Supplementary material for
"Spin and Charge Correlations in Quantum Dots: An Exact Solution"

I.S. Burmistrov,[1,2] Yuval Gefen,[3] and M.N. Kiselev[4]

[1] *L.D. Landau Institute for Theoretical Physics RAS, Kosygina street 2, 119334 Moscow, Russia*
[2] *Department of Theoretical Physics, Moscow Institute of Physics and Technology, 141700 Moscow, Russia*
[3] *Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*
[4] *International Center for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy*
(Dated: March 26, 2010)

In the present note we expand on some important steps of the derivation. For the convenience of the reader we include here a guide, relating specific comments/steps in the main text to certain equations in the Supplementary Material.

1. Canonical partition functions: see Eqs. (32), (38).
2. Wei-Norman-Kolokolov (WNK) representation: see Section III.
3. Differences of our $\bar{\chi}$ from previous works: see Section VI.
4. Full distribution function of $\boldsymbol{S}^2$: see Section VII.
5. Steps in the derivation of the TDOS and the partition function: see Sections I-V.

I. CHARGE AND SPIN SEPARATION

Employing a Hubbard-Stratonovich transformation of the original action $S_{\rm tot}$, the one-particle Green function ($\tau_1 > \tau_2$) can be written as

$$G_{\alpha,\sigma}(\tau_1,\tau_2) = -\frac{1}{Z}\int \mathcal{D}[\bar{\Psi}\,\Psi\,\phi\,\boldsymbol{\Phi}]\,\psi_{\alpha,\sigma}(\tau_1)\,\bar{\psi}_{\alpha,\sigma}(\tau_2)e^{-\int_0^\beta d\tau\mathcal{L}}, \qquad Z = \int \mathcal{D}[\bar{\Psi}\,\Psi\,\phi\,\boldsymbol{\Phi}]\,e^{-\int_0^\beta d\tau\mathcal{L}} \tag{1}$$

(2)

where we remind

$$\mathcal{L} = \sum_\alpha \bar{\Psi}_\alpha\left[\partial_\tau - \epsilon_\alpha + \mu + i\phi + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Phi}}{2}\right]\Psi_\alpha + \frac{\boldsymbol{\Phi}^2}{4J} + \frac{\phi^2}{4E_c} - iN_0\phi. \tag{3}$$

First, we split the field $\phi(\tau)$ as[1]

$$\phi(\tau) = \tilde{\phi}(\tau) + 2\pi mT + \phi_0, \qquad |\phi_0| \leqslant \pi T, \qquad \int_0^\beta d\tau\tilde{\phi}(\tau) = 0. \tag{4}$$

Then, $\tilde{\phi}(\tau) + 2\pi mT$ can be gauge away by unitary transformation of the fermionic fields. The one-particle Green function becomes

$$G_{\alpha,\sigma}(\tau_1,\tau_2) = \int_{-\pi T}^{\pi T}\frac{d\phi_0}{2\pi T}\,\frac{\mathcal{Z}(\phi_0)}{Z}D(\tau_{12},\phi_0)\mathcal{G}_{\alpha,\sigma}(\tau_{12},\phi_0), \qquad Z = \int_{-\pi T}^{\pi T}\frac{d\phi_0}{2\pi T}\,D(0,\phi_0)\mathcal{Z}(\phi_0). \tag{5}$$

Here $\tau_{12} = \tau_1 - \tau_2$ and the so-called Coulomb-boson propogator reads[2,3]

$$D(\tau,\phi_0) = \left[\int \mathcal{D}[\tilde{\phi}]e^{-\frac{1}{4E_c}\int_0^\beta d\tau\tilde{\phi}^2(\tau)}\right]^{-1}\int \mathcal{D}[\tilde{\phi}]e^{-\frac{1}{4E_c}\int_0^\beta d\tau\tilde{\phi}^2(\tau)}e^{i\int_0^\tau d\tau\tilde{\phi}(\tau)}\sum_{m\in\mathbb{Z}}e^{-\frac{\pi^2 T}{E_c}(m+\frac{\beta\phi_0}{2\pi})^2}e^{2\pi iN_0(m+\frac{\beta\phi_0}{2\pi})}e^{-i2\pi mT\tau}$$

$$= e^{-E_c|\tau|}\sum_{k\in\mathbb{Z}}e^{i\phi_0(\beta k+\tau)-\beta E_c(k-N_0)^2-2E_c(k-N_0)\tau}. \tag{6}$$

