Lectures on General Relativity
II. Black holes and cosmology

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Abstract

Two important applications of Einstein gravity are discussed.

Contents

1 Schwarzschild Metric

Reading: *LL §100*

It is generally hard to find analytic solutions of a nonlinear system of equations, like those of Einstein

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1}$$

This is a hard problem even in vacuum, where $T_{\mu\nu} = 0$. Symmetries usually simplify the task. In particular, assuming spherical symmetry allows us to find a simple and yet very important analytic solution. Spherical symmetry means that we can write the metric as

$$ds^2 = -h(t,r)dt^2 + 2k(t,r)dtdr + l(t,r)dr^2 + r^2 d\Omega^2, \tag{2}$$

where $d\Omega^2$ is the metric of a unit round sphere

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{3}$$

Here we have chosen one of the coordinates (r) to describe the area of the symmetric spheres ($4\pi r^2$). This is must be familiar from the spherical coordinates in $3d$ Euclidean geometry where

$$ds^2 = dr^2 + r^2 d\Omega^2. \tag{4}$$

However, it is a special feature of $\mathbf{R}^3$ that r also measures the distance to the origin. For instance, the unit 3-sphere metric can be written as

$$ds^2 = \frac{dr^2}{1-r^2} + r^2 d\Omega^2. \tag{5}$$

Here $4\pi r^2$ determines the area of 2-spheres at coordinate r, but it does not measure the distance between them and the $r = 0$ point. One can think of the coordinate system (5) as picking an arbitrary point (say the North pole) on the 3-sphere and foliating the manifold with 2-spheres at constant distance from the North pole. These are a higher dimensional analogs of the latitudes on the surface of the Earth; so I'll call them such.

1. Show that the above metric is equivalent to

$$ds^2 = d\chi^2 + \sin^2\chi d\Omega^2, \tag{6}$$

and find the distance between a point at r and the North pole ($r = 0$).

There is a coordinate singularity at $r = 1$ in (5). This is the analog of the equator. The 3-sphere is not singular there, it is just the area of the latitudes that is not a good coordinate for the entire manifold because it ranges over the same values on the northern and southern hemispheres. Similarly, the coordinates in (2) might not cover the entire manifold.

We can set $g_{tr} = 0$ in (2) by the coordinate transformation

$$t \to f(t,r) \qquad \text{with} \qquad \partial_r f(t,r) = \frac{k(t,r)}{h(t,r)}. \tag{7}$$

The g_{tt} and g_{rr} components will change under this transformation, but they remain functions of t and r. It is convenient to parametrize the new metric as

$$ds^2 = -e^{\nu(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + r^2 d\Omega^2. \tag{8}$$

2. Find the transformation that takes the metric

$$ds^2 = -dt^2 + 2t dt dr + dr^2 \tag{9}$$

to one with $g_{tr} = 0$, and $g_{tt} = -e^{2r}$. What is the new g_{rr}?

It is relatively easy to compute the Riemann tensor and hence $R_{\mu\nu}$, R and $G_{\mu\nu}$ for the metric (8). For instance

$$G_{tr} = \frac{\dot{\lambda}}{r}, \tag{10}$$

$$G_{tt} = e^{\nu-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{e^{\nu}}{r^2}, \tag{11}$$

$$G_{rr} = \left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{e^{\lambda}}{r^2}, \tag{12}$$

where dot denotes d/dt and prime d/dr. In vacuum $G_{\mu\nu} = 0$. Therefore, from (10) we learn that λ is a function of r only. From the combination

$$0 = e^{\lambda-\nu}G_{tt} + G_{rr} = \frac{\nu' + \lambda'}{r} \tag{13}$$

we learn that

$$\nu(t,r) = -\lambda(r) + \tilde{\nu}(t). \tag{14}$$

But we can set $\tilde{\nu}(t) = 0$ by a coordinate redefinition of $t \to \tilde{t}$:

$$\tilde{t} = \int^t e^{\tilde{\nu}(s)/2} ds. \tag{15}$$

Note that this does not involve r and hence does not change g_{tr} or g_{rr}. I will drop the tilde and call the new coordinate t, in terms of which $\nu = -\lambda$. Finally, we can integrate (11) to find

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} + r^2 d\Omega^2, \tag{16}$$

where r_g (the gravitational radius) is an integration constant.

The fact that spherical symmetry uniquely fixes the vacuum solution up to one parameter r_g (also known as the Birkhoff's theorem) is reminiscent of electromagnetism, where the role of r_g is played by the electric charge Q. This is the consequence of the fact that there is no spherically symmetric electromagnetic or gravitational waves, which in turn results from the nonzero helicity of photons and gravitons.

Newtonian limit. The Schwarzschild metric describes the spacetime outside spherically symmetric matter distribution. As a simple model for the matter distribution consider a non-relativistic, static star. This means that there is a choice of coordinates where $T_{i0} = 0$ and

$$T_{00} = \rho(r), \qquad \rho(r) \underset{r\to\infty}{=} 0. \tag{17}$$

In these coordinates we continue to have $g_{tr} = 0$ and $\dot{\nu} = \dot{\lambda} = 0$ inside the star. In the non-relativistic limit, the spacetime is close to Minkowski, so we can linearize in ν and λ, moreover $T_{ij} \approx 0$. We can now use

$$8\pi G\rho = G_{tt} = \frac{1}{r^2}(\lambda + r\lambda') + \mathcal{O}(\nu^2, \lambda^2, \nu\lambda), \tag{18}$$

and the trace of the Einstein equation

$$R = -8\pi G T^\mu_\mu \approx 8\pi G e^{-\nu}\rho, \tag{19}$$

where using $\dot{\nu} = \dot{\lambda} = 0$, and to linear order in ν, λ

$$R^{(1)} = \frac{2\lambda}{r^2} - \frac{1}{r}(r\nu'' + 2\nu' - 2\lambda'). \tag{20}$$

Eliminating λ gives

$$\nu'' + 2\frac{\nu'}{r} \approx 8\pi G\rho \tag{21}$$

which is the Poisson equation for the Newtonian potential $\phi = \frac{\nu}{2}$. Its solutions is

$$\phi = -\frac{GM}{r}, \qquad M \equiv \int_0^\infty 4\pi r^2 \rho(r) dr. \tag{22}$$

3. What is the necessary condition on the radius of the star for the non-relativistic approximation to be valid?

Orbits. Free test particles move on geodesics, described by

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}, \tag{23}$$

where τ is the proper time for massive particles and the affine parameter for the massless ones.

