau H_F(\tau), \quad \mathcal{A}_0 = \sum_j \sum_{k=1}^N \int_0^\beta d\tau \bar{a}_k(j,\tau)(\partial_\tau + \mu(j)) a_k(j,\tau) \tag{15}$$

and the fermionic representation of SU(N) generators (10) is applied.

Let us first consider the case $n_c = 1$. We denote the corresponding distribution by $P_{N,m}(\mu(j))$, where m is the number of particles in the SU(N) orbital, or in other words, $1 \leq m < N$ labels the different fundamental representations of SU(N).

$$n_j = \sum_{k=1}^N \bar{a}_k(j) a_k(j) = m \tag{16}$$

To satisfy this requirement, the minimal set of chemical potentials and the corresponding form of $P_{N,m}(\mu(j))$ are to be derived.

To derive the distribution function, we use the following identity for the constraint (16) expressed in terms of Grassmann variables

$$\delta_{n_j,m} = \frac{1}{N} \sin\left(\pi(n_j - m)\right) / \sin\left(\frac{\pi(n_j - m)}{N}\right) \tag{17}$$

Substituting this identity into (12) and comparing with (14) one gets

$$P_{N,m}(\mu(j)) = \frac{1}{N} \sum_{k=1}^N \exp\left(\frac{i\pi m}{N}(2k-1)\right) \delta(\mu(j) - \mu_k), \tag{18}$$

where

$$\mu_k = -\frac{i\pi T}{N}(2k-1). \tag{19}$$

Since the Hamiltonian is symmetric under the exchange of particles and holes when the sign of the Lagrange multiplier is also changed simultaneously, we can simplify (18) to

$$P_{N,m}(\mu(j)) = \frac{2i}{N}\sum_{k=1}^{\lfloor N/2\rfloor}\sin\left(\pi m\frac{2k-1}{N}\right)\delta(\mu(j)-\mu_k) \tag{20}$$

where $\lfloor N/2\rfloor$ denotes the integer part of $N/2$. As shown below, this is the minimal representation of the distribution function corresponding to the minimal set of the discrete imaginary Lagrange multipliers. Another distributions function different from (20) can be constructed when the sum is taken from $k = N/2+1$ to N. Nevertheless, this DF is different from (20) only by the sign of imaginary Lagrange multipliers $\tilde{\mu}_k = \mu_k^* = -\mu_k$ and thus is supplementary to (20).

Particularly interesting for even N is the case when the SU(N) orbital is half–filled, $m = N/2$. Then all Lagrange multipliers carry equal weight

$$P_{N,N/2}(\mu(j)) = \frac{2i}{N}\sum_{k=1}^{N/2}(-1)^{k+1}\delta\left(\mu(j)-\mu_k\right). \tag{21}$$

Taking the limit $N\to\infty$ one may replace the summation in expression (21) in a suitable way by integration. Note, that while taking $N\to\infty$ and $m\to\infty$ limits, we nevertheless keep the ratio $m/N = 1/2$ fixed. Then, the following limiting distribution function can be obtained:

$$P_{N,N/2}(\mu(j)) \overset{N\to\infty}{\longrightarrow} \frac{\beta}{2\pi i}\exp\left(-\beta\mu(j)\frac{N}{2}\right) \tag{22}$$

resulting in the usual continuous representation of the local constraint for the simplest case $n_c = 1$

$$Z_S = Tr(\exp\left(-\beta H_F\right)\delta\left(n_j - \frac{N}{2}\right)) \tag{23}$$

We note the obvious similarity of the limiting DF (22) with the *Gibbs canonical distribution* provided that the Wick rotation from the imaginary axis of the Lagrange multipliers μ to the real axis of energies E is performed and thus $\mu(j)N/2$ has a meaning of energy.

Up to now, the representation we discussed was purely fermionic and expressed in terms of usual Grassmann variables when the path integral formalism is applied. The only difference from slave fermionic approach is that imaginary Lagrange multipliers are introduced to fulfill the constraint. Nevertheless, by making

the replacement

$$a_k(j,\tau)) \to a_k(j,\tau)\exp\left(\frac{i\pi\tau}{\beta}\frac{2k-1}{N}\right), \quad \bar{a}_k(j,\tau) \to \bar{a}_k(j,\tau)\exp\left(-\frac{i\pi\tau}{\beta}\frac{2k-1}{N}\right) \tag{24}$$

we arrive at the generalized Grassmann (semi-fermionic) boundary conditions

$$a_k(j,\beta) = a_k(j,0)\exp\left(i\pi\frac{2k-1}{N}\right), \quad \bar{a}_k(j,\beta) = \bar{a}_k(j,0)\exp\left(-i\pi\frac{2k-1}{N}\right) \tag{25}$$

This leads to a temperature diagram technique for the Green's functions

$$\mathcal{G}^{\alpha\beta}(j,\tau) = -\langle T_\tau a_\alpha(j,\tau)\bar{a}_\beta(j,0)\rangle \tag{26}$$

of semi-fermions with Matsubara frequencies different from both Fermi and Bose representations (see Fig.2).

The exclusion principle for this case is illustrated on Fig.1, where the $S = 1/2$ representation for the first two groups SU(2) and SU(4) are shown. The first point to observe is that the spin Hamiltonian does not distinguish the n particle and the n hole (or $N-n$ particle) subspace. Eq. (19) shows that the two phase factors $\exp(\beta\mu n)$ and $\exp(\beta\mu(N-n))$ accompanying these subspaces in Eq. (20) add up to a purely imaginary value within the same Lagrange multiplier, and the empty and the fully occupied states are always canceled. In the case of $N \geq 4$, where we have multiple Lagrange multipliers, the distribution function $P(\mu)$ linearly combines these imaginary prefactors to select out the desired physical subspace with particle number $n = m$.

In Fig.1, we note that on each picture, the empty and fully occupied states are canceled in their own unit circle. For SU(2) there is a unique chemical potential $\mu = \pm i\pi T/2$ which results in the survival of single occupied states. For SU(4) there are two chemical potentials (see also Fig.2). The cancellation of single and triple occupied states is achieved with the help of proper weights for these states in the distribution function whereas the states with the occupation number 2 are doubled according to the expression (21). In general, for SU(N) group with $n_c = 1$ there exists $N/2$ circles providing the realization of the exclusion principle.