The one-particle Green function and the partition function

$$\mathcal{G}_{\alpha,\sigma}(\tau_{12},\phi_0) = -\frac{1}{\mathcal{Z}(\phi_0)}\int \mathcal{D}[\bar{\Psi}\,\Psi\,\mathbf{\Phi}]\,\psi_{\alpha,\sigma}(\tau_1)\bar{\psi}_{\alpha,\sigma}(\tau_2)\,e^{-\mathcal{S}}, \qquad \mathcal{Z}(\phi_0) = \int \mathcal{D}[\bar{\Psi}\,\Psi\,\mathbf{\Phi}]\,e^{-\mathcal{S}} \tag{7}$$

are taken with respect to the action

$$\mathcal{S} = \int_0^\beta d\tau \left[\sum_\alpha \bar{\Psi}_\alpha(\tau)\left(\partial_\tau + \epsilon_\alpha - \mu + i\phi_0 + \frac{\boldsymbol{\sigma}\cdot\mathbf{\Phi}(\tau)}{2}\right)\Psi_\alpha(\tau) + \frac{\mathbf{\Phi}^2(\tau)}{4J}\right]. \tag{8}$$

Performing integration over $\mathbf{\Phi}$, we find that the one-particle Green function $\mathcal{G}_{\alpha,\sigma}(\tau_{12},\phi_0)$ and the partition function $\mathcal{Z}(\phi_0)$ corresponds to the problem with hamiltonian

$$\mathcal{H} = \sum_{\alpha,\sigma}\tilde{\epsilon}_\alpha a^\dagger_{\alpha,\sigma}a_{\alpha,\sigma} - J\boldsymbol{S}^2 \tag{9}$$

where $\tilde{\epsilon}_\alpha = \epsilon_\alpha - \mu + i\phi_0$.

II. WEI-NORMAN-KOLOKOLOV REPRESENTATION

The equation for time evolution operator for the time dependent Hamiltonian $H(t)$

$$i\frac{dU}{dt} = H(t)U(t) \tag{10}$$

with initial condition $U(0) = 1$ can be represented by a finite product of n exponential operators[4]. The index n is a dimension of the Lie algebra generated by $H(t)$ where the Hamiltonian is assumed to be linearly dependent on the group generators. The general formalism of time evolution operator construction is known as Wei-Norman method[4]. The particular parametrization of the time-evolution operator for $SU(2)$ group is due to Kolokolov[5].

Consider a time-evolution operator for the system with linear realization of dynamical $SU(2)$ symmetry described by the Hamiltonian

$$\hat{H}(t) = \vec{\Theta}(t)\cdot\vec{S} \tag{11}$$

The solution of (10) can be parameterized as

$$\hat{U}(t) = \exp(f(t)S^-)\exp(h(t)S^z)\exp(g(t)S^+) \tag{12}$$

where $S^\pm = S^x \pm iS^y$ and S^z are three generators of $SU(2)$ group. Functions $f(t)$, $h(t)$ and $g(t)$ satisfy the system of differential equations

$$\begin{cases} i\dot{f} &= \frac{1}{2}\Theta^+ - \Theta^z f - \frac{1}{2}\Theta^- f^2 \\ i\dot{h} &= \Theta^z + \Theta^- f \\ i\dot{g} &= \frac{1}{2}\Theta^- e^{-h} \end{cases} \tag{13}$$