4. Derive the geodesic equation by varying the point particle action with respect to x^μ

$$S_{\rm pp} = -m \int d\sigma \sqrt{-g_{\mu\nu}(x)\, \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}. \tag{24}$$

At large distances, the orbits of the Schwarzschild metric are well approximated by Keplerian orbits, however when $r \sim r_g$ there are significant differences. For instance, there is an innermost circular orbit as we will see next.

Without loss of generality, we can choose the circular geodesic to be in the $\theta = \frac{\pi}{2}$ plane. Then

$$u^\mu = (\dot t, 0, 0, \dot\varphi), \qquad \dot t \equiv \frac{dt}{d\tau}, \quad \dot\varphi \equiv \frac{d\varphi}{d\tau}. \tag{25}$$

Note that we changed the definition of dot from d/dt to $d/d\tau$.

Since the metric is independent of t and φ, we have two conservation laws:

$$\frac{d}{d\tau} u_t = 0 \Rightarrow \dot t = \frac{e}{1 - \frac{r_g}{r}}, \tag{26}$$

$$\frac{d}{d\tau} u_\varphi = 0 \Rightarrow \dot\varphi = \frac{\ell}{r^2}, \tag{27}$$

where the constants e and ℓ are, respectively, the energy and angular momentum per unit mass. The normalization of the 4-velocity $u^\mu u_\mu = -1$ implies

$$e^2 = \left(1 - \frac{r_g}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right). \tag{28}$$

Finally, the r component of (23) implies

$$-\frac{r_g}{r^2}\dot t^2 + 2r\dot\varphi^2 = 0. \tag{29}$$

Substituting the above expressions for $\dot t, \dot\varphi$, and e gives

$$\ell^2 = \frac{r_g r}{2\left(1 - \frac{3r_g}{2r}\right)}. \tag{30}$$

The minimum value for r for this to have a solutions is $r_{\rm min} = \frac{3}{2} r_g$.

2 Extension of Schwarzschild metric

Reading: *Wald 6.4*

We saw that the Schwarzschild metric describes the metric outside a spherically symmetric matter distribution. Inside matter, the metric will differ from Schwarzschild, so for instance the t, r coordinate system is perfectly suitable to describe the interior of a non-relativistic star. What if the matter distribution gets so compact that at $r = r_g$ we are still in vacuum? The Schwarzschild metric is singular at $r = r_g$, but one can check that invariants made of curvature (e.g. $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$) and its derivatives remain finite at this point. This singularity is due to the failure of t, r coordinates to cover the entire manifold. Below we will find an alternative.

1. How long does it take a free-falling observer who starts at rest at $r = 2r_g$ to arrive at $r = r_g$
   (a) in proper time?
   (b) in coordinate time t?

This exercise hints at a way of extending the geometry in the future beyond r_g, namely to use the time measured by infalling observers rather than t, which is the time measured by the asymptotic observer (i.e. one at $r \gg r_g$). This is the idea behind the Painlevé coordinates.

Instead, here we follow Kruskal and Szekeres and consider radially moving null rays. From $ds^2 = 0$ (for a null ray) we obtain

$$\frac{dr}{dt} = \pm\left(1 - \frac{r_g}{r}\right). \tag{31}$$

Let's define the *tortoise coordinate*

$$r_* = r + r_g \log\frac{r - r_g}{r_g}, \tag{32}$$

which ranges from $r_* = -\infty$ at $r = r_g$ to $r_* = \infty$ at $r = \infty$. In terms of r_* the $t - r$ part of the Schwarzschild metric becomes *conformally flat*

$$ds^2 = \left(1 - \frac{r_g}{r}\right)(-dt^2 + dr_*^2) + r^2 d\Omega^2, \tag{33}$$

where r should be thought of as a function of r_*. The solutions to (31) in these coordinates are the outgoing rays

$$t - r_* = u, \tag{34}$$

and the infalling rays

$$t + r_* = v, \tag{35}$$

where $-\infty < u, v < \infty$ are constants that label the null rays. As we approach $r = r_g$ along an infalling null ray ($v =$ constant) $u \to \infty$ and as we approach it by moving backward along an outgoing null ray ($u =$ constant) $v \to -\infty$.

2. Find the affine time $\Delta\sigma$ and the coordinate time Δt of an infalling null ray to reach r_g starting from $2r_g$, assuming $\frac{dt}{d\sigma}(2r_g) = 1$.

3. Repeat the same exercise for an outgoing ray.

Next we switch to the null coordinates u, v. This is a common trick to simplify $2d$ metrics, and here we are concerned with the $t - r$ part of the metric. We can use $v - u = 2r_*$ to solve for the singular factor $1 - r_g/r$, and obtain

$$ds^2 = -\frac{r_g}{r} e^{-r/r_g} e^{\frac{v-u}{2r_g}} du dv + r^2 d\Omega^2, \tag{36}$$

where again r has to be thought of as an implicit function of u and v. We can now define the new coordinates

$$U = -2r_g e^{-u/2r_g}, \qquad V = 2r_g e^{v/2r_g}, \tag{37}$$

with the range $-\infty < U < 0$ and $0 < V < \infty$ as u and v vary over the real line. Reaching $r = r_g$ along infalling and outgoing null rays corresponds to reaching $U = 0$ and $V = 0$ respectively. In terms of U, V

$$ds^2 = -\frac{r_g}{r} e^{-r/r_g} dU dV + r^2 d\Omega^2. \tag{38}$$

Now we see that nothing is singular if we extend U and V along the entire real line, as long as $r > 0$. One can check that $r = 0$ is a real singularity of the manifold, where curvature invariants diverge. We determine its location in terms of the U, V coordinates by using