1.2 SU(2) group

We consider now the generalization of the SU(2) algebra for the case of spin S. Here, the most convenient fermionic representation is constructed with the help of a $2S+1$ component Fermi field $a_k(j)$ provided that the generators of SU(2) satisfy the following equations:

$$S^+ = \sum_{k=-S}^{S-1} \sqrt{S(S+1) - k(k+1)}\, a^\dagger_{k+1}(j)a_k(j),$$

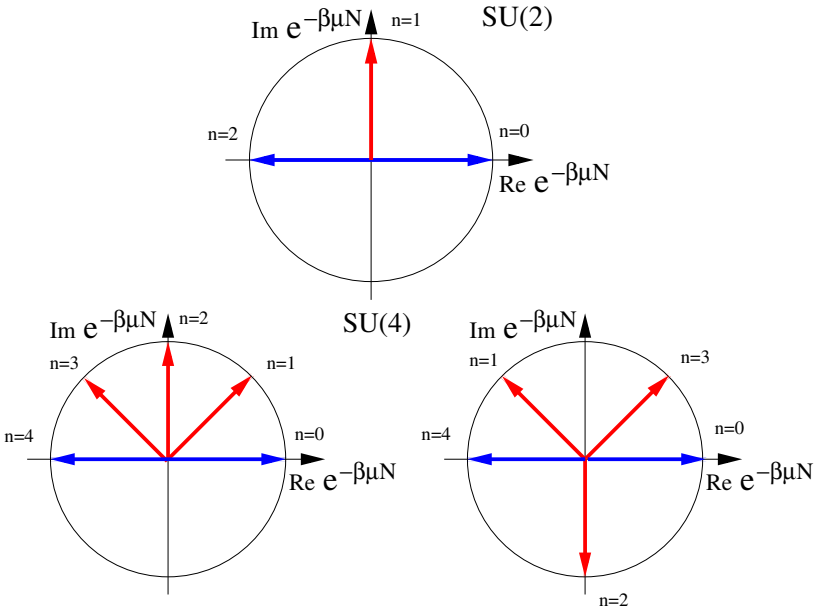

Figure 1: Graphical representation of exclusion principle for SU(N) semi-fermionic representation with even N, $n_c = 1$ (we use $\mu = i\pi T/2$ for SU(2) and $\mu_1 = i\pi T/4$, $\mu_2 = 3i\pi T/4$ for SU(4)).

$$S^- = \sum_{k=-S+1}^{S} \sqrt{S(S+1) - k(k-1)}\, a_{k-1}^\dagger(j) a_k(j),$$

$$S^z = \sum_{k=-S}^{S} k a_k^\dagger(j) a_k(j) \tag{27}$$

such that $dim H_F = 2^{2S+1}$ whereas the constraint reads as follows

$$n_j = \sum_{k=-S}^{k=S} a_k^\dagger(j) a_k(j) = l = 1 \tag{28}$$

Following the same routine as for SU(N) generators and using the occupancy condition to have $l = 1$ (or $2S$) states of the $(2S + 1)$ states filled, one gets the following distribution function, after using the particle–hole symmetry of the Hamiltonian H_S:

$$P_{2S+1,1}(\mu(j)) = \frac{2i}{2S+1} \sum_{k=1}^{\lfloor S+1/2 \rfloor} \sin\left(\pi \frac{2k-1}{2S+1}\right) \delta(\mu(j) - \mu_k) \tag{29}$$

where the Lagrange multipliers are $\mu_k = -i\pi T(2k-1)/(2S+1)$ and $k = 1, ..., \lfloor S + 1/2 \rfloor$, similarly to Eq.(19). In the particular case of the SU(2) model for some chosen values of spin S the distribution functions are given by the following expressions

$$P_{2,1}(\mu(j)) = i\, \delta\left(\mu(j) + \frac{i\pi T}{2}\right)$$

for $S = 1/2$

$$P_{3,1}(\mu(j)) = P_{3,2}(\mu(j)) = \frac{i}{\sqrt{3}}\, \delta\left(\mu(j) + \frac{i\pi T}{3}\right)$$

for $S = 1$.

This result corresponds to the original Popov-Fedotov description restricted to the $S = 1/2$ and $S = 1$ cases. A limiting distribution function corresponding to Eq. (22) for the constraint condition with arbitrary l is found to be

$$P_{\infty,l}(\mu(j)) \stackrel{S\to\infty}{\longrightarrow} \frac{\beta}{2\pi i} \exp(-\beta l \mu(j)). \tag{30}$$

For the case $l = m = N/2 \to \infty$ and $S = (N-1)/2 \to \infty$ the expression for the limiting DF $P_{\infty,l}(\mu(j))$ coincides with (23). We note that in $S \to \infty$ (or $N \to \infty$) limit, the continuum "chemical potentials" play the role of additional U(1) fluctuating field whereas for finite S and N they are characterized by fixed and discrete values.

When S assumes integer values, the minimal fundamental set of Matsubara frequencies is given by the table in Fig.2.

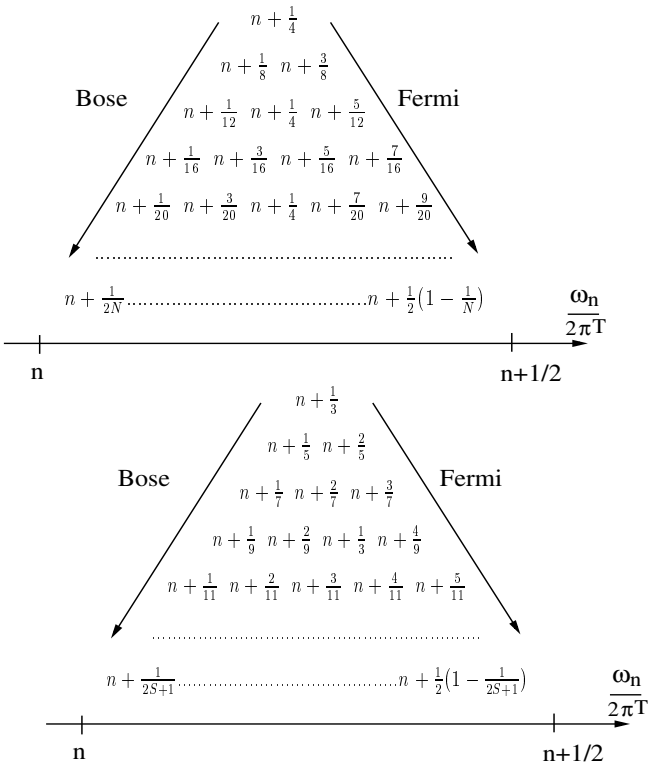

Figure 2: The minimal set of Matsubara frequencies for a) $SU(N)$ representation with even N/ $SU(2)$ representation for half-integer value of the spin. b) $SU(2)$ representation for integer values of the spin and $l = 1$.