with initial condition $f(0) = h(0) = g(0) = 0$. This system can be easily obtained with the help of Hausdorf formula[4]. The solution of the system (13) depends on the solution of the single Riccati equation (first equation in (13)). Parametrization of (13) by three new functions $\kappa^\pm_t, \rho_t$ defined as

$$\kappa^+_t = -f(t), \quad \kappa^-_t = -i\dot{g}(t)e^{h(t)}, \quad \rho_t = -i\dot{h}(t). \tag{14}$$

leads to the Kolokolov[5] representation of the time-evolution operator

$$\hat{U}(t) = e^{-\kappa^+_t S^-}e^{iS^z\int_0^t \rho_{t_1}dt_1}\exp\left(iS^+\int_0^t \kappa^-_{t_1}e^{-i\int_0^{t_1}\rho_{t2}dt_2}dt_1\right) \tag{15}$$

The initial condition $f(0) = 0$ is translated to $\kappa^+(0) = 0$. The initial conditions $g(0) = h(0) = 0$ are satisfied by construction of functions κ^-_t and ρ_t.

III. EVALUATION OF $\mathcal{K}_{\alpha\sigma}(t_+,t_-)$ AND $\mathcal{Z}$

The evolution operator $\mathcal{K}_{\alpha\sigma}(t_+,t_-)$ can be written as (see the main text)

$$\mathcal{K}_{\alpha\sigma}(t_+,t_-)=\prod_{p=\pm}\int\mathcal{D}[\rho_p,\kappa_p^p]e^{-\frac{ip}{4J}\int_0^{t_p}dt(\rho_p^2-4ip\dot\kappa_p^p\kappa_p^{-p})}e^{\frac{ip}{2}\int_0^{t_p}dt\rho_p(t)}\mathcal{C}_{\alpha\sigma}(t_+,t_-)\prod_{\gamma\neq\alpha}\mathcal{B}_\gamma(t_+,t_-),\tag{16}$$

where $\mathcal{C}_{\alpha\sigma}$ and $\mathcal{B}_\gamma$ are given in terms of single-particle traces:

$$\begin{aligned}\mathcal{C}_{\alpha\sigma}&=\operatorname{tr}\Big[e^{-i\tilde\epsilon_\alpha\hat n_\alpha t_+}\mathcal{A}_\alpha^{(+)}(t_+)a^\dagger_{\alpha\sigma}e^{i\tilde\epsilon_\alpha\hat n_\alpha t_-}\mathcal{A}_\alpha^{(-)}(t_-)a_{\alpha\sigma}\Big],\\ \mathcal{B}_\gamma&=\operatorname{tr}\Big[e^{-i\tilde\epsilon_\gamma\hat n_\gamma t_+}\mathcal{A}_\gamma^{(+)}(t_+)e^{+i\tilde\epsilon_\gamma\hat n_\gamma t_-}\mathcal{A}_\gamma^{(-)}(t_-)\Big].\end{aligned}\tag{17}$$

The expression for $\mathcal{Z}$ can be obtained from Eq. (16) by the substitution of $\mathcal{B}_\alpha$ for $\mathcal{C}_{\alpha\sigma}$:

$$\mathcal{Z}=\prod_{p=\pm}\int\mathcal{D}[\rho_p,\kappa_p^p]e^{-\frac{ip}{4J}\int_0^{t_p}dt(\rho_p^2-4ip\dot\kappa_p^p\kappa_p^{-p})}e^{\frac{ip}{2}\int_0^{t_p}dt\rho_p(t)}\prod_\gamma\mathcal{B}_\gamma(t_+,t_-).\tag{18}$$

