Scalar and vector Keldysh models in the time domain

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(Dated: January 15, 2009)

The exactly solvable Keldysh model of disordered electron system in a random scattering field with extremely long correlation length is converted to the time-dependent model with extremely long relaxation. The dynamical problem is solved for the ensemble of two-level systems (TLS) with fluctuating well depths having the discrete Z_2 symmetry. It is shown also that the symmetric TLS with fluctuating barrier transparency may be described in terms of the planar Keldysh model with dime-dependent random planar rotations in xy plane having continuous $SO(2)$ symmetry. The case of simultaneous fluctuations of the well depth and barrier transparency is subject to non-abelian algebra. Application of this model to description of dynamic fluctuations in quantum dots and optical lattices is discussed.

PACS numbers: 73.21.La, 73.23.Hk, 85.35.Gv

The model with infinite correlation range of fluctuating fields $V(\mathbf{r})$ proposed by L.V. Keldysh[1] is one of few exactly solvable problems in the theory of disordered electron systems. The approximation

$$D(\mathbf{r}-\mathbf{r}') = \langle V(\mathbf{r})V(\mathbf{r}')\rangle = W^2 \quad (1)$$

makes identical all diagrams for the electron Green function (GF) in the order V^{2n}. As a result summation of diagrammatic series in the "cross technique"[2] reduces to the problem of calculation of combinatoric coefficient A_n (number of pairwise coupling of scattering vertices). In fact $A_n = (2n-1)!!$ is the total number of identical diagrams in the order $2n$. The perturbation series is summed exactly[1,3], and one deals with averaging of ensemble of samples with constant V but the magnitude of this field randomly changes from realization to realization. In momentum space the correlation function (1) transforms into $D(\mathbf{q}) = (2\pi)^3 W^2\delta(\mathbf{q})$. The electron GF in Keldysh model is averaged with Gaussian distribution function characterized by the variance W^2.

This model is not widely used in current literature because it is difficult to propose an experimental device, where the conjecture (1) could be realized (see, however,[4]). In the present paper we discuss a realization of Keldysh model in *time domain*. In this case the analog of infinite spatial correlation is the long memory effect, which can be realized in many physical situations (see below). The structure of perturbation series in time-dependent Keldysh model (TDKM) is the same as in original one, and the long characteristic times of dynamical correlation play the same part as infinite range spatial correlation of static random potentials. Since the time axis is the only coordinate in this problem, its effective dimension is "0+1". Moreover, the TDKM admits natural generalization of original Keldysh model. We will show that the dynamical fluctuations in time domain may be both of scalar and of vector character. The kinematic constraint existing in the vector TDKM results in elimination of essential part of diagrams in cross technique, but the summation of perturbation series is still exact. It results in 2D Gaussian averaging for the GF in dynamical random field.

Leaving for the last section the discussion of real systems, where TDKM arises as a description of generic disorder, we start with a toy model of an ensemble of non-interacting two-level systems (TLS) in a randomly fluctuating environment. In standard realization of TLS, namely a double-valley well, particles are distinguished not only by conventional quantum numbers but also by their position in the well characterized by the index $j = l, r$ of the left (l) or right (r) valley. The barrier between the valleys is characterized by the tunneling matrix element Δ_0. The Hamiltonian of isolated TLS has the form

$$H^{(0)}_{\rm TLS} = \sum_j \left(\varepsilon_j n_j + U n_j^2\right) - \Delta_0(c^\dagger_l c_r + \text{H.c.}). \quad (2)$$

Here $n_j = c^\dagger_j c_j$ is the particle occupation number, ε_j is the discrete energy level in the valley j and U is the interaction parameter for two particles in the same valley. The condition $U \gg \Delta$ is usually assumed. We consider spinless particles, having in mind that the theory can be applied both to interacting bosons and fermions (electrons) with frozen spin degrees of freedom. To be specific we discuss tunneling electrons as an example.