$$r_* = \frac{v - u}{2} = r_g \log \frac{-UV}{4r_g^2}, \tag{39}$$

using (32) we get

$$\frac{r}{r_g} = \log \frac{UV}{4r_g(r_g - r)}. \tag{40}$$

Note that whenever we cross $r = r_g$ either U or V switch sign, so the argument of log is always positive. At $r = 0$, we get

$$UV|_{r=0} = 4r_g^2, \tag{41}$$

which comprise a past singularity $V, U < 0$ and a future singularity $V, U > 0$. Finally, we can also reintroduce timelike and spacelike variables $U = T - X$, $V = T + X$, to find

$$ds^2 = \frac{r_g}{r} e^{-r/r_g} (-dT^2 + dX^2) + r^2 d\Omega^2, \tag{42}$$

where $r = r(T, X)$. The resulting geometry is the maximally extended Schwarzschild geometry. It is shown in figure 1-left. I will next review some of the features of this geometry.

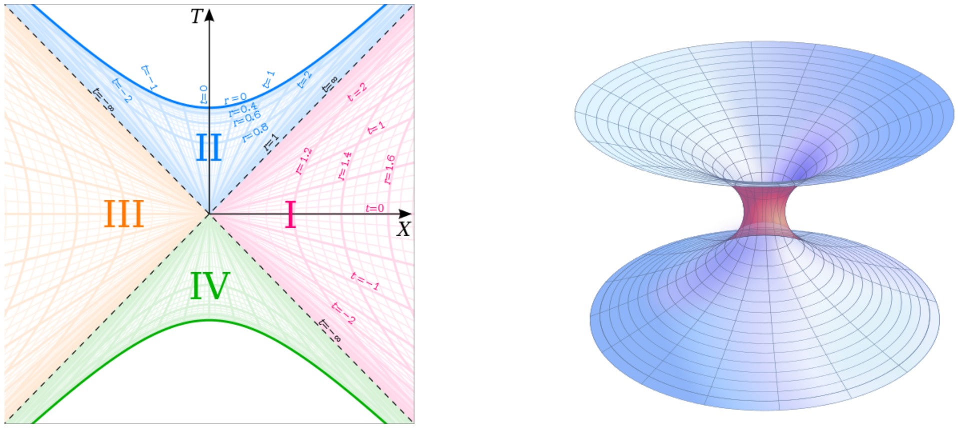


Figure 1: Left: The maximally extended Schwarzschild geometry. Right: A spatial slice of the geometry (with spheres represented by circles), that connects the asymptotic regions. It is called the Einstein-Rosen bridge. (Pictures from Wikipedia.)

Exterior regions. There are two identical exterior regions I and III with $r_g < r < \infty$ (or $UV < 0$) that asymptote to Minkowski as $r \to \infty$. They are causally disconnected (no signal can be sent from one to the other).

Interior regions. The two regions II and IV with $0 < r < r_g$ ($0 < UV < 4r_g^2$) are respectively the black hole and white hole regions. No signal can escape from the black hole, and no signal can enter the white hole.

Horizon. This causal structure leads us to the definition of horizons as the boundaries of black hole and white hole regions at $r = r_g$ ($UV = 0$). These are null hypersurfaces since either $U = 0$ or $V = 0$ and constant U and constant V correspond to radially moving null geodesics. In this case radial motion does not mean going to larger or smaller r — these are null rays that are stuck at $r = r_g$. These geodesics are called the *generators* of the horizon.

Spatial wormhole (or Einstein-Rosen bridge). There are spatial sections of the extended geometry that look like a wormhole, a $3d$ geometry along which r shrinks from infinity to a minimum and expands again. We can imagine a slight modification of the geometry where the asymptotics $r \to \infty$ of region I and III are identified. In this case the wormhole connects two far away points in the same spacetime. However, as discussed above, this wormhole is *nontraversable*.

Real black holes. The maximally extended Schwarzschild geometry is an abstract model. There is no evidence that such a thing exists in our Universe, but black holes do exist. Astrophysical black holes are formed from the gravitational collapse of matter. Under the assumption of spherical symmetry, the Schwarzschild solution would then describe the metric outside matter, but inside matter r smoothly shrinks to 0. This eliminates regions III and IV and modifies parts of regions I and II. Nevertheless, the future singularity at $r = 0$ is inevitable once matter distribution collapses below $r = r_g$. In region II r is timelike and falling toward $r = 0$ cannot be avoided. Penrose singularity theorem proves the existence of future singularities under relatively mild assumptions, even if there is no spherical symmetry.

3 Conformal Compactification

It is often desirable to draw a diagram that shows the causal structure of Lorentzian manifolds. In $2d$, or when there is enough symmetry that one can focus on a $2d$ part of the metric (as in the case of Schwarzschild) there is a way to do this. Let's start from the example of $4d$ Minkowski metric, which in spherical reads

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \qquad -\infty < t < \infty, \quad 0 < r < \infty. \tag{43}$$

The idea is to suppress the spherical part and focus on the $t - r$ part of the metric. Then we use the same trick of transforming to null coordinates as in the last chapter

$$u = t - r, \qquad v = t + r, \tag{44}$$

with the range $-\infty < u, v < \infty$ subject to the constraint

$$v \geq u. \tag{45}$$

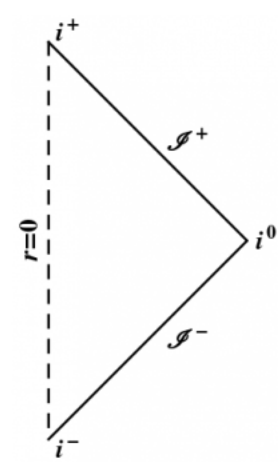
To compactify the $u - v$ part of the metric, we define

$$\bar{u} = \tanh u, \qquad \bar{v} = \tanh v \tag{46}$$

which range over $-1 < \bar{u}, \bar{v} < 1$ subject to the constraint (45) which implies $\bar{v} > \bar{u}$. The metric now reads

$$ds^2 = -\frac{d\bar{u}d\bar{v}}{(1-\bar{u}^2)(1-\bar{v}^2)} + r^2 d\Omega^2. \tag{47}$$

As long as we are interested in radial motion (i.e. motion in $\bar{u}, \bar{v}$) the causal structure is the same as the $2d$ Minkowski space $ds^2 = -d\bar{u}d\bar{v}$ except for the constraint on the range of the variables. So we obtain



Every point of this diagram should be thought of as a 2-sphere. At every point the lightcone (in radial direction) is made of $\bar{u}$ = constant and $\bar{v}$ = constant lines that pass through that point.