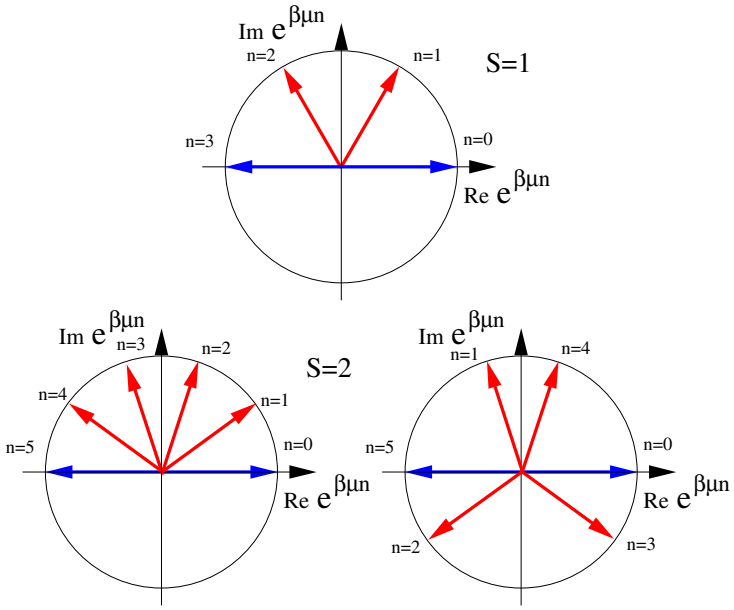

Figure 3: Graphical representation of exclusion principle for SU(2) semi-fermionic representation for $S = 1$ and $S = 2$. For any arbitrary integer value of spin there exists S circle diagrams corresponding to the S different chemical potentials and providing the realization of the exclusion principle.

The exclusion principle for SU(2) in the large spin limit can be also understood with the help of Fig.1 and Fig.3. One can see that the empty and the fully occupied states are canceled in each given circle similarly to even-N SU(N) algebra. The particle-hole (PH) symmetry of the representation results in an equivalence of single occupied and $2S$ occupied states whereas all the other states are canceled due to proper weights in the distribution function (29). In accordance with PH symmetry being preserved for each value of the chemical potential all circle diagrams (see Fig.3, Fig.5) are invariant with respect to simultaneous change $\mu \leftrightarrow -\mu$ and $n_{particle} \leftrightarrow n_{holes}$.

1.3 From SU(2) to SO(4)

We have shown that the general rectangular Young Tableau of size $n_c * m$ is represented by $N * n_c$ component fermionic field with n_c diagonal constraints and $n_c(n_c-1)/2$ off-diagonal constraints. However, the fully symmetric representation (one row) requires only $n_c + 1 = 2S + 1$ component field. The general scheme of projected representation for SU(N) group is given in [18]. We illustrate this idea on a simple example of $S = 1$.

We start with $2 * n_c = 4$ - field representation

$$(a_{11},\ a_{12},\ a_{21},\ a_{22}) \tag{31}$$

There are two diagonal and two off-diagonal constraints which read as follows:

$$a^\dagger_{11}a_{11} + a^\dagger_{21}a_{21} = 1, \quad a^\dagger_{12}a_{12} + a^\dagger_{22}a_{22} = 1. \tag{32}$$

$$a^\dagger_{11}a_{12} + a^\dagger_{21}a_{22} = 0, \quad a^\dagger_{12}a_{11} + a^\dagger_{22}a_{21} = 0 \tag{33}$$

and generators of $SU(2)$ group are given by

$$S^- = S^1_2 = a^\dagger_{11}a_{21} + a^\dagger_{12}a_{22}, \quad S^+ = S^2_1 = a^\dagger_{21}a_{11} + a^\dagger_{22}a_{12}$$

$$2S^z = S^2_2 - S^1_1 = a^\dagger_{21}a_{21} + a^\dagger_{22}a_{22} - a^\dagger_{11}a_{11} - a^\dagger_{12}a_{12} \tag{34}$$

Combining definition (34) with constraint (33) we reach the following equations:

$$\begin{aligned} S^- &= a^\dagger_{11}(a_{21} + a_{12}) + (a^\dagger_{12} + a^\dagger_{21})a_{22}, \\ S^+ &= (a^\dagger_{21} + a^\dagger_{12})a_{11} + a^\dagger_{22}(a_{12} + a_{21}), \\ S^z &= a^\dagger_{22}a_{22} - a^\dagger_{11}a_{11} \end{aligned} \tag{35}$$

Therefore, we conclude that the antisymmetric (singlet) combination $a_{12} - a_{21}$ does not enter the expression for spin $S = 1$ operators. Thus, three (out of four) component Fermi-field is sufficient for the description of $S = 1$ $SU(2)$ representation. Defining new fields as follows

$$a_{11} = f_{-1}, \quad a_{22} = f_1, \quad \frac{1}{\sqrt{2}}(a_{12} + a_{21}) = f_0, \quad \frac{1}{\sqrt{2}}(a_{12} - a_{21}) = s. \tag{36}$$

where fermions f_1, f_0, f_{-1} stand for $S^z = 1, 0 - 1$ projections of the triplet state and fermion s determines the singlet state, we come to standard $S = 1$ $SU(2)$ representation (c.f 6)

$$S^+ = \sqrt{2}(f_0^\dagger f_{-1} + f_1^\dagger f_0), \quad S^- = \sqrt{2}(f_{-1}^\dagger f_0 + f_0^\dagger f_1), \quad S_z = f_1^\dagger f_1 - f_{-1}^\dagger f_{-1}, \tag{37}$$

with the constraint

$$n_1 + n_0 + n_{-1} + n_s = 2 \tag{38}$$

where $n_\alpha = f_\alpha^\dagger f_\alpha$.

Nevertheless, the constraint (38) transforms to a standard SU(2) $S = 1$ constraint in both cases $n_s = 0$ and $n_s = 1$ since there is no singlet/triplet mixing allowed by SU(2) algebra.

To demonstrate the transformation of the local constraint let's first consider the case $n_s = 0$. The constraint reads as follows

$$n_1 + n_0 + n_{-1} = 2S \qquad \Longleftrightarrow \qquad \mathbf{S}^2 = S(S+1). \tag{39}$$

On the other hand, the the states with $2S$ occupation are equivalent to the states with single occupation due to particle-hole symmetry. Thus, the constraint (38) might be written as

$$\tilde{n}_1 + \tilde{n}_0 + \tilde{n}_{-1} = 1 \tag{40}$$

where $\tilde{n}_\alpha = 1 - n_\alpha$. The latter case corresponds to $n_s = 1$.