We start with the singly occupied TLS, where the constraint $N = \sum_i n_i = 1$ is imposed on the Hamiltonian, introduce pseudospin operators $\sigma^+ = c^\dagger_l c_r, \sigma^- = c^\dagger_r c_l, 2\sigma_z = n_l - n_r$, and reduce (2) to

$$H^{(0)}_{\rm TLS} = -\delta_0\sigma_z - \Delta_0\sigma_x - \mu_0(N-1). \quad (3)$$

in the pseudospin subspace. Here the asymmetry parameter $\delta_0 = \varepsilon_r - \varepsilon_l$ play the role of effective "magnetic" field, the Lagrange parameter μ_0 controls constraint.

The scalar fluctuation field is introduced as random fluctuations of TLS asymmetry, namely as a time dependent field $\delta_\rho(t)=\delta_0+h_\rho(t)$ determined by its moments

$$\overline{h_\rho(t)} = 0, \qquad \overline{h_\rho(t)h_\rho(t+\tau)} = D(\tau). \quad (4)$$

Here the overline stands for the ensemble average. Thus we reduced the original model to the effective spin Hamiltonian in magnetic field with random time-dependent component. The problem can be reformulated as a study of propagation of fermions along the time axis in the presence of time-dependent random scalar potential $\delta_\rho(t)(n_r - n_l)/2$, and the cross technique may be used in calculation of the propagators[5].

The analog of Keldysh conjecture in time domain is a slowly varying random field $\sim\exp(-\gamma t)$. A very long relaxation time $\tau_{rel}\sim 1/\gamma$ with small γ is presumed, so that the noise correlation function is given by

$$D(\omega) = \lim_{\gamma\to 0} \frac{2\zeta^2\gamma}{\omega^2+\gamma^2} = 2\pi\zeta^2\delta(\omega) \qquad (5)$$

(the noise correlation function (5) is normalized in such a way that the corresponding vertices are dimensionless). In this limit the averaged spin propagator describes the ensemble of states with a field $\delta = const$ in a given state, but this constant is random in each realization. The "planar-type" TDKM may be derived in a similar way. For this sake one should introduce a random component in the tunneling matrix element, $\Delta = \Delta_0 + \Delta_\rho(t)$ with $\overline{\Delta_\rho(t)} = 0$ and make similar conjecture (5) about the correlation function $F(\tau) = \overline{\Delta_\rho(t)\Delta_\rho(t+\tau)}$, namely approximate its Fourier transform by

$$F(\omega) = \lim_{\gamma\to 0} \frac{4\xi^2\gamma}{\omega^2+\gamma^2} = 4\pi\xi^2\delta(\omega) \qquad (6)$$

Thus, we treat our toy Hamiltonian (3) in the following way. First we study the limiting case $\delta_0 \gg \Delta_0$, were the interdot tunneling is considered as a perturbation to the *longitudinal* term affected by stochastization in the scalar TDKM. This problem may be solved by means of the standard technique[1,3,6] generalized for the time-domain case. Then we turn to the case of *transversal* random tunneling potential and solve this problem by means of correspondingly modified "planar" TDKM.

We start with the scalar TDKM and treat the term $H_{\|}^{(0)} = \delta_0(n_r - n_l)/2$ as a zero order approximation with electron occupying the level ε_l in the ground state. Without stochastic perturbation the role of the tunneling term $H_\perp = -\Delta_0\sigma_x$ is in admixing a charge transfer exciton to the ground state with the corresponding energy level shifts, $\varepsilon_{l,r} \to \varepsilon_{l,r} \mp \Delta_0^2/\delta_0$ for the ground and excited states, respectively. The time-dependent perturbation stochastisizes this simple picture.

Let us introduce the retarded propagators for the scalar TDKM

$$G^R_{j,s}(t-t') = \langle c_j(t)c_j^\dagger(t')\rangle_R = -i\langle[c_j(t)c_j^\dagger(t')]_+\rangle. \qquad (7)$$

and consider their evolution on the time axis under the influence of random component $h_j(t)$ ("random longitudinal magnetic field" in pseudospin notation), first assuming $\gamma\to 0$, $\Delta_0/\gamma \to 0$. After averaging $G^R_j(t-t')$ in accordance with (4) and making the Fourier transformation by means of (5), we come to the series

$$G^R_{j,s}(\varepsilon) = g_j(\varepsilon)\left[1 + \sum_{n=1}^{\infty} A_n\zeta^{2n} g_j^{2n}(\varepsilon)\right] \qquad (8)$$

Here $g_l(\varepsilon) = (\varepsilon + i\eta)^{-1}$ and $g_r = (\varepsilon + \delta_0 + i\eta)^{-1}$ are the bare propagators, $A_n = (2n-1)!!$ is the above mentioned combinatoric coefficient (see Fig. 1, where several first irreducible diagrams are shown).