The interesting parts of this diagram are $u = v$, corresponding to $r = 0$, $u = v = \pm 1$ corresponding to future and past timelike infinity ($i^{\pm}$), $v = 1$, which is future null infinity $\mathscr{I}^+$, and $u = -1$, which is past null infinity $\mathscr{I}^-$.

1. (a) Mark the point corresponding to $r \to \infty$ and t = constant.

   (b) Draw the worldline of an observer at constant r.

   (c) Draw the full trajectory of a null ray that passes through the origin.

2. (a) Show that every $2d$ metric can be transformed in the neighborhood of any point into a conformally flat metric

$$ds^2 = -\Omega^2(u, v)dudv. \tag{48}$$

   (b) How many independent components are there in the Riemann tensor in $2d$?

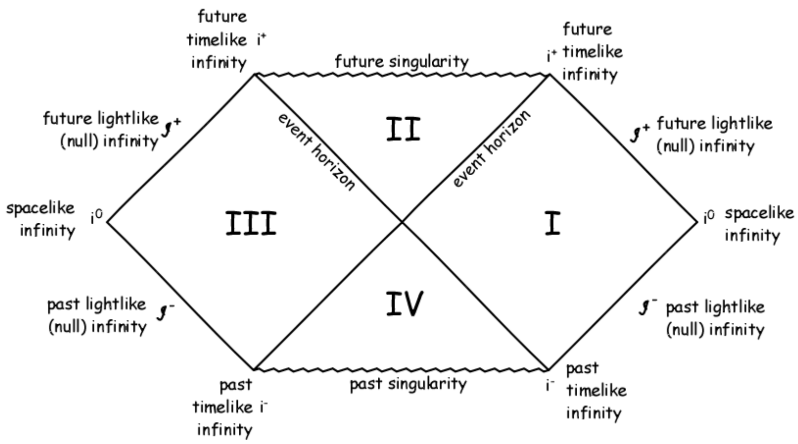
Schwarzschild. Now we apply the same logic to the $U - V$ part of the extended Schwarzschild metric

$$\bar{u} = \tanh U, \qquad \bar{v} = \tanh V, \tag{49}$$

with the range $-1 < \bar{u}, \bar{v} < 1$, but subject to the constraint

$$UV < 4r_g^2, \tag{50}$$

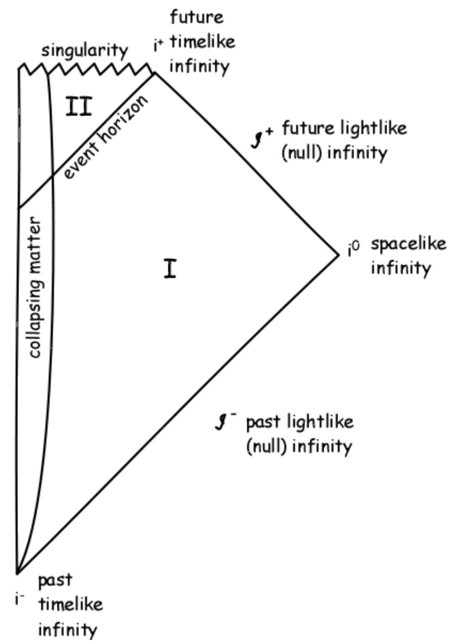
to avoid the $r = 0$ singularity. The resulting Penrose diagram is



The significance of Penrose diagrams is to illustrate the causal structure. The timelike and spacelike curves can be deformed by change of variables $\bar{u} \to f(\bar{u})$, $\bar{v} \to g(\bar{v})$, as long as f and g are monotonic functions in the range $(-1, 1)$. Hence in the above diagram the $r = 0$ singularities, which are spacelike hypersurfaces, are drawn horizontally even though from (49) we would obtain curves that bulge out.

3. Alice lives in region I and Bob in region III of the extended Schwarzschild geometry. How can they meet? How big should be r_g for this to be a safe trip?

The Penrose diagram of a black hole formed from the collapse of spherically symmetric matter looks like



4. The surface temperature of a neutron star of radius $5r_g$ is 6×10^5 K. As the star starts collapsing into a black hole, the surface falls freely. What is the temperature measured by a distant observer as a function of time t?

4 Rotating Black Holes

Reading: *Wald 12.3, 12.4*

We have learned that total energy and total momentum is defined in asymptotically flat spacetimes, leading to a well-defined notion of mass, called the ADM mass (see section 11 of part I). Under the same conditions the expression for the angular momentum is self-evident

$$J^i = \varepsilon^{ijk} \int d^3x \; x^j \tau^{0k}, \tag{51}$$

where ε^{ijk} is the totally antisymmetric tensor on the spatial slice where we are computing the integral and $\tau^{\mu\nu}$ is the pseudo-tensor of energy and momentum. As in the case of total energy and momentum, this expression is useful because it reduces to a boundary integral, which is invariant under linearized diffeomorphisms. Therefore, as long as we can choose a coordinate frame where linearizing in $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is justified when $r \to \infty$, J^i is well-defined.

On a spherically symmetric spacetime (like Schwarzschild) $J^i = 0$, but we could ask what happens if we drop a particle with nonzero angular momentum in a Schwarzschild black hole, or what is the end result of the gravitational collapse of a rotating star. This is known to be a rotating black hole, described by the **Kerr metric**:

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\varphi \\ &+ \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \end{aligned} \tag{52}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - 2GMr, \tag{53}$$

and $a < GM$.