Evaluation of the single-particle traces yields

$$\mathcal{C}_{\alpha\uparrow}=e^{-2i\tilde\epsilon_\alpha t_+}\sum_{p=\pm}e^{i\tilde\epsilon_\alpha t_p}e^{\frac{ip}{2}\int_0^{t_p}dt\rho_p(t)}\tag{19}$$

and

$$\begin{aligned}\mathcal{B}_\gamma=1+e^{-2i\tilde\epsilon_\gamma(t_+-t_-)}+2e^{-i\tilde\epsilon_\gamma(t_+-t_-)}\cos\left[\sum_{p=\pm}\int_0^{t_p}dt\frac{\rho_p(t)}{2}\right]+\prod_{p=\pm}e^{-ip\tilde\epsilon_\gamma t_p}e^{\frac{ip}{2}\int_0^{t_p}dt\rho_p(t)}\\ \times\left[p\kappa_p^p(t_p)+i\int_0^{t_{-p}}dt\kappa_{-p}^p(t)e^{ip\int_0^t dt'\rho_{-p}(t')}\right].\end{aligned}\tag{20}$$

The expression for $\mathcal{B}_\gamma$ is bilinear form of the fields κ_p^+ and κ_p^-. Due to specific form of the initial conditions, they have simple dynamics and can be integrated out exactly[5]. The resulting functional integral over fields ρ_p is of the Feynman-Kac type. After introducing the following variables

$$\xi_p(t)=ip\int_0^t dt'\rho_p-\frac{i}{2}\sum_{p'}p'\int_0^{t_{p'}}dt'\rho_{p'},\tag{21}$$

it can be written as

$$\begin{aligned}\mathcal{K}_{\alpha\uparrow}(t_+,t_-)=\frac14 e^{-\frac{\beta J}{4}}e^{-i\tilde\epsilon_\alpha t_+}\prod_{\gamma\neq\alpha}\left(-\oint_{|z_\gamma|=1}\frac{dz_\gamma}{2\pi i z_\gamma^2}\right)\frac{e^{-w}}{v}\int_{-\infty}^{\infty}d\xi_+d\xi_-\exp\left(-\frac{1}{4v}e^{-\frac{\varsigma_++\varsigma_-}{2}}-2v\cosh\frac{\xi_+-\xi_-}{2}\right)\\ \times\left[e^{\xi_+/2}+e^{-i\tilde\epsilon_\alpha(t_+-t_-)}e^{\xi_-/2}\right]\langle\xi_+|e^{-iH_Jt_+}e^{-3\xi/2}e^{iH_Jt_-}|\xi_-\rangle.\end{aligned}\tag{22}$$

Here we have introduced

$$w=\sum_{\gamma\neq\alpha}z_\gamma(1+e^{-2i\tilde\epsilon_\gamma(t_+-t_-)}),\qquad v=\sum_{\gamma\neq\alpha}z_\gamma e^{-i\tilde\epsilon_\gamma(t_+-t_-)}.\tag{23}$$

The hamiltonian

$$H_J=-J\frac{\partial^2}{\partial\xi^2}+\frac{J}{4}e^{-\xi}\tag{24}$$

of the one-dimensional quantum mechanics is exactly solvable. Its eigenfunctions are spanned by modified Bessel functions $K_{2i\nu}$:

$$\langle\nu|\xi\rangle=\frac{2}{\pi}\sqrt{\nu\sinh(2\pi\nu)}K_{2i\nu}(e^{-\xi/2})\tag{25}$$

where ν is real parameter and corresponding eigenvalues are equal to $J\nu^2$. In a similar way the expression (18) for the partition function becomes

$$\mathcal{Z} = \frac{1}{4} e^{-\frac{\beta J}{4}} e^{-i\tilde{\epsilon}_\alpha t_+} \prod_{\gamma\neq\alpha} \left(- \oint\limits_{|z_\gamma|=1} \frac{dz_\gamma}{2\pi i z_\gamma^2} \right) \frac{e^{-w}}{v} \int_{-\infty}^{\infty} d\xi_+ d\xi_- \exp\left(-\frac{1}{4v} e^{-\frac{\zeta_+ + \zeta_-}{2}} - 2v \cosh \frac{\xi_+ - \xi_-}{2} \right) \times \langle \xi_+ | e^{-iH_J t_+} e^{-\xi} e^{iH_J t_-} | \xi_- \rangle. \quad (26)$$