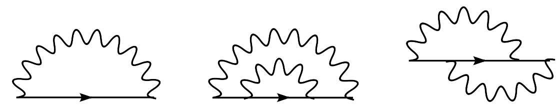

FIG. 1: Irreducible Feynman diagrams for scalar TDQM. Solid and wavy lines stand for $g_i(\varepsilon)$ and $D(\omega)$.

Like in the real space Keldysh model[1], the series (8) may be summed by means of the integral representation[7] for $(2n-1)!!$. Then changing the order of summation and integration (Borel summation), one comes to the following equation for the left valley GF

$$G^R_{l,s}(\varepsilon) = \frac{1}{\zeta\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2\zeta^2}\frac{dz}{\varepsilon - z + i\eta} \qquad (9)$$

Remarkably, the single electron GF in this model has no poles, singularities or branch cuts. Similar procedure may be applied to the Green function G^R_r. As a result of this Gaussian averaging the "magnetization" $\overline{\sigma}_z$ is reduced and the corresponding response to transversal field is modified accordingly.

Next, we formulate the "planar" TDKM for the bare Hamiltonian $H^{(0)}_{\text{TLS}}$ (2) with impenetrable barrier $\Delta_0 \to 0$ and symmetric valleys, $\varepsilon_l = \varepsilon_r$, isolated from each other. The transverse random perturbation is introduced by

$$H_\rho(t) = \Delta_\rho(t)\sigma^+ + \Delta^*_\rho(t)\sigma^- \qquad (10)$$

so that the inter-valley tunneling is stochastisized by means of averaging in time-domain with correlation function (6). The tunnel matrix element Δ is transformed as $\Delta_\rho\to\Delta_\rho e^{i(\varphi_r-\varphi_l)}$ under the gauge transformation $c_j \to c_j e^{i\varphi_j}$, and we presume Δ_ρ to be a complex variable.

Unlike the scalar TDKM, the noticeable part of diagrams in the perturbation series disappears due to the kinematical restrictions $\sigma^+\sigma^+ = \sigma^-\sigma^- = 0$ (see also[9]). Only the diagrams with pseudospin operators ordered as $\ldots\sigma^+\sigma^-\sigma^+\sigma^-\ldots$ survive in the expansion for the GF of planar model

$$G^R_{j,p}(\varepsilon) = g_j(\varepsilon) + \sum_{n=1}^{\infty} B_n(\sqrt{2}\xi)^{2n} g_j^{2n+1}(\varepsilon). \qquad (11)$$

The vertices in the cross technique are now "colored" in accordance with two terms entering the random potential. The vertices with different colors have to be ordered

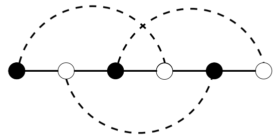

FIG. 2: First non-vanishing vertex correction to the Green function self-energies in the planar Keldysh model. Black and white sites correspond to two terms in the Hamiltonian (**??**); transversal pseudospin correlation functions $F(\omega)$ (6) are represented by dashed lines.

in alternating way, and the correlation lines connect only the vertices of opposite color (see Fig. 2.)

As a result of the above kinematic restrictions, the combinatoric coefficient $B_n = n!$. Then we use the integral representation[10] for $n!$ and transform the series (11) into

$$G^R_{j,p}(\epsilon) = g_i(\epsilon)\left\{1 + 2\sum_n \int_0^\infty tdt\left[t\sqrt{2}\xi g_j(\epsilon)]^{2n}\right] e^{-t^2}\right\}$$

Here we substituted t^2 for the variable z. Then changing the order of summation and integration, we transform $G^R_j(\epsilon)$ into the integral

$$G^R_{j,p}(\epsilon) = \int_0^\infty 2tdt \frac{g_j(\epsilon)}{1 - 2t^2\xi^2 g_j^2(\epsilon)} e^{-t^2} \qquad (12)$$