1. The Kerr metric is asymptotically flat. Use (51) to show

$$J = Ma. \tag{54}$$

The $a \to 0$ limit of the Kerr metric is the Schwarzschild metric. At finite a the geometry has less symmetries than Schwarzschild. It is stationary (invariant under time translations) and axisymmetric (invariant under rotations along the z axis). These symmetries lead to two isometries, $\chi = \partial_t$ and $\psi = \partial_\varphi$. **Isometries** are vector fields such that infinitesimal diffeomorphisms generated by them leave the metric unchanged:

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \qquad \xi \text{ an isometry.} \tag{55}$$

2. Show that if ξ^μ is an isometry, then $\xi^\mu u_\mu$ =constant along geodesics.

Horizon. The Kerr metric has singularities at

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2}, \tag{56}$$

where $g_{rr} = \infty$. Approaching from infinity, one first encounters the r_+ singularity. This is a coordinate singularity just like the singularity at $r = 2GM$ in Schwarzschild. One can show that an infalling observer whose size is small compared to the black hole will cross $r = r_+$ after a finite proper time and without any drama.

While at $r > r_+$ constant-r hypersurfaces are timelike (i.e. they include timelike curves) the $r = r_+$ hypersurface is null. This can be seen by considering the normalization of a 4-velocity in the $t - \varphi$ direction $u^\mu = (u^t, 0, 0, u^\varphi)$:

$$g_{\mu\nu}(r_+)u^\mu u^\nu = \frac{\sin^2\theta}{\Sigma}(au^t - (r_+^2 + a^2)u^\varphi)^2 \geq 0, \tag{57}$$

while for a timelike curve we must have $u^\mu u_\mu = -1$ (adding a u^θ component doesn't help as it makes (57) more positive). On the other hand, this hypersurface includes the null curves

$$\ell^\mu = (1, 0, 0, a/(r_+^2 + a^2)) = \chi + \frac{a}{r_+^2 + a^2}\psi. \tag{58}$$

One can show that these curves are null geodesics (for instance by taking the limit $\epsilon \to 0$ of null geodesics passing at $r = r_+ + \epsilon$, with $u^r \propto \epsilon$). One can also show that the $r = r_+$ hypersurface is **achronal**, namely no two points on it are timelike separated. This is the same property that the future (or past) lightcone possesses. As a result the $r = r_+$ hypersurface is the boundary of the black hole exterior from which signals can escape to infinity and the black hole interior. As in the case of Schwarzschild, the maximal extension of the geometry includes also a white hole region and therefore two $r = r_+$ hypersurfaces corresponding to the black hole and the white hole horizons.

From the point of view of the asymptotic observer (at large r), the null generators of the horizon have a nonzero angular velocity $d\varphi/dt = \ell^\varphi/\ell^t$

$$\Omega_H \equiv \frac{d\varphi}{dt} = \frac{a}{r_+^2 + a^2}, \tag{59}$$

which is identified as the angular velocity of the horizon.

Ergosphere. In the region

$$r_+ < r < GM + \sqrt{G^2M^2 - a^2\cos^2\theta} \tag{60}$$

the tt component of the metric flips sign, $g_{tt} > 0$. As a result, causal curves must be corotating with the black hole,

$$u^\varphi u^t > 0 \Rightarrow \frac{u^\varphi}{u^t} > 0. \tag{61}$$

This region is called ergosphere. Recall that the conserved energy measured at infinity is

$$E = -p_t = -\chi^\mu p_\mu. \tag{62}$$

Any particle that can escape to infinity must have $E \geq m$ (the rest mass) and hence $E > 0$. On the other hand, since in the ergosphere χ^μ becomes spacelike, there can be particles in this region with $p_t > 0$. Of course they won't be able to escape to infinity. This allows a mechanism to extract energy from the rotating black hole, known as the **Penrose Process**. Imagine we drop particle a from infinity, and inside the ergosphere $a \to b + c$. At the moment of decay

$$p_a^\mu = p_b^\mu + p_c^\mu. \tag{63}$$

In particular $p_t^a = p_t^b + p_t^c$. Suppose particle c has $p_t^c > 0$ and falls into the black hole but particle b escapes to infinity. Then we will have

$$E_b > E_a. \tag{64}$$

This process spins down the black hole as we will see next. Firstly, any particle that falls through the Kerr horizon with momentum p must have

$$\ell^\mu p_\mu < 0, \tag{65}$$

because p^μ and ℓ^μ are future directed vectors (in particular, ℓ is parametrized such that it co-rotates with the black hole). Equation (65) can be written as an inequality between the energy and angular momentum of the particle as measured from infinity. We expect the black hole to relax to a new Kerr solution after a while with δM and δJ given by those of the particle. It follows that

$$\delta M - \Omega_H \delta J > 0. \tag{66}$$

So if $\delta M < 0$, as in the Penrose process, then δJ must also be negative. It is impossible to extract energy from a non-rotating black hole.

Superradiance. So far we saw the Penrose process as a mere logical possibility. Black hole superradiance is a clever setup that efficiently realizes this possibility. I will illustrate the idea following the original work of Zel'dovich.[1] Suppose you have a medium in which waves of a certain kind dissipate (electromagnetic waves in a piece of Copper is an example). As a simple model consider a massive Klein-Gordon field. In vacuum $\Box\psi - m^2\psi = 0$. The wave equation in the rest frame of the dissipative medium takes the form $\Box\psi - a\partial_t\psi - m^2\psi = 0$ with $a > 0$ a constant. If

[1] I highly recommend reading his two-page paper `http://jetpletters.ru/cgi-bin/articles/download.cgi/1604/article_24607.pdf`.

the medium is moving then the dissipation term transforms to

$$a\partial_t\psi \to a\gamma(\partial_t\psi + v\partial_x\psi), \qquad \gamma = 1/\sqrt{1-v^2}. \tag{67}$$

Now suppose we have an axisymmetric configuration like a Copper cylinder of radius R spinning with angular velocity Ω, and shine on it a wave of frequency ω and angular momentum n along the axis of symmetry, $\psi = f(r)e^{-i\omega t+in\varphi}$. At the surface of the cylinder we replace $v \to \Omega R$ and $\partial_x \to \frac{1}{R}\partial_\varphi$ in (67), which turns it into

$$-ia\gamma(\omega - n\Omega)\psi. \tag{68}$$

When $\omega < n\Omega$, which is called the superradiance condition, this flips sign and the waves get amplified.