We start now with definition of $SO(4)$ group obeying the following commutation relations

$$[S_j, S_k] = ie_{jkl}S_l, \qquad [P_j, P_k] = ie_{jkl}S_l, \qquad [P_j, S_k] = ie_{jkl}P_l \tag{41}$$

where 6 generators of SO(4) group, namely vectors $\mathbf{S}$ and $\mathbf{P}$ are represented by the matrices

$$S^+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S^- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S^z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P^+ = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$P^- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P^z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{42}$$

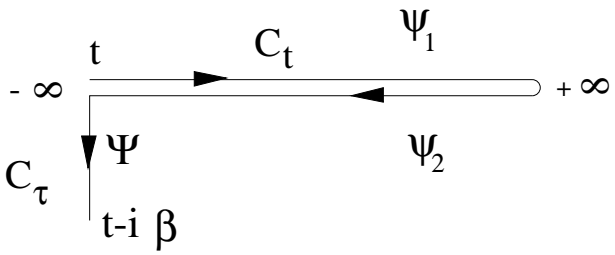

Figure 4: The Keldysh contour going from $-\infty \to \infty \to -\infty$ in real time. The boundary conditions on the imaginary time segment determine the generalized distribution functions for quasiparticles.

With the Casimir operator

$$\mathbf{S} \cdot \mathbf{P} = 0, \qquad \mathbf{S}^2 + \mathbf{P}^2 = 3.$$

Unlike SU(2) group, the singlet/triplet transitions are allowed in SO(4) group and determined by $\mathbf{P}$ operators. Using the definition of singlet/triplet fermions one comes to following representation

$$S^+ = \sqrt{2}(f_0^\dagger f_{-1} + f_1^\dagger f_0), \quad S^- = \sqrt{2}(f_{-1}^\dagger f_0 + f_0^\dagger f_1), \quad S_z = f_1^\dagger f_1 - f_{-1}^\dagger f_{-1}, \tag{43}$$

$$P^+ = \sqrt{2}(f_1^\dagger s - s^\dagger f_{-1}), \quad P^- = \sqrt{2}(s^\dagger f_1 - f_{-1}^\dagger s), \quad P_z = -(f_0^\dagger s + s^\dagger f_0). \tag{44}$$

with the only constraint

$$n_1 + n_0 + n_{-1} + n_s = 1$$

whereas the orthogonality condition is fulfilled automatically.

1.4 Real-time formalism

We discuss finally the real-time formalism based on the semi-fermionic representation of SU(N) generators. This approach is necessary for treating the systems out of equilibrium, especially for many component systems describing Fermi (Bose) quasiparticles interacting with spins. The real time formalism [24], [25] provides an alternative approach for the analytical continuation method for equilibrium problems allowing direct calculations of correlators whose analytical properties as function of many complex arguments can be quite cumbersome.

To derive the real-time formalism for SU(N) generators we use the path integral representation along the closed time Keldysh contour (see Fig.4). Following the standard route [26], we can express the partition function of the problem containing SU(N) generators as a path integral over Grassmann variables $\psi_l =$

$(a_{l,1}(j),...,a_{l,N}(j))^T$ where $l=1,2$ stands for upper and lower parts of the Keldysh contour, respectively,

$$\mathcal{Z}/\mathcal{Z}_0 = \int D\bar{\psi}D\psi \exp(i\mathcal{A})/\int D\bar{\psi}D\psi \exp(i\mathcal{A}_0) \tag{45}$$

where the actions $\mathcal{A}$ and $\mathcal{A}_0$ are taken as an integral along the closed-time contour $C_t + C_\tau$ which is shown in Fig.4. The contour is closed at $t = -\infty + i\tau$ since $\exp(-\beta H_0) = T_\tau \exp\left(-\int_0^\beta H_0 d\tau\right)$. We denote the ψ fields on upper and lower sides of the contour C_t as ψ_1 and ψ_2 respectively. The fields Ψ stand for the contour C_τ. These fields provide the matching conditions for $\psi_{1,2}$ and are excluded from the final expressions. Taking into account the semi-fermionic boundary conditions for generalized Grassmann fields (25) one gets the matching conditions for $\psi_{1,2}$ at $t = \pm\infty$,

$$\psi^\mu_{1,\alpha}|_k(-\infty) = \exp\left(i\pi\frac{2k-1}{N}\right)\psi^\mu_{2,\alpha}|_k(-\infty),$$
$$\psi^\mu_{1,\alpha}|_k(+\infty) = \psi^\mu_{2,\alpha}|_k(+\infty) \tag{46}$$

for $k = 1,...,\lfloor N/2 \rfloor$ and $\alpha = 1,...,N$. The correlation functions can be represented as functional derivatives of the generating functional

$$Z[\eta] = \mathcal{Z}_0^{-1}\int D\bar{\psi}D\psi \exp\left(i\mathcal{A} + i\oint_C dt(\bar{\eta}\sigma^z\psi + \bar{\psi}\sigma^z\eta)\right) \tag{47}$$

where η represents sources and the σ^z matrix stands for "causal" and "anti-causal" orderings along the contour.

The on-site Green's functions (GF) which are matrices of size $2N\times 2N$ with respect to both Keldysh (lower) and spin-color (upper) indices are given by

$$G^{\alpha\beta}_{\mu\nu}(t,t') = -i\frac{\delta}{i\delta\bar{\eta}^\alpha_\mu(t)}\frac{\delta}{i\delta\eta^\beta_\nu(t')}Z[\eta]|_{\bar{\eta},\eta\to 0}. \tag{48}$$

To distinguish between imaginary-time (26) and real-time (48) GF's, we use different notations for Green's functions in these representations.

After a standard shift-transformation [26] of the fields ψ the Keldysh GF of free semi-fermions assumes the form

$$G^\alpha_0(\epsilon) = G^{R,\alpha}_0\begin{pmatrix} 1-f_\epsilon & -f_\epsilon \\ 1-f_\epsilon & -f_\epsilon \end{pmatrix} - G^{A,\alpha}_0\begin{pmatrix} -f_\epsilon & -f_\epsilon \\ 1-f_\epsilon & 1-f_\epsilon \end{pmatrix},$$

where the retarded and advanced GF's are

$$G^{(R,A)\alpha}_0(\epsilon) = (\epsilon \pm i\delta)^{-1}, \quad f_\epsilon = f^{(N,k)}(\epsilon), \tag{49}$$

with equilibrium distribution functions

$$f^{(N,k)}(\epsilon) = T\sum_n \frac{e^{i\omega_{n_k}\tau|_{+0}}}{i\omega_{n_k}-\epsilon} = \frac{1}{e^{i\pi(2k-1)/N}\exp(\beta\epsilon)+1}. \tag{50}$$