Taking into account the explicit form of the free propagator $g_j(\epsilon)$, we change the integration variable once more, $t = u/\sqrt{2}\xi$, and transform (12) into

$$G^R_{j,p}(\epsilon) = \int_0^\infty \frac{udu}{2\xi^2}\left(\frac{1}{\epsilon - u + i\eta} + \frac{1}{\epsilon + u + i\eta}\right) e^{-u^2/2\xi^2} \qquad (13)$$

Now we introduce the "cartesian" coordinates, $x = u\cos\phi, y = u\sin\phi$, so that $u = \sqrt{x^2+y^2}$ and $dxdy = udud\phi$. The angle independent integral (13) may be rewritten as

$$G^R_{j,p}(\epsilon) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dy e^{-y^2/2\xi^2}}{\xi\sqrt{2\pi}} \left[\frac{1}{\epsilon - \sqrt{x^2+y^2} + i\eta} + \frac{1}{\epsilon + \sqrt{x^2+y^2} + i\eta}\right]. \qquad (14)$$

This result is a natural generalization of the one-dimensional Gaussian averaging (9) characteristic for the scalar TDKM to the two-dimensional Gaussian averaging of planar random field with purely transversal (xy) fluctuations. Only the modulus of random field $r = \sqrt{x^2+y^2}$ is averaged, whereas the angular variable remains irrelevant due to the in-plane isotropy of the system. Like in the scalar model, the averaged GF has no singularities.

Although the GF lost its pole structure, the standard Feynman rules for construction of irreducible parts (Figs. 1, 2) and corresponding skeleton diagrams[2,3] are still valid. However, the important reservation should be made: the self energy cannot be treated as a renormalization of bare pole because the bare and dressed GF are connected by non-local integral operators [see Eqs. (9),(13)]. Nevertheless, the ordinary differential equation connecting the GF and its derivative over energy can be obtained for both versions of TDKM. This equation was found for the scalar Keldysh model in[3]. Here we derive this equation without appealing to the Ward identity and then generalize the derivation procedure for the planar model.

To calculate the derivative $dG(\varepsilon)/d\varepsilon$ for the scalar TDKM, we start with expansion (8) (index s is omitted below for the sake of brevity). It is convenient to count the energy off the position of chemical potential μ_0 [see Eq. (3)] in the middle between the levels $\varepsilon_{l,r}$. So, we shift the energies $\varepsilon \to \varepsilon_\pm = \varepsilon \mp \delta_0/2$ for the left and right GR, respectively.

The same procedure, which leads to Eq. (9) for GR gives the following equation for its derivative:

$$\frac{dG}{d\varepsilon} = -\frac{g^2(\varepsilon_\alpha)}{\zeta\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{z^2}{2\zeta^2}} \left[1 + \frac{1}{\zeta^2}\frac{z^3}{\varepsilon_\alpha - z + i\eta}\right] dz.$$

($\alpha = \mp$). Calculating the integral and substituting $g^2(\varepsilon_\alpha) = \varepsilon_\alpha^{-2}$ we come to the differential equation

$$\zeta^2 \frac{dG(\varepsilon_\alpha)}{d\varepsilon} = 1 - \varepsilon_\alpha G(\varepsilon_\alpha) \qquad (15)$$

similar to that obtained for the real space scalar Keldysh model[3,6].

Generalization of this procedure for the symmetric planar model ($\delta_0 = 0$) is more cumbersome. We start with differentiating the series (11) over the energy. Then the analog of Eq. (12) for the derivative has the form

$$\frac{dG}{d\varepsilon} = -g^2(\varepsilon)\left[1 + \int_0^\infty 2tdt \frac{2(2t^2-1)t^2\xi^2 g^2(\varepsilon)}{1 - 2t^2\xi^2 g^2(\varepsilon)} e^{-t^2}\right]$$

The subsequent variable change which gave Eq. (13) for the GF gives for its derivative the following equation

$$\frac{dG}{d\varepsilon} = -g^2(\varepsilon)\left[1 + \frac{1}{2}\left(\frac{J_4}{\xi^4} - \frac{J_2}{\xi^2}\right)\right] \qquad (16)$$

where

$$J_n = \int_0^\infty dz z^n \exp\left(-\frac{z^2}{2\xi^2}\right)\left[g(\varepsilon - z) + (-1)^{n+1} g(\varepsilon + z)\right]$$