Similarly, a non-rotating black hole absorbs all sorts of waves, but a rotating one will amplify the modes that satisfy the superradiance condition. Now we can imagine surrounding a Kerr black hole with mirrors and sending in such a mode. The amplification continues and the rotation slows down until getting to $\Omega = \omega/n$. Admittedly this arrangement is still out of reach at the current level of human development. However, for light fields the mass term can play the role of a natural mirror and looking for the signatures of astrophysical black holes undergoing such a process is an active area of research.

5 Black Hole Thermodynamics

Reading: *Wald 12.5*

There is good evidence that after formation black holes relax to stationary configurations, characterized by mass, angular momentum, and gauge charges. This is a negligible amount of data compared to the many ways we can form black holes, suggesting that they ought to be thought of as thermodynamic entities. If so we need to identify their temperature and entropy and ask if they satisfy the laws of thermodynamics. This was a great discovery by Bekenstein and Hawking who showed black holes are thermal only after quantum effects are taken into account and their entropy is the horizon area/$4G$.

To motivate these answers, let us return to the Kerr solution and consider the area of a constant t section of the horizon. The induced metric is

$$ds^2 = \Sigma d\theta^2 + \frac{(r_+^2 + a^2)^2}{\Sigma} \sin^2\theta \; d\varphi^2, \tag{69}$$

and the resulting area is

$$A = 4\pi(r_+^2 + a^2) = 8\pi(G^2M^2 + \sqrt{G^4M^4 - G^2J^2}). \tag{70}$$

In the last step we used $J = Ma$ and the definition of r_+ to write A in terms of black hole mass and spin. Under an infinitesimal change dM and dJ, we obtain

$$dM = \Omega_H dJ + \frac{\sqrt{G^4M^4 - G^2J^2}}{2G^2MA} dA. \tag{71}$$

This equation is reminding of the **first law of thermodynamics** $dE = \mu dQ + TdS$ (in a system with rotation symmetry, Ω is the chemical potential associated to the conserved charge J). It suggests identifying the horizon area with black hole entropy

$$S_{BH} \propto A, \tag{72}$$

though we need a different calculation to determine the temperature and hence the (positive) proportionality constant. Nevertheless, we can already see that (72) passes a nontrivial consistency check. We learned in the last lecture that anything falling into the black hole satisfies $\delta M - \Omega_H \delta J > 0$, which using (71) and (72), implies

$$\delta S_{BH} > 0. \tag{73}$$

Namely, the **second law of thermodynamics** holds, even in the Penrose process for which $\delta M < 0$.

In fact, there is a much more general **area theorem** in classical GR that proves the area of sections of the event horizon can never decrease as long as the null energy condition is satisfied:

$$k^\mu k^\nu T_{\mu\nu} \geq 0, \qquad \text{for all null } k^\mu. \tag{74}$$

(Notice that we used a particular version of this inequality, i.e. $\ell^\mu p_\mu < 0$ for objects falling through the Kerr horizon, to arrive at (73).) This strongly supports (72) for big black holes, for which quantum effects are small.

Black hole temperature

Hawking showed that on the black hole background particles are created quantum mechanically and with an effective temperature $T = 1/4\pi r_g$. Particle production in a background field is familiar from the **Schwinger process** in QED. The idea is that in vacuum virtual electron-positron pairs are constantly created and annihilated. By energy conservation these can never materialize as on-shell particles that escape to infinity and hit the detectors. However, a background electric field can provide the work needed to put the particles on-shell. The typical distance of the virtual pairs is of the order of the Compton wavelength $\lambda_C \sim 1/m_e$. When

$$eE\lambda_C > m_e \Rightarrow E > \frac{m_e^2}{e}, \tag{75}$$

pair production becomes very efficient and the vacuum acts as a conductor, discharging the electric field.

In a similar fashion, **Hawking process** results in the creation of physical particles near the horizon of black holes. Here, a virtual particle-anti-particle pair need to separate enough for one of them to end up inside the black hole and the other to escape to infinity. Below I will give a simple argument that motivates the idea.

Thermal correlators. Let us first see that thermal correlators are periodic in Euclidean time. To fix the notations, start from a massless scalar field in $4d$ Minkowski spacetime. In terms of creation and annihilation operators

$$\phi(t, \vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2|k|}}(a_{\vec{k}} e^{-i|k|t + i\vec{k}\cdot\vec{r}} + \text{h.c.}) \tag{76}$$

where h.c. means Hermitian conjugate, and

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3\delta^3(\vec{k} - \vec{k}'). \tag{77}$$

The 2-point correlator of ϕ at fixed position is

$$\langle\phi(t,0)\phi(0,0)\rangle = \frac{-1}{4\pi^2(t - i\epsilon)^2}. \tag{78}$$

Lorentz symmetry implies

$$\langle\phi(t_1,\vec{r}_1)\phi(t_2,\vec{r}_2)\rangle = \frac{-1}{4\pi^2[(t_1-t_2-i\epsilon)^2-|\vec{r}_1-\vec{r}_2|^2]}. \tag{79}$$

Now suppose we are at finite temperature $T=1/\beta$. Then we will have nonzero thermal occupation number

$$\left\langle a^\dagger_{\vec{k}} a_{\vec{k}'}\right\rangle_\beta = \frac{1}{e^{\beta|k|}-1}(2\pi)^3\delta^3(\vec{k}-\vec{k}'). \tag{80}$$

(In the presence of gravity, it makes more sense to consider the thermal system in finite volume, where the momenta would be discrete. Then, say in a cubic box of size L, we would have $(2\pi)^3\delta^3(\vec{k}-\vec{k}')\to L^3\delta_{\vec{k},\vec{k}'}$ and $\int\frac{d^3\vec{k}}{(2\pi)^3}\to L^{-3}\sum_{\vec{k}}$.)