Evaluating the integrals over ξ_+ and ξ_- in Eqs. (22) and (26) with the help of the following identity

$$\exp\left(-\frac{1}{4v} e^{-\frac{\zeta_+ + \zeta_-}{2}} - 2v \cosh \frac{\xi_+ - \xi_-}{2} \right) = \frac{16}{\pi^2} \int_0^\infty d\nu\, \nu \sinh(2\pi\nu) K_{2i\nu}(e^{-\xi_+/2}) K_{2i\nu}(e^{-\xi_-/2}) K_{2i\nu}(2v), \quad (27)$$

we obtain

$$\mathcal{K}_{\alpha\uparrow}(t_+, t_-) = \frac{1}{\sqrt{\pi}} [iJ(t_+ - t_-)]^{-3/2} \exp\left(-\frac{iJ(t_+ - t_-)}{4} - 2i\tilde{\epsilon}_\alpha t_+ \right) \int\limits_{-\infty}^{\infty} dh \sinh h \prod_{\gamma\neq\alpha} \prod_{\sigma=\pm} \left[1 + e^{\sigma h + \tilde{\epsilon}_\gamma (t_+ - t_-)} \right] \times \sum_{s=\pm} (2h + isJt_{-s}) \exp\left(i\tilde{\epsilon}_\alpha t_s + \frac{isJt_s}{4} - \frac{(2h + isJt_s)^2}{4i(t_+ - t_-)J} \right) \quad (28)$$

and

$$\mathcal{Z} = \frac{2}{\sqrt{\pi}} [iJ(t_+ - t_-)]^{-3/2} \int_{-\infty}^{\infty} dh\, h \sinh h \prod_{\gamma} \prod_{\sigma=\pm} \left[1 + e^{\sigma h + \tilde{\epsilon}_\gamma (t_+ - t_-)} \right] e^{-h^2/\beta J}. \quad (29)$$

IV. EVALUATION OF THE PARTITION FUNCTION Z

The partition function Z is given as (see the main text)

$$Z = \int_{-\pi T}^{\pi T} \frac{d\phi_0}{2\pi T} \sum_{k\in\mathbb{Z}} e^{ik\beta\phi_0 - \beta E_c (k - N_0)^2} \mathcal{Z}(\phi_0) \quad (30)$$

where

$$\mathcal{Z}(\phi_0) = \frac{2}{\sqrt{\pi} (\beta J)^{3/2}} \int_{-\infty}^{\infty} dh\, h \sinh h\, e^{-h^2/\beta J} \prod_{\gamma} \prod_{\sigma=\pm} \left[1 + e^{\sigma h - \beta(\epsilon_\gamma - \mu + i\phi_0)} \right]. \quad (31)$$

Performing first integration over h in Eq. (30) with the help of the following idenitity[3]

$$\prod_\gamma \left(1 + e^{i\theta} e^{-\beta(\epsilon_\gamma - \mu)} \right) = \sum_{N=0}^{\infty} Z_N e^{\beta\mu N} e^{iN\theta}, \qquad Z_N = \oint \frac{dz}{2\pi i} z^{-N-1} \prod_\gamma \left(1 + z e^{-\beta\epsilon_\gamma} \right), \quad (32)$$

and then integration over ϕ_0 we obtain Eq.(4) of the main text.