After some manipulations, these integrals are represented via the GF for the planar model (13):

$$J_2 = 2\varepsilon\xi^2 G - 2\xi^2, \quad J_4 = -4\xi^4 + \varepsilon^2 J_2 \qquad (17)$$

Substituting these integrals in Eq. (16), we come eventually to the differential equation

$$\xi^2 \frac{dG(\varepsilon)}{d\varepsilon} = 1 - \varepsilon G(\varepsilon)\left(1 - \frac{\xi^2}{\varepsilon^2}\right) \qquad (18)$$

which is obviously the generalization of Eq. (15). These two equation may be rewritten in a unified way:

$$\varepsilon_\alpha G_{\alpha,s} - 1 = \zeta^2 G_{\alpha,s}^2 \frac{d}{d\varepsilon} G_{\alpha,s}^{-1}$$
$$\varepsilon G_p - 1 = \xi^2 G_p^2 \left[\frac{1}{\varepsilon}\frac{d}{d\varepsilon}\left(\varepsilon G_p^{-1}\right)\right] \qquad (19)$$

The solutions of these equations satisfying the boundary condition $G(\epsilon \to \infty) = \epsilon^{-1}$ is given by (9, 13). It is worth noting that the differential operator in the right hand side of the second equation is nothing but div_ε in polar coordinates. This form reflects effective two-dimensionality of Gaussian averaging in the planar TDKM. Now one may introduce vertex parts using the analogy with the Ward identities

$$\Gamma_s = \frac{d}{d\varepsilon} G_s^{-1}, \quad \Gamma_p = \frac{1}{\varepsilon}\frac{d}{d\varepsilon}(\varepsilon G_p^{-1}). \qquad (20)$$

These vertices, together with equations (19) will be useful for calculation of response functions of our TLS (see below). Like in the self energy parts (Figs. 1, 2), the planar TDKM lacks most of diagrams of scalar model due to the kinematic restrictions: the sites in the vertices of triangle are of the same color, black and white sites alternate, and dashed lines connect sites of opposite colors. First nonvanishing vertices for both models are shown in Fig. 3

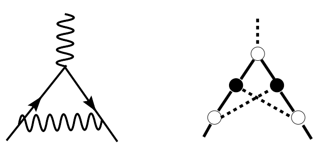

FIG. 3: First non-vanishing vertex diagrams for the Γ in scalar (left) and planar (right) TDKM.

The density of states (DoS) in stochastisized TLS is given by the imaginary parts of GF (9), (13) for scalar and planar TDKM, respectively:

$$\nu_s(\varepsilon) = \frac{2}{\zeta\sqrt{2\pi}} \exp\left(-\frac{\varepsilon^2 + \delta_0^2}{2\zeta^2}\right) \cosh\left(\frac{\varepsilon\delta_0}{\zeta^2}\right), \qquad (21)$$

$$\nu_p(\varepsilon) = \frac{1}{\xi^2}|\varepsilon| \exp\left(-\frac{\varepsilon^2}{2\xi^2}\right). \qquad (22)$$

In the scalar model $\nu_s(\varepsilon)$ is a superposition of two Gaussians centered around ε_l and ε_r, respectively. In the planar model $\nu_p(\varepsilon)$ is represented by a single Gaussian with a dip "burnt" around zero energy (Fig. 4).

Switching on the tunneling term $H_\perp$ in scalar model without random field, we come to the picture of two levels mutually repulsed due to coherent interdot tunneling and broadened due to incoherent time-dependent intradot fluctuations. The spectrum is still gapful at small enough ratio ζ/δ_0, If the $\zeta \lesssim \delta_0$, the DoS merges into

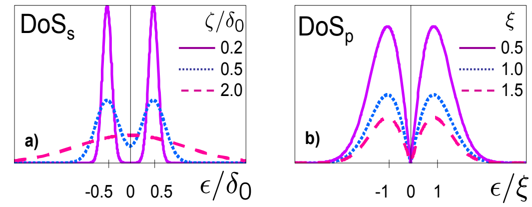

FIG. 4: Density of states in scalar (left) and planar (right) model (all units are arbitrary).

double hump Gaussian structure. The information about position of electron in right or left valley is completely lost at $\zeta/\delta_0 > 1$. In the planar model for symmetric TLS instead two-peak structure due to avoided crossing characteristic for coherent tunneling, we get a pseudogap around zero energy due to stochastic tunneling, which survives at any variance ξ.