The thermal 2-point function is

$$\langle\phi(t,0)\phi(0,0)\rangle_\beta = \int\frac{d^3\vec{k}}{(2\pi)^3 2|k|}\left[e^{-i|k|t}+\frac{1}{e^{\beta|k|}-1}\left(e^{-i|k|t}+e^{i|k|t}\right)\right]. \tag{81}$$

We can expand $\frac{1}{e^{\beta|k|}-1}=\sum_{n=1}^\infty e^{-\beta n|k|}$ to write the integrand as a sum of exponentials whose integral can be taken easily (as in Minkowski)

$$\langle\phi(t,0)\phi(0,0)\rangle_\beta = \frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(\beta n+it)^2} = -\frac{1}{4\beta^2\sinh^2\frac{\pi t}{\beta}}. \tag{82}$$

As a consistency check, in the zero temperature limit, $\beta\to\infty$, this matches (78). We also see that as a function the Euclidean time $t_E=it$ the thermal correlator is periodic, with periodicity

$$t_E\sim t_E+\beta. \tag{83}$$

Rindler observer. Next consider an accelerated observe in Minkowski, with proper acceleration α. We choose the origin such that the trajectory is

$$t(\tau)=\frac{1}{\alpha}\sinh(\alpha\tau),\qquad x(\tau)=\frac{1}{\alpha}\cosh(\alpha\tau). \tag{84}$$

Suppose the observer has a detector coupled to the same massless field ϕ. The 2-point correlator measured as a function of the Rindler proper time τ is simply obtained from (78), with $(t,\vec{r})$ substituted as functions of τ:

$$\langle\phi(\tau)\phi(0)\rangle_R = -\frac{\alpha^2}{4\pi^2[\sinh^2(\alpha\tau)-(\cosh(\alpha\tau)-1)^2]} = -\frac{\alpha^2}{16\pi^2\sinh^2\frac{\alpha\tau}{2}}. \tag{85}$$

This is a thermal correlator with temperature

$$T_R=\frac{\alpha}{2\pi}. \tag{86}$$

Hence, the accelerated observer detects thermal radiation in Minkowski vacuum. This phenomenon is called **Unruh radiation**.

Let us see a geometric way of determining T_R, by writing the $t-x$ part of the Minkowski metric in Rindler coordinates (the other "transverse" coordinates just go for a ride):

$$t = y \sinh(\alpha\tau), \qquad x = y \cosh(\alpha\tau). \tag{87}$$

Note that τ corresponds to the proper time of a Rindler observer at $y = 1/\alpha$ (it is common to set $\alpha = 1$ by a rescaling of τ). The $t-x$ part of the Minkowski metric becomes

$$ds^2 = -y^2\alpha^2 d\tau^2 + dy^2, \tag{88}$$

which covers the Rindler wedge ($x > |t|$). Under Wick rotation $\tau \to -i\tau_E$ this becomes

$$ds_E^2 = y^2\alpha^2 d\tau_E^2 + dy^2. \tag{89}$$

This is the metric of the Euclidean plane in radial coordinates, with y playing the role of the radius and $\alpha\tau_E$ the angular variable. However, unless we impose the periodicity $\tau_E \sim \tau_E + 2\pi/\alpha$ there will be a conical singularity at the origin. This periodicity coincides with the inverse of the Rindler temperature (86). The reason is that the Minkowski vacuum state can be understood as being *prepared* by a path integral over the Euclidean half-plane with no conical singularity at the origin.

Hawking temperature. Finally, let us zoom in the near horizon region of Schwarzschild spacetime, by writing

$$r = r_g + x, \tag{90}$$

and keeping the leading nontrivial terms in x. The Schwarzschild metric becomes

$$ds^2 \approx -\frac{x}{r_g}dt^2 + \frac{r_g}{x}dx^2 + r_g^2 d\Omega^2. \tag{91}$$

In terms of the new radial variable

$$y = 2\sqrt{r_g x}, \tag{92}$$

the metric looks like

$$ds^2 \approx -\frac{y^2}{4r_g^2}dt^2 + dy^2 + r_g^2 d\Omega^2. \tag{93}$$

The $t-y$ part of this metric has the same form as the Rindler metric. Hence if we prepare the of the state of the system by a Euclidean path integral, in order to have a non-singular horizon, we should as before impose periodicity on the Euclidean time $t_E = it$

$$t_E \sim t_E + 4\pi r_g, \tag{94}$$

corresponding to the Hawking temperature

$$T_H = \frac{1}{4\pi r_g}. \tag{95}$$

Black holes evaporate via Hawking radiation and as they do so their temperature increases. Small black holes explode. For astrophysical black holes, the temperature is extremely small and the Hawking radiation is of no phenomenological significance. On the other hand, it does constrain black hole remnants from earlier stages of cosmic evolution. These black holes are called **Primordial Black Holes**, and they are sometimes considered as a candidate for dark matter.

Returning to the first law of black hole thermodynamics (71), we can now use our result (95) to verify the proportionality constant in **Bekenstein-Hawking entropy**

$$S_{BH} = \frac{A}{4G}. \tag{96}$$

This turns out to be valid for all large black holes, even if charged or rotating.

6 FRW Cosmology

Reading: *Wald 5*

Friedmann-Robertson-Walker cosmology is a model inspired by the large scale homogeneity and isotropy of our universe. It is a spacetime in which there is an infinite set of preferred observers (called *comoving* observers) who all go through the same experience, and every spacetime point is passed by the worldline of one such observer. This means that after synchronizing their clocks at some initial time, there are preferred, maximally symmetric time-slices on which their clocks all show the same time t. Up to an overall rescaling, there are three possible choices for the metric of these $3d$ spatial manifolds, depending on whether their scalar Ricci $R^{(3)}$ is zero, positive, or negative. Maximal symmetry means that there is no preferred point or direction, so $R^{(3)}$ fully specifies the curvature. The full metric can be written as

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{97}$$

where $a(t)$ is called the scale factor, and by an appropriate redefinition of $a(t)$ and r, k can be taken to be 0 if the spatial slices are "flat" ($3d$ Euclidean), 1 if they are "closed" (spherical/positively curved), or -1 if they are "open" (hyperbolic/negatively curved).