At temperatures $\delta \ll T \ll \mu \ln J_\star / T$, we find

$$\prod_\gamma \prod_{\sigma=\pm} \left[1 + e^{\sigma h - \beta(\epsilon_\gamma - \mu + i\phi_0)} \right] \approx e^{-\beta h^2/\delta} \prod_\gamma \left[1 + e^{-\beta(\epsilon_\gamma - \mu + i\phi_0)} \right]^2, \quad (33)$$

where

$$\frac{1}{\delta} = -T \frac{\partial^2}{\partial \mu^2} \sum_\gamma \ln \left[1 + e^{-\beta(\epsilon_\gamma - \mu + i\phi_0)} \right]. \quad (34)$$

Performing integration over h and, then, over ϕ_0 in Eqs. (31) and (30), we find

$$Z = \sum_{n\in\mathbb{Z}} e^{-\beta E_c (n - N_0)^2} \prod_\gamma \left[1 + e^{-\beta(\epsilon_\gamma - \mu)} \right]^2 \left(\frac{J_\star}{J} \right)^{3/2} e^{\beta(J_\star - J)/4}. \quad (35)$$

V. EVALUATION OF THE TDOS $\nu(\varepsilon)$

The exact one-particle Green function in the imaginary time is given as (see the main text)

$$G_{\alpha\uparrow}(\tau>0) = -\frac{1}{Z}\int_{\pi T}^{\pi T}\frac{d\phi_0}{2\pi T}e^{i\phi_0\tau}\sum_{k\in\mathbb{Z}}e^{ik\beta\phi_0}e^{-\beta E_c(k-N_0)^2}e^{-2E_c(k-N_0+\frac{1}{2})\tau}\mathcal{K}_{\alpha\uparrow}(\tau,\phi_0) \tag{36}$$

where

$$\mathcal{K}_{\alpha\uparrow}(\tau) = \frac{1}{\sqrt{\pi}(\beta J)^{3/2}}e^{-\beta J/4}e^{J\tau(1-T\tau)/4}e^{-(\epsilon_\alpha-\mu+i\phi_0)\tau}\int_{-\infty}^{\infty}dh\sinh h\, e^{-h^2/\beta J}\prod_{\gamma\neq\alpha}\prod_{\sigma=\pm}\Big(1+e^{\sigma h-\beta(\epsilon_\gamma-\mu+i\phi_0)}\Big)$$
$$\times\Big[e^{-hT\tau}(2h+J\tau-J\beta)+e^{-(\epsilon_\alpha-\mu+i\phi_0)\beta}e^{h(T\tau-1)}(2h-J\tau)\Big]. \tag{37}$$

Performing first integration over h in Eq. (37) with the help of the following idenitity[3]

$$\prod_{\gamma\neq\alpha}\Big(1+e^{i\theta}e^{-\beta(\epsilon_\gamma-\mu)}\Big)=\sum_{N=0}^{\infty}Z_N(\epsilon_\alpha)e^{\beta\mu N}e^{iN\theta},\qquad Z_N(\epsilon_\alpha)=\oint\frac{dz}{2\pi i}z^{-N-1}\prod_{\gamma\neq\alpha}\left(1+ze^{-\beta\epsilon_\gamma}\right) \tag{38}$$

and then integration over ϕ_0 we obtain

$$G_{\alpha\uparrow}(\tau>0) = -\frac{1}{2Z}\sum_{n_{\uparrow,\downarrow}\in\mathbb{Z}}e^{-\beta E_c(n-N_0)^2+\beta\mu n+\beta Jm(m+1)}\exp\left[-\tau\left(\epsilon_\alpha-\mu+J\left(m+\frac{1}{4}\right)+2E_c\left(n-N_0+\frac{1}{2}\right)\right)\right]$$
$$\times\left[2m\Big(Z_{n_\uparrow}(\epsilon_\alpha)Z_{n_\downarrow}(\epsilon_\alpha)-Z_{n_\uparrow+1}(\epsilon_\alpha)Z_{n_\downarrow-1}(\epsilon_\alpha)\Big)+e^{-\beta\epsilon_\alpha}(2m+1)\Big(Z_{n_\uparrow-1}(\epsilon_\alpha)Z_{n_\downarrow}(\epsilon_\alpha)-Z_{n_\uparrow}(\epsilon_\alpha)Z_{n_\downarrow-1}(\epsilon_\alpha)\Big)\right] \tag{39}$$