Let us allow now the fluctuations of both longitudinal and transverse components which corresponds to simultaneous fluctuations of the well depth and barrier transparency for the case of symmetric TLS ($\delta_0 = 0$). We solve this problem by means of path integral formalism. The Lagrangian and corresponding action are defined on the Keldysh contour K (see e.g.[11])

$$\mathcal{L}(t) = \sum_{j=l,r} \bar{c}_j i\partial_t c_j - H, \qquad S_K = \int_K \mathcal{L}(t)dt. \qquad (23)$$

Here $\bar{c}_j, c_j$ are Grassmann variables describing the electron. The time-dependent gauge transformation $c_j(t) \to c_j(t)e^{i\varphi_j(t)}$, converts the fluctuation of the well depth to the fluctuation of the *phase* of the tunnel matrix element under the choice

$$\varphi_j(t) = \int_{-\infty}^t h_j(t')dt'. \qquad (24)$$

We therefore identify the longitudinal and transverse noise with phase fluctuations of the barrier transparency and fluctuations of the modulus of the tunnel matrix element, respectively and unify them in the path integral description.

The ensemble averaging

$$\langle ... \rangle_{noise} = \int dh_\rho P_l(h_\rho) \int d\Delta_\rho^* d\Delta_\rho P_{tr}(\Delta_\rho^*, \Delta_\rho)...$$

is done with the help of probability distribution functions for longitudinal and transverse fluctuations

$$P_l = \frac{1}{\zeta\sqrt{2\pi}} \exp\left(-\int_K dt dt' h_\rho(t) D^{-1}(t-t') h_\rho(t')\right)$$

$$P_{tr} = \frac{1}{2\pi\xi^2} \exp\left(-\int_K dt dt' \Delta_\rho^*(t) F^{-1}(t-t') \Delta_\rho(t')\right).$$

The GF's can be calculated by means of generating functional corresponding to Keldysh action S_K in a standard way[12],[13]. In the "infinite memory" limit (5), (6) we easily express the GF of the electron in the symmetric double well potential

$$G^R_{j,v}(\epsilon) = \frac{1}{\zeta\xi^2(2\pi)^{3/2}}\int_{-\infty}^{\infty} dz e^{-z^2/2\zeta^2} \int dw^* dw e^{-|w|^2/2\xi^2}\frac{\epsilon \pm z}{(\epsilon+i\eta)^2 - z^2 - |w|^2} \quad (25)$$

Noticing that the GFs do not depend on the well index j and performing the integration over angles in spherical coordinate system we get

$$G^R_v(\epsilon) = \frac{1}{2\xi}\int_0^{\infty} d\rho\rho \exp\left(-\frac{\rho^2}{2\xi^2}\right) \frac{\mathrm{erf}\left(\rho\sqrt{\frac{\xi^2-\zeta^2}{2\xi^2\zeta^2}}\right)}{\sqrt{\xi^2-\zeta^2}}\left(\frac{1}{\epsilon-\rho+i\eta}+\frac{1}{\epsilon+\rho+i\eta}\right). \quad (26)$$

The Eqs (25, 26) generalize (9) and (14) for the case of anisotropic vector Keldysh model. The three-dimensional Gaussian averaging in (25) stands for vector character of the random field distributed on an ellipsoid. Typical size of semi-axes is defined by the variances of longitudinal and transverse noises. Like in the scalar an planar models, the averaged GF has no singularities. The angle ϕ dependence is absent in (25) due to the in-plane isotropy of the model $P_{tr} = P_{tr}(|\Delta|^2)$ which is preserved here. The limits of strong easy axis $\xi \to 0$ and easy plane $\zeta \to 0$ anisotropy correspond to scalar and planar models, correspondingly.