A straightforward calculation of $f^{(N,k)}$ for the case of even N leads to the following expression

$$f^{(N,k)}(\epsilon) = \frac{\sum_{l=1}^{N}(-1)^{l-1}\exp\left(\beta\epsilon(N-l)\right)\exp\left(-\frac{i\pi l(2k-1)}{N}\right)}{\exp(N\beta\epsilon)+1}, \qquad (51)$$

where $k = 1, ..., N/2$. The equilibrium distribution functions (EDF) $f^{(2S+1,k)}$ for the auxiliary Fermi-fields representing arbitrary S for $SU(2)$ algebra are given by

$$f^{(2S+1,k)}(\epsilon) = \frac{\sum_{l=1}^{2S+1}(-1)^{l-1}\exp\left(\beta\epsilon(2S+1-l)\right)\exp\left(-\frac{i\pi(2k-1)}{2S+1})\right)}{\exp((2S+1)\beta\epsilon)+(-1)^{2S+1}} \qquad (52)$$

for $k = 1, ..., \lfloor S+1/2 \rfloor$. Particularly simple are the cases of $S = 1/2$ and $S = 1$,

$$f^{(2,1)}(\epsilon) = n_F(2\epsilon) - i\frac{1}{2\cosh(\beta\epsilon)}$$

$$f^{(3,1)}(\epsilon) = \frac{1}{2}n_B(\epsilon) - \frac{3}{2}n_B(3\epsilon) - i\sqrt{3}\frac{\sinh(\beta\epsilon/2)}{\sinh(3\beta\epsilon/2)} \qquad (53)$$

Here, the standard notations for Fermi/Bose distribution functions $n_{F/B}(\epsilon) = [\exp(\beta\epsilon) \pm 1]^{-1}$ are used. For $S = 1/2$ the semi-fermionic EDF satisfies the obvious identity $|f^{(2,1)}(\epsilon)|^2 = n_F(2\epsilon)$.

In general the EDF for half-integer and integer spins can be expressed in terms of Fermi and Bose EDF respectively. We note that since auxiliary Fermi fields introduced for the representation of SU(N) generators do not represent the true quasiparticles of the problem, helping only to treat properly the constraint condition, the distribution functions for these objects in general do not have to be real functions. Nevertheless, one can prove that the imaginary part of the EDF does not affect the physical correlators and can be eliminated by introducing an infinitesimally small real part for the chemical potential. In spin problems, a uniform/staggered magnetic field usually plays the role of such real chemical potential for semi-fermions.

2 Application of semi-fermionic representation

In this section we illustrate some of the applications of SF representation for various problems of strongly correlated physics.

2.1 Heisenberg model: FM, AFM and RVB

The effective nonpolynomial action for Heisenberg model with ferromagnetic (FM) coupling has been investigated in [10]. The model with antiferromagnetic (AFM)

interaction has been considered by means of semi-fermionic representation in [16] and [17] (magnon spectra) and in [11] for resonance valence bond (RVB) excitations. The Hamiltonian considered is given as

$$H_{int} = -\sum_{<ij>} J_{ij} \left(\vec{S}_i \vec{S}_j - \frac{1}{4} \right) \tag{54}$$

- Ferromagnetic coupling $J = I_{FM} > 0$

The exchange $\vec{S}_i \vec{S}_j$ is represented as four-semi-fermion interaction. Applying the Hubbard-Stratonovich transformation by the *local vector* field $\vec{\Phi}_i(\tau)$ the effective nonpolynomial action is obtained in terms of vector c-field. The FM phase transition corresponds to the appearance at $T \leq T_c$ of the nonzero average $\langle \Phi^z(q=0,0) \rangle$ which stands for the nonzero magnetization, or in other words, corresponds to the Bose condensation of the field Φ^z.

$$\Phi^z(\vec{k},\omega) = \mathcal{M}(\beta N)^{1/2} \delta_{\vec{k},0} \delta_{\omega,0} + \tilde{\Phi}^z(\vec{k},\omega). \tag{55}$$

In one loop approximation the standard molecular field equation can be reproduced

$$\mathcal{M} = I_{FM}(0) \tanh(\beta \mathcal{M}/2). \tag{56}$$

The saddle point (mean-field) effective action is given by well-known expression

$$\mathcal{A}_0[\mathcal{M}] = -N \left[\frac{\beta \mathcal{M}^2}{4 I_M(0)} - \ln \left(2 \cosh \left(\frac{\beta \mathcal{M}}{2} \right) \right) \right], \tag{57}$$

and the free energy per spin f_0 is determined by the standard equation:

$$\beta f_0 = -\ln Z_S = \frac{\beta \mathcal{M}^2}{4 I_M(0)} - \ln \left(2 \cosh \left(\frac{\beta \mathcal{M}}{2} \right) \right) \tag{58}$$

Calculation of the second variation of $\mathcal{A}_{eff}$ gives rise to the following expression

$$\begin{aligned} \delta \mathcal{A}_{eff} = & -\frac{1}{4} \sum_{\vec{k}} \Phi^z(\vec{k},0) \left[I_M^{-1}(\vec{k}) - \frac{\beta}{2 \cosh^2(\beta \Omega)} \right] \Phi^z(\vec{k},0) \\ & - \frac{1}{4} \sum_{\vec{k},\omega \neq 0} I_M^{-1}(\vec{k}) \Phi^z(\vec{k},\omega) \Phi^z(\vec{k},\omega) - \sum_{\vec{k},\omega} \Phi^+(\vec{k},\omega) \left[I_M^{-1}(\vec{k}) - \frac{\tanh(\beta\Omega)}{2\Omega - i\omega} \right] \Phi^-(\vec{k},\omega) \end{aligned} \tag{59}$$

where $\Omega = (g \mu_B H + \mathcal{M})/2$. The magnon spectrum ($T \leq T_c$) is determined by the poles of $\langle \Phi^+ \Phi^- \rangle$ correlator, $\omega = \lambda \mathbf{k}^2$.