Here $n=n_\uparrow+n_\downarrow$ and $m=(n_\uparrow-n_\downarrow)/2$. For the symmetry reasons, the expression for $G_{\alpha\downarrow}(\tau>0)$ is also given by Eq. (39). By using general expression[6]

$$\nu(\varepsilon) = -\frac{1}{\pi}\cosh\frac{\varepsilon}{2T}\sum_{\alpha,\sigma=\pm}\int_{-\infty}^{\infty}dt\, e^{i\varepsilon t}G_{\alpha\sigma}\left(it+\frac{\beta}{2}\right) \tag{40}$$

we find Eq.(3) of the main text. At temperatures $\delta\ll T\ll\mu/\ln(J_\star/T)$ we can use approximation similar to Eq. (33) for $Z_n(\varepsilon_\alpha)$. Performing integration over h in Eq. (37) and, then, over ϕ_0 in Eq. (36), we obtain Eq.(3) of the main text.

VI. COMPARISON OF THE SPIN SUSCEPTIBILITY $\overline{\chi}$ WITH THE PREVIOUS RESULTS

The spin susceptibility $\overline{\chi}$ can be found from the partition function

$$\overline{\chi} = \frac{\langle \boldsymbol{S}^2\rangle}{3T} = \frac{1}{3}\frac{\partial\ln Z}{\partial J} \tag{41}$$

By using Eq. (35), for $\delta\ll T\ll\mu/\ln(J_\star/T)$ we obtain (see Eq.(5) of the main text):

$$\overline{\chi} = \frac{1}{2}\frac{1}{\delta-J}+\frac{1}{12T}\frac{\delta^2}{(\delta-J)^2}-\frac{1}{12T}. \tag{42}$$

It implies the following result for the average total spin:

$$\langle\boldsymbol{S}^2\rangle = \frac{3}{2}\frac{T}{\delta-J}+\frac{1}{4}\frac{\delta^2}{(\delta-J)^2}-\frac{1}{4}. \tag{43}$$

The average total spin $\langle\boldsymbol{S}^2\rangle$ has been calculated in Ref.[7] near the Stoner instability, $\delta-J\ll\delta$. In our notations, the result of Ref.[7] at $T\gg J_\star$ becomes (see Eqs.(4.8), (4.13b), (4.15) of Ref.[7])

$$\langle\boldsymbol{S}^2\rangle = \frac{c_0T}{\delta-J}\left[1+c_1\frac{\sqrt{J_\star}}{\sqrt{T}}+c_2\frac{J_\star}{T}+\dots\right] \tag{44}$$

where numerical coefficients $c_0 = 1$, $c_1 = \sqrt{\pi}/4$, and $c_2 \approx 0.238$ for unitary ensemble and $c_0 = 1$, $c_1 = \sqrt{2\pi}/4$, and $c_2 \approx 0.227$ for orthogonal ensemble. The result (44) of Ref.[7] contradicts our result (43) in which $c_0 = 3/2$, $c_1 = 0$ and $c_2 = 1/6$ are independent of the ensemble statistics of the single-particle levels. The reason for this disrepancy is not clear now. According to Ref.[7], at $T = 0$, (see Eq.(4.19) of Ref.[7])

$$\langle \boldsymbol{S}^2 \rangle \propto \frac{\delta^2}{(\delta - J)^2}. \tag{45}$$

As one can see from Eq. (43), our result for $T \ll J_\star$ smoothly interpolates into result of Ref.[7] for $T = 0$.

Our result for $\overline{\chi}$ implies that the magnetic field tends to zero first (before, e.g., temperature). The result found by Schechter[8] is valid in the limit of vanishing temperature but finite magnetic field (provided an additional coarse graining is performed). Generalization of Eq. (42) to finite magnetic field resembles the result of Schechter at magnetic fields larger than temperature[9].