We notice that the DoS for vector TDKM is also characterized by a pseudogap. The energy dependence $\nu_v(\varepsilon) \sim \varepsilon^2$ for small energies should be contrasted with $\nu_s(\varepsilon) \sim const$ for scalar $\nu_p(\varepsilon) \sim \varepsilon$ for planar model behavior. This dependence reflects the fact that the probability to remain close to initial level position decreases with increase of effective dimensionality of the problem due to stochastization of the complex tunneling matrix element.

One should specially mention the degenerate case of the isotropic vector TDKM, characterized by $\xi = \zeta = \lambda$ and describing rotation of pseudospin on a Bloch sphere. Performing simple algebra we get an equation for GF

$$\varepsilon G_v - 1 = \lambda^2 G_v^2 \left[\frac{1}{\varepsilon^2}\frac{d}{d\varepsilon}\left(\varepsilon^2 G_v^{-1}\right)\right] \quad (27)$$

and the Ward Identity

$$\Gamma_v = \frac{1}{\varepsilon^2}\frac{d}{d\varepsilon}(\varepsilon^2 G_v^{-1}). \quad (28)$$

The DoS for the isotropic model is given by the expression

$$\nu_v(\varepsilon) = \frac{1}{\lambda^3\sqrt{2\pi}}\varepsilon^2 \exp\left(-\frac{\varepsilon^2}{2\lambda^2}\right). \quad (29)$$

Our next task is calculation of response functions. In scalar TDKM the longitudinal susceptibility is given by the correlation function $\chi_{\|}(\omega) = i\int dt \exp(i\omega t)\langle\sigma_z(t)\sigma_z(0)\rangle_R$, which is represented by two loops with vertex corrections (Fig. 5). Here both solid lines correspond either to $j = l$ or to $j = r$. We confine ourselves with calculation of static susceptibility, $\omega \to 0$. In order to work at finite temperatures we turn to Matsubara Green functions functions $\mathcal{G}(i\epsilon_n)$ (similar calculation can be done on the Keldysh contour in the real time path integral formalism) and susceptibility

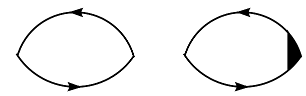

FIG. 5: Diagrams for bare loop (transverse susceptibility) and loop with dressed vertex (longitudinal susceptibility).

$$\chi(i\omega_m) = T\sum_{n,j}\mathcal{G}_j(i\omega_m + i\epsilon_n)\mathcal{G}_j(i\epsilon_n)\Gamma(i\epsilon_n, i\omega_m) \quad (30)$$

Then the first of Ward identities (20) provides us with the exact equation for the vertex

$$\zeta^2(G^R_{j,s})^2\Gamma^R_{j,s}(\epsilon,0) = \epsilon G^R_{j,s} - 1, \quad (31)$$

giving an access to the exact evaluation of $\chi(0)$. Combining (30) with (31) and analytical continuation of (9), we find

$$\chi_{\|}(0) = -\sum_{\alpha=\mp}\int_{-\infty}^{\infty}\frac{y dy e^{-y^2/2}}{\sqrt{2\pi}\zeta} n_F\left(\frac{(2y-\alpha\delta_0)\zeta}{2T}\right) \quad (32)$$

Here $n_F(x)$ is the Fermi distribution function. The asymptotic behavior of static susceptibility $\chi(0)$ is

$$\chi(0) \sim \begin{cases} 1/T, & T \gg (\zeta, \delta_0) \\ 1/\zeta, & \zeta \gg (T, \delta_0) \end{cases} \quad (33)$$

There is no vertex correction to transverse susceptibility $\chi_\perp(0) = \langle\sigma_+, \sigma_-\rangle_R$. It is given by the bare loop formed by the left and right GF. In case of symmetric TQD ($\delta_0 = 0$) and big variance $\zeta \gg \Delta_0$ this function is as smooth as $\chi_{\|}(0)$ with changing ζ and T. Its asymptotic behavior is given by the same Eq. (33). The physical sense of these results in obvious: the information about position of electron in a given well is lost at strong enough stochastization $\zeta \gg T$.