- Antiferromagnetic coupling $J = I_{AFM} < 0$. *Néel solution*

The AFM transition corresponds to formation of the staggered condensate

$$\Phi^z(\vec{k},\omega) = \mathcal{N}(\beta N)^{1/2}\delta_{\vec{k},\vec{Q}}\delta_{\omega,0} + \tilde{\Phi}^z(\vec{k},\omega) \tag{60}$$

The one-loop approximation leads to standard mean-field equations for the staggered magnetization

$$\mathcal{N} = -I_{AFM}(Q)\tanh(\beta\mathcal{N}/2),$$
$$\mathcal{A}_0[\mathcal{N}] = N\left[\frac{\beta\mathcal{N}^2}{4I_{AFM}(Q)} + \ln\left(2\cosh\left(\frac{\beta\mathcal{N}}{2}\right)\right)\right]. \tag{61}$$

After taking into account the second variation of $\mathcal{A}_{eff}$, the following expression for the effective action is obtained [(see e.g. [16],[17]):

$$\delta\mathcal{A}_{eff} = \frac{1}{4}\sum_{\vec{k}}\Phi^z(\vec{k},0)\left[I^{-1}_{AFM}(\vec{k}) + \frac{\beta}{2\cosh^2(\beta\tilde{\Omega})}\right]\Phi^z(\vec{k},0)$$
$$+\frac{1}{4}\sum_{\vec{k},\omega\neq 0} I^{-1}_{AFM}(\vec{k})\Phi^z(\vec{k},\omega)\Phi^z(\vec{k},\omega)$$
$$+\sum_{\vec{k},\omega}\Phi^+(\vec{k},\omega)\left[I^{-1}_{AFM}(\vec{k}) + \frac{2\tilde{\Omega}\tanh(\beta\tilde{\Omega})}{4\tilde{\Omega}^2+\omega^2}\right]\Phi^-(\vec{k},\omega)$$
$$-\sum_{\vec{k},\omega}\Phi^+(\vec{k}+\vec{Q},\omega)\frac{i\omega}{4\tilde{\Omega}^2+\omega^2}\Phi^-(\vec{k},\omega). \tag{62}$$

The AFM magnon spectrum $\omega = c|\mathbf{k}|$.

- Antiferromagnetic coupling. *Resonance Valence Bond solution*

The four-semi-fermion term in (54) is decoupled by *bilocal scalar* field Λ_{ij}. The RVB spin liquid (SL) instability in 2D Heisenberg model corresponds to Bose-condensation of exciton-like [27] pairs of semi-fermions:

$$\Delta_0 = -\sum_{\mathbf{q}}\frac{I_{\mathbf{q}}}{I_0}\tanh\left(\frac{I_{\mathbf{q}}\Delta_0}{T}\right), \quad \mathcal{A}_0 = \frac{\beta|I|\Delta_0^2}{2} - \sum_{\mathbf{q}}\ln\left[2\cosh(\beta I_{\mathbf{q}}\Delta_0)\right] \tag{63}$$

where $\Delta_0 = \Delta(\mathbf{q}=0)$ is determined by the modulus of Λ_{ij} field

$$\Lambda_{<ij>}(\vec{R},\ \vec{r}) = \Delta(\vec{r})\exp\left(i\vec{r}\vec{A}(\vec{R})\right) \tag{64}$$

whereas the second variation of $\delta\mathcal{A}_{eff}$ describes the fluctuations of phase Λ_{ij}

$$\mathcal{A}_{eff} = \sum_{\mathbf{k},\omega} A_\alpha(\mathbf{k},\omega)\pi^{\alpha\beta}_{\mathbf{k},\omega}A_\beta(\mathbf{k},\omega),$$
$$\pi^{\alpha\beta}_{\mathbf{k},\omega} = Tr(p^\alpha p^\beta(G_{p+k}G_p + G_{p+k}G_p) + \delta_{\alpha\beta}f(I_{\mathbf{p}}\Delta_0)) \tag{65}$$

The spectrum of excitation in uniform SL is determined by zeros of π^R and is purely diffusive [28]-[29].

2.2 Kondo lattices: competition between magnetic and Kondo correlations

The problem of competition between Ruderman-Kittel-Kasuya-Yosida (RKKY) magnetic exchange and Kondo correlations is one of the most interesting problem of the heavy fermion physics. The recent experiments unambiguously show, that such a competition is responsible for many unusual properties of the integer valent heavy fermion compounds e.g. quantum critical behavior, unusual antiferromagnetism and superconductivity (see references in [19]). We address the reader to the review [30] for details of complex physics of Kondo effect in heavy fermion compounds. In this section we discuss the influence of Kondo effect on the competition between local (magnetic, spin glass) and non-local (RVB) correlations. The Ginzburg-Landau theory for nearly antiferromagnetic Kondo lattices has been constructed in [19] using the semi-fermion approach. We discuss the key results of this theory.

The Hamiltonian of the Kondo lattice (KL) model is given by

$$H = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + J \sum_j \left(\mathbf{S}_j \mathbf{s}_j + \frac{1}{4} N_j n_j \right) \tag{66}$$

Here the local electron and spin density operators for conduction electrons at site j are defined as

$$n_j = \sum_{j\sigma} c^\dagger_{j\sigma} c_{j\sigma}, \quad \mathbf{s}_j = \sum_{\sigma} \frac{1}{2} c^\dagger_{j\sigma} \hat{\tau}_{\sigma\sigma'} c_{j\sigma'}, \tag{67}$$

where $\hat{\tau}$ are the Pauli matrices and $c_{j\sigma} = \sum_k c_{k\sigma} \exp(ikj)$. The spin glass (SG) freezing is possible if an additional quenched randomness of the inter-site exchange I_{jl} between the localized spins arises. This disorder is described by

$$H' = \sum_{jl} I_{jl} (\mathbf{S}_j \mathbf{S}_l). \tag{68}$$

We start with a perfect Kondo lattice. The spin correlations in KL are characterized by two energy scales, i.e., $I \sim J^2/\varepsilon_F$, and $\Delta_K \sim \varepsilon_F \exp(-\varepsilon_F/J)$ (the inter-site indirect exchange of the RKKY type and the Kondo binding energy, respectively). At high enough temperature, the localized spins are weakly coupled with the electron Fermi sea having the Fermi energy ε_F, so that the magnetic response of a rare-earth sublattice of KL is of paramagnetic Curie-Weiss type. With decreasing temperature either a crossover to a strong-coupling Kondo singlet regime occurs at $T \sim \Delta_K$ or the phase transition to an AFM state occurs at $T = T_N \sim zI$ where z is a coordination number in KL. If $T_N \approx \Delta_K$ the interference between two trends results in the decrease of both characteristic temperatures or in suppressing one of them. The mechanism of suppression is based on the screening effect due to Kondo interaction. As we will show, the Kondo correlations screen the local order parameter, but leave nonlocal correlations intact. The mechanism

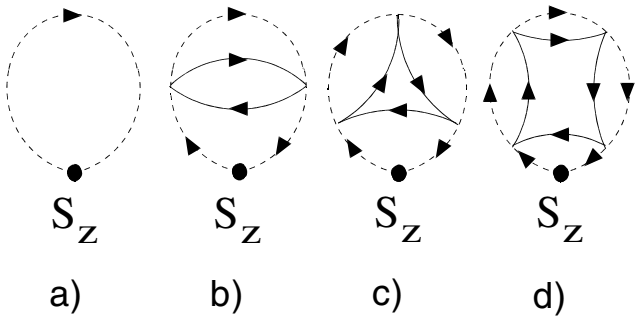

Figure 5: Kondo-screening of local moment by conduction electrons (solid line).