VII. THE DISTRIBUTION FUNCTION FOR $\langle \boldsymbol{S}^2 \rangle$

The average moments of the total spin can be found from the partition function Z (see Eq.(4) of the main text) as

$$\langle [\boldsymbol{S}^2]^k \rangle = \frac{T^k}{Z} \frac{\partial^k Z}{\partial J^k}. \tag{46}$$

It can be characterized by the distribution function $\mathcal{P}_{\boldsymbol{S}^2}(x)$:

$$\langle [\boldsymbol{S}^2]^k \rangle = \int_0^\infty dx\, x^k \mathcal{P}_{\boldsymbol{S}^2}(x). \tag{47}$$

If we explicitly indicate the dependence of Z on J then the distribution function is given as

$$\mathcal{P}_{\boldsymbol{S}^2}(x) = \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{-iyx} \frac{Z(J + iyT)}{Z(J)}. \tag{48}$$

By using the result (35) which is valid at temperatures $\delta \ll T \ll \mu/\ln(J_\star/T)$, we find near the Stoner instability, $\delta - J \ll \delta$:

$$\mathcal{P}_{\boldsymbol{S}^2}(x) = 2\sqrt{\frac{\beta\delta^4}{\pi J_\star^3}} e^{-\beta J_\star/4} \sinh(\beta\delta\sqrt{x})\, e^{-\beta\delta^2 x/J_\star}. \tag{49}$$

To characterize this asymmetric non-Gaussian distribution we present the results for skewness

$$\gamma_1 = \frac{\langle (\boldsymbol{S}^2 - \langle \boldsymbol{S}^2 \rangle)^3 \rangle}{\langle (\boldsymbol{S}^2 - \langle \boldsymbol{S}^2 \rangle)^2 \rangle^{3/2}} = 3\frac{\sqrt{T}(J_\star + 2T)}{(J_\star + 3T)^{3/2}} = \begin{cases} 3\sqrt{2T/J_\star}, & T \ll J_\star, \\ 2\sqrt{2/3}, & J_\star \ll T, \end{cases} \tag{50}$$

and excess kurtosis

$$\gamma_2 = \frac{\langle (\boldsymbol{S}^2 - \langle \boldsymbol{S}^2 \rangle)^4 \rangle}{\langle (\boldsymbol{S}^2 - \langle \boldsymbol{S}^2 \rangle)^2 \rangle^2} = 12\frac{T(2J_\star + 3T)}{(J_\star + 3T)^2} = \begin{cases} 24T/J_\star, & T \ll J_\star, \\ 4, & J_\star \ll T. \end{cases} \tag{51}$$

---

[1] K.B. Efetov and A. Tschersich, Phys. Rev. B **67**, 174205 (2003).
[2] A. Kamenev and Y. Gefen, Phys. Rev. B **54**, 5428 (1996).
[3] N. Seldmayr, I.V. Yurkevich, I.V. Lerner, Europhys. Lett. **76**, 109 (2006).
[4] J. Wei and E. Norman, J. Math. Phys **4**, 575 (1963).
[5] I.V. Kolokolov, Ann. Phys. (N.Y.) **202**, 165 (1990); M. Chertkov and I. Kolokolov, Phys. Rev. B **51**, 3974 (1994); Sov. Phys. JETP **79**, 1063 (1994).
[6] K.A. Matveev and A.V. Andreev, Phys. Rev. B **66**, 045301 (2002).
[7] I.L. Kurland, I.L. Aleiner, B.L. Altshuler, Phys. Rev. B **62**, 14886 (2000).
[8] M. Schechter, Phys. Rev. B **70**, 024521 (2004).
[9] I. Burmistrov, Y. Gefen, M. Kiselev, and L. Medvedovsky, to be published.