Next, we calculate the susceptibility for planar model in case of symmetric TLS with $\delta_0 = 0$. In this case $\chi_\perp(0)$ differs from $\chi_{\|}(0)$ by factor 2, so that it is enough to calculate the latter one. Now we appeal to the second equation from (19) with Γ_p defined in (20). Then like in previous case the calculation of $\chi_{\|}(0)$ is reduced to finding the combination $i\epsilon_n\mathcal{G}_p(i\epsilon_n) - 1$. Straightforward computation gives

$$\chi_{\|}(0) = \frac{1}{\xi}\int_0^{\infty} y^2 dy e^{-y^2/2}\tanh\left(\frac{y\xi}{2T}\right). \quad (34)$$

The behavior of $\chi_{\|}$ as a function of T and ξ is close to that for scalar model, including the asymptotic dependence

(33). The equations for static susceptibilities in isotropic vector TDKM can be easily found with help of (25, 26)

$$\chi(0) = \frac{1}{\lambda\sqrt{2\pi}}\int_0^\infty y^3 dy e^{-y^2/2}\tanh\left(\frac{y\lambda}{2T}\right). \quad (35)$$

These susceptibilities are characterized by the same asymptotic behavior as those for scalar/planar models.

The toy model of noninteracting TLS under dynamical stochastization demonstrate some generic properties of TDKM: (i) the loss of characteristic spin or pseudospin behavior at variance exceeding temperature; (ii) the effective two-dimensionality of Gaussian averaging in planar TDKM as its main distinction from scalar model; (iii) the effective three-dimensionality of vector TDKM. These features survive also in more realistic situations. One of possible applications of this theory is the problem of electron tunneling through double quantum dot in a regime of strong Coulomb blockade, where the source of stochastization is a random time-dependent gate voltage applied to one of the valleys[5]. The case of $N = 2$ was considered, where the starting Hamiltonian $H^{(0)}$ is that of Eq. (2) with added spin index. In this case the scalar TDKM may be used in the limit of slow fluctuations (5), the double quantum dot looses its spin characteristics at low T, and the Kondo-type zero bias anomaly is smeared accordingly[8]. The important difference between spinful and spinless TLS models is in their symmetry. The symmetry of TLS with N=2 considered in[8] is $SO(5)$, which is reduced to $SO(3)$ for low-energy part of excitation spectrum, so the Lie algebra is non-abelian. However, it was possible to introduce scalar TDKM due to abelian character of time-dependent random gauge field.

The symmetry of the scalar TLS with N=1 is given by the discrete group Z_2 with abelian algebra. The gauge transformation allows to identify the fluctuations of the scalar model as $U(1)$ fluctuations of the phase of tunnel matrix element. In the planar model one deals only with the planar (xy) rotations, so the relevant symmetry is $SO(2)$ with still abelian algebra. The phase fluctuation in that case are discrete Z_2 (ϕ=0,π to provide the condition $\bar{\Delta}_\rho$=0), while modulus fluctuations determine effective 2d behavior. The symmetry of isotropic vector TDKM is $SU(2)$ and corresponding algebra is non-abelian.

One may mention several more physical systems, where the scalar and/or planar TDQM is useful. One of such models is the big quantum dot with charge fluctuations accompanied by longitudinal and transversal spin fluctuations[14]. The class of Gaussian ensembles corresponds to infinite-range correlations in the charge and spin sectors of the model. Both spin and charge interactions contain stochastic component[15], leaving a room for original formulation of the Keldysh model. The gauge field theory, based on functional bosonization being formulated in the time-domain, opens a possibility of stochastic treatment of dynamic processes. As is shown in[14], the transverse spin correlation function for anisotropic spin exchange contains both short-time and long-time correlation parts. While the short-time (white noise) correlations dominate away from the Stoner instability, the (infinitely) long-time correlations become important as one approaches the regime of strong fluctuations of the magnetization. The long-time part of the model is equivalent to the planar TDKM.

Another interesting object is the optical superlattice consisting of biased double wells[16]. The bias is random, but the number of atoms in the same in all TLS in this experimental setup due to the "interaction blockade". One may expect that the well population in these TLS could be stochastized in accordance with Fig. 4, provided the Keldysh-type fluctuations (5) or (6) with long relaxation times were realized experimentally.

We thank Boris Altshuler, Jan von Delft, Yuri Galperin, Yuval Gefen, Jean Richert and Wilhelm Zwerger for helpful discussions.

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