of Kondo screening for single-impurity Kondo problem is illustrated on Fig.5. The magnetization of local impurity in the presence of Kondo effect is given by:

$$\mathcal{M}(H) = S(g\mu_B)T\sum_{\omega}\left(\mathcal{G}_{\uparrow}(\omega) - \mathcal{G}_{\downarrow}(\omega)\right) = S(g\mu_B)\tanh\left(\frac{H\beta}{2}\right)\left[1 - \frac{1}{\ln(T/T_K)} - \frac{\ln(\ln(T/T_K))}{2\ln^2(T/T_K)} + ...\right]. \quad (69)$$

To take into account the screening effect in the lattice model we apply the semi-fermionic representation of spin operators. In accordance with the general path-integral approach to KL's, we first integrate over fast (electron) degrees of freedom. The Kondo exchange interaction is decoupled by auxiliary field ϕ [32] with statistics complementary to that of semi-fermions which prevents this field from Bose condensation except at $T = 0$. As a result, we are left with an effective bosonic action describing low-energy properties of KL model at high $T > T_K$ temperatures.

- Kondo screening of the Néel order

To analyze the influence of Kondo screening on formation of AFM order, we adopt the decoupling scheme for the Heisenberg model discussed in Section II.A. Taking into account the classic part of Néel field, we calculate the Kondo-contribution to the effective action which depends on magnetic order parameter $\mathcal{N}$:

$$\mathcal{A}_{\phi} = 2\sum_{\mathbf{q},n}\left[\frac{1}{\widetilde{J}} - \Pi(\mathcal{N})\right]|\phi_n(\mathbf{q})|^2. \quad (70)$$

where a polarization operator $\Pi(\mathcal{N})$ casts the form

$$\Pi(\mathcal{N}) = \rho(0)\ln\left(\frac{\epsilon_F}{T}\right) + \left[\frac{\pi}{2}\left(\frac{1}{\cosh(\beta\mathcal{N})} - 1\right) + O\left(\frac{\mathcal{N}^2}{T\epsilon_F}\right)\right], \quad (71)$$

where $\rho(0)$ is the density of states of conduction electrons at the Fermi level and the Kondo temperature $T_K = \epsilon_F \exp(-1/(\rho(0)J))$. Minimizing the effective action $\mathcal{A}(\phi, \mathcal{N})$ with respect to classic field $\mathcal{N}$, the mean field equation for Néel transition is obtained (c.f. with (56))

$$\mathcal{N} = \tanh\left(\frac{I_{\mathbf{Q}}\mathcal{N}}{2T}\right)\left[1 - \frac{a_N}{\ln(T/T_K)}\frac{\cosh^2(\beta I_{\mathbf{Q}}\mathcal{N}/2)}{\cosh^2(\beta I_{\mathbf{Q}}\mathcal{N})}\right]. \tag{72}$$

As a result, Kondo corrections to the molecular field equation reduce the Néel temperature

- Kondo enhancement of RVB correlations

Applying the similar procedure to nonlocal RVB correlations, we take into account the influence of Kondo effect on RVB correlations

$$\Pi(I_{\mathbf{q}}\Delta) = \rho(0)\ln\left(\frac{\epsilon_F}{T}\right) + \sum_{\mathbf{k}}\left[\frac{1}{\cosh\beta(I_{\mathbf{k}}\Delta)} - 1 + I_{\mathbf{k}}\Delta\tanh(\beta I_{\mathbf{k}}\Delta)\right]\frac{1}{\xi^2_{\mathbf{k}+\mathbf{q}} + (\pi/2\beta)^2}\,. \tag{73}$$

Here $\xi_k = \epsilon(\mathbf{k}) - \epsilon_F$. Minimizing the effective action with respect to Δ we obtain new self-consistent equation to determine the non-local semi-fermion correlator.

$$\Delta = -\sum_{\mathbf{q}}\frac{I_{\mathbf{q}}}{I_0}\left[\tanh\left(\frac{I_{\mathbf{q}}\Delta}{T}\right) + a_{sl}\frac{I_{\mathbf{q}}\Delta}{T\ln(T/T_K)}\right]. \tag{74}$$

It is seen that unlike the case of local magnetic order, the Kondo scattering favors transition into the spin-liquid state, because the scattering means the involvement of the itinerant electron degrees of freedom into the spinon dynamics.

- Kondo effect and quenched disorder

Let's assume that the RKKY interactions are random (e.g. due to the presence of non-magnetic impurities resulting in appearance of random phase in the RKKY indirect exchange). In this case the spin glass phase should be considered. As it has been shown in [15] and [19], the influence of static disorder on Kondo effect in models with Ising exchange on fully connected lattices (Sherrington-Kirkpatrick model) can be taken into account by the mapping KL model with quenched disorder onto the single impurity Kondo model in random (depending on replicas) magnetic field. It allows for the self-consistent determination of the Edwards-Anderson q_{EA}order parameter given by the following set of self-consistent equations

$$\tilde{q} = 1 - \frac{2c}{\ln(T/T_K)} - O\left(\frac{1}{\ln^2(T/T_K)}\right),$$

$$q = \int_x^G \tanh^2\left(\frac{\beta I x\sqrt{q}}{1 + 2c(\beta I)^2(\tilde{q} - q)/\ln(T/T_K)}\right) + O\left(\frac{q}{\ln^2(T/T_K)}\right). \tag{75}$$

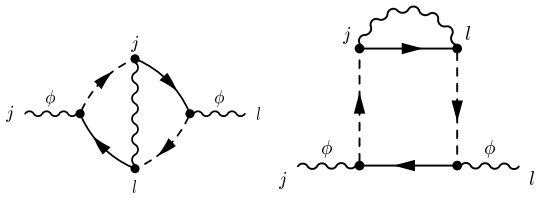

Figure 6: Feynman diagrams for nonlocal excitations associated with the overlap of Kondo clouds.

Here $q = q_{EA}$ and $\tilde{q}$ are nondiagonal and diagonal elements of Parisi matrix respectively. Therefore, the Kondo-scattering results in the depression of the freezing temperature due to the screening effects in the same way as the magnetic moments and the one-site susceptibility are screened in the single-impurity Kondo problem (c.f. Fig.5) when Ising and Kondo interactions are of the same order of magnitude. Let's now briefly discuss the fluctuation effects in Kondo lattices. The natural way to construct the fluctuation theory is to consider the non-local dynamical Kondo correlations described by the field $\phi(\mathbf{q}, \omega)$ (see Fig.6). In fact, the non-locality of the "semi-Bosonic" field is associated with an overlap of Kondo clouds [19] and responsible for a crossover from the localized magnetism to the itinerant-like fluctuational spin-liquid magnetism. The temperature dependence of static magnetic susceptibility becomes nonuniversal in spite of the fact that we are in a region of critical AFM fluctuations which is consistent with recent experimental observations.

2.3 Kondo effect in quantum dots

The single electron tunneling through the quantum dot [33] has been studied in great details during the recent decade. Among many interesting phenomena behind the unusual transport properties of mesoscopic systems, the Kondo effect in quantum dots, recently observed experimentally, continues to attract an attention both of experimental and theoretical communities. The modern nanoscience technologies allow one to produce the highly controllable systems based on quantum dot devices and possessing many of properties of strongly correlated electron systems. The quantum dot in a semiconductor planar heterostructure is a confined few-electron system (see Fig.7) contacted by sheets of two-dimensional gas (leads). Junctions between dot and leeds produce the exchange interaction between the spins of the dot and spins of itinerant 2D electron gas. Measuring the dc $I - V$ characteristics, one can investigate the Kondo effect in quantum dots under various conditions.

Various realizations of Kondo effect in quantum dots were proposed both

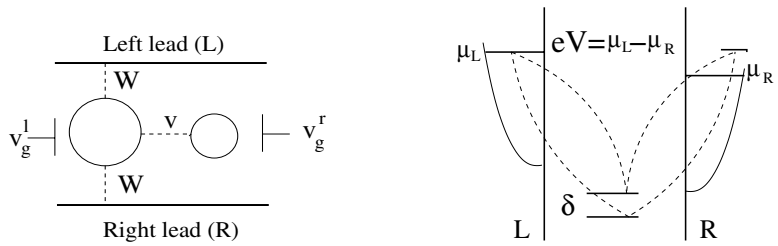

Figure 7: (a) Double quantum dot in a side-bound configuration (b) co-tunneling processes in biased DQD responsible for the resonance Kondo tunneling.

theoretically and experimentally in recent publications (see e.g. [34] for review). In order to illustrate the application of semi-fermionic approach we discuss briefly electric field induced Kondo tunneling in double quantum dot (DQD). As was noticed in [35], quantum dots with even $\mathcal{N}$ possess the dynamical symmetry $SO(4)$ of spin rotator in the Kondo tunneling regime, provided the low-energy part of its spectrum is formed by a singlet-triplet (ST) pair, and all other excitations are separated from the ST manifold by a gap noticeably exceeding the tunneling rate γ. A DQD with even $\mathcal{N}$ in a side-bound configuration where two wells are coupled by the tunneling v and only one of them (say, l) is coupled to metallic leads (L, R) is a simplest system satisfying this condition [35]. Such system was realized experimentally in Ref.[36]. As it was shown in [20] the Shrieffer-Wolff (SW) transformation, when applied to a spin rotator results in the following effective spin Hamiltonian

$$H_{int} = \sum_{kk',\alpha\alpha'=L,R} J^S_{\alpha\alpha'} f_s^\dagger f_s c^\dagger_{k\alpha\sigma} c_{k'\alpha'\sigma} + \sum_{kk',\alpha\alpha'\Lambda\Lambda'} \left(J^T_{\alpha\alpha'} \hat{S}^d_{\Lambda\Lambda'} + J^{ST}_{\alpha\alpha'} \hat{P}^d_{\Lambda\Lambda'} \right) \tau^d_{\sigma\sigma'} c^\dagger_{k\alpha\sigma} c_{k'\alpha'\sigma'} f^\dagger_\Lambda f_{\Lambda'} \quad (76)$$

where the c-operators describe the electrons in the leads and f-operators stand for the electrons in the dot. The matrices $\hat{S}^d$ and $\hat{P}^d$ (d=x,y,z) are 4×4 matrices defined by relations (41) (see Section I.C) and $J^S = J^{SS}$, $J^T = J^{TT}$ and J^{ST} are singlet, triplet and singlet-triplet coupling SW constants, respectively.

Applying the semi-fermionic representation of SO(4) group introduced in Section I.C we started with perturbation theory results analyzing the most divergent Feynman diagrams for spin-rotator model [20]. Following the "poor man's scaling" approach we derive the system of coupled renormalization group equations for effective couplings responsible for the transport through DQD. As a result, the differential conductance $G(eV, T)/G_0 \sim |J^{ST}_{LR}|^2$ is shown to be the universal

function of two parameters T/T_K and V/T_K, $G_0 = e^2/\pi\hbar$:

$$G/G_0 \sim \ln^{-2}\left(\max[(eV-\delta), T]/T_K\right) \tag{77}$$

Thus, the tunneling through singlet DQDs with $\delta = E_T - E_S \gg T_K$ exhibits a peak in differential conductance at $eV \approx \delta$ instead of the usual zero bias Kondo anomaly which arises in the opposite limit, $\delta < T_K$. Therefore, in this case the Kondo effect in DQD is induced by a strong external bias. The scaling equations can also be derived in Schwinger-Keldysh formalism (see [11] and also [18]) by applying the "poor man's scaling" approach directly to the dot conductance. The detailed analysis of the model (76) in a real-time formalism is a subject for a separate publication.

3 Epilogue and perspectives

In this paper, we demonstrated several examples of the applications of semi-fermionic representation to various problems of condensed matter physics. The list of these applications is not exhaustive. We did not discuss, e.g., the interesting development of SF approach for the Hubbard model with repulsive [14] and attractive [13] interaction, Dicke model, 2D Ising model in transverse magnetic field, application of SF formalism to mesoscopic physics [21] etc. Nevertheless, we would like to point out some problems of strongly correlated physics where the application of SF representation might be a promising alternative to existing field-theoretical methods.

Heavy Fermions

- Crossover from localized to itinerant magnetism in Kondo lattices
- Quantum critical phenomena associated with competition between local and nonlocal correlations
- Nonequilibrium spin liquids
- Effects of spin impurities and defects in spin liquids
- Crystalline Electric Field excitations in spin liquids
- Dynamic theory of screening effects in Kondo spin glasses.

Mesoscopic systems

- Nonequilibrium Kondo effect in Quantum Dots
- Two-channel Kondo in complex multiple dots
- Spin chains, rings and ladders

- Nonequilibrium spin transport in wires

Summarizing, we constructed a general concept of semi-fermionic representation for SU(N) groups. The main advantage of this representation in application to the strongly correlated systems in comparison with another methods is that the local constraint is taken into account exactly and the usual Feynman diagrammatic codex is applicable. The method proposed allows us to treat spins on the same footing as Fermi and Bose systems. The semi-fermionic approach can be helpful for the description of the quantum systems in the vicinity of a quantum phase transition point and for the nonequilibrium spin systems.

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