By definition, comoving observers are at fixed spatial coordinates (which are called comoving coordinates) and they are moving along geodesics, with 4-velocity $u^\mu = (1,0,0,0)$. Consider two nearby observers at comoving distance χ. The length of the shortest path that is restricted to a $t =$constant slice and connects the two (often called the physical distance) is

$$x_{\rm ph} = a\chi, \tag{98}$$

and hence

$$\dot{x}_{\rm ph} = \frac{\dot{a}}{a} x_{\rm ph}. \tag{99}$$

This relation between velocity and distance is called the "Hubble law". The Hubble parameter is thus defined by

$$H \equiv \frac{\dot{a}}{a}. \tag{100}$$

The homogeneity and isotropy of the FRW model restricts the stress-energy tensor to be a function only of t and diagonal

$$T^0_0 = -\rho(t), \qquad T^i_j = p(t)\ \delta^i_j. \tag{101}$$

This is how the stress-tensor of a perfect fluid with energy density ρ and pressure p looks like, though the matter content of the universe might not be an actual fluid.

The ultimate goal in cosmology is to figure out the history of the universe. In the FRW model this corresponds to finding $a(t)$, $\rho(t)$ and $p(t)$ given the matter content and the current state of the universe. One equation that governs this evolution is the energy-momentum conservation $\nabla_\nu T^\nu_\mu = 0$. The only nontrivial component in this case is $\mu = 0$ component, which gives

$$\dot{\rho} = -3H(\rho + p). \tag{102}$$

Another equation follows from the tt component of the Einstein equation, and is called the **Friedmann equation**

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{103}$$

The other components are either trivial or derivable from (102) and (103). Finally, we need a relation between p and ρ to solve the system. This is called the **equation of state**, and depends on the microscopic details of the theory. In general the relation can be quite complicated, but it is useful and often a sufficiently accurate approximation to consider a mixture of a few components each with a linear equation of state

$$p_i = w_i \rho_i. \tag{104}$$

If these components are decoupled, then their stress-tensors are separately conserved and (102) holds for each of them separately. Under this assumption, we can integrate $\dot{\rho}_i = -3H(1 + w_i)\rho_i$ to get

$$\rho_i \propto a^{-3(1+w_i)}. \tag{105}$$

Let's consider some important examples.

- A thermal gas of relativistic particles has $p = \rho/3$. It is commonly denoted as **radiation**

$$\rho_r \propto a^{-4}. \tag{106}$$

- Non-raltivistic matter has negligible pressure, $w = 0$. In cosmology, **matter** usually means non-relativistic matter

$$\rho_m \propto a^{-3}. \tag{107}$$

- Cosmological constant if interpreted as a part of the stress-energy tensor corresponds to $T_{\mu\nu} = -\Lambda g_{\mu\nu}$. Hence, on FRW metric it has $\rho_\Lambda = -p_\Lambda = \Lambda$:

$$\rho_\Lambda = \text{const}. \tag{108}$$

Our universe contains all of these three components. It is seen that in an expanding universe radiation dominates at earlier times, while cosmological constant at late times.

There are theoretical reasons to believe not all equations of state can have a physical microscopic description. These lead to various energy conditions that are imposed on components of $T_{\mu\nu}$. **Null Energy Condition** is one of the weakest conditions, and it is expected to be satisfied classically.

It says that $k^\mu k^\nu T_{\mu\nu} \geq 0$ for all null vectors k^μ. In FRW cosmology, this implies

$$\rho + p \geq 0 \Rightarrow w \geq -1. \tag{109}$$

Cosmological constant saturates the bound, and hence nothing can dominate a Λ-dominated cosmology in the future.

One common way to specify the energy content of the universe is in terms of the fractional contribution to the Friedmann equation at time t_0 (which is usually taken to be the present):

$$\Omega_i \equiv \frac{8\pi G \rho_i(t_0)}{3H_0^2}. \tag{110}$$

Also defining

$$\Omega_K \equiv -\frac{k}{a_0^2 H_0^2}, \tag{111}$$

we have

$$\sum_i \Omega_i + \Omega_K = 1. \tag{112}$$

Hence, it is enough to know Ω_i and H_0 to determine k, and also a_0 if $k \neq 0$.

1. Find the age of a matter dominated universe $\Omega_m = 1$, with Hubble H_0.
   Solution: The Friedmann equation is in this case

   $$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{a_0}{a}\right)^3. \tag{113}$$

   Since $k = 0$, we can rescale the comoving coordinates to set $a_0 = 1$. Integrating the above equation we find

   $$a(t) = \left(\frac{3}{2} H_0 t\right)^{2/3}, \tag{114}$$

   where we chose the integration constant such that $a(0) = 0$. The Hubble rate is

   $$H = \frac{2}{3t}, \tag{115}$$

   and therefore the age of the universe $t_0 = 2/3H_0$. At $t = 0$ the Hubble rate diverges and there is a curvature singularity. This is called the **Big Bang** singularity.

Conformal time. We have seen that transforming the metric into a form that is conformal to Minkowski (in all directions or at least in two directions) is useful in understanding the causal structure of the spacetime. In such coordinates, null rays travel at 45°. This is even more relevant in cosmology because we observe the universe through such null signals, i.e. photons, and recently also gravitons. Hence we introduce the conformal time as

$$\tau(t) = \int^t \frac{dt'}{a(t')}, \tag{116}$$

in terms of which the metric looks like

$$ds^2 = a^2(\tau)(-d\tau^2 + \partial\chi^2 + r^2(\chi)d\Omega^2) \tag{117}$$

where $r(\chi) = \sin\chi, \chi, \sinh\chi$ for $k = 1, 0, -1$, respectively. Picking our conformal coordinate as the origin, $\chi = 0$, our past lightcone becomes simply $\tau + \chi = \tau_0$, where τ_0 is the present value of the conformal time. As an example, in matter-dominated universe

$$\tau = 3\left(\frac{2}{3H_0}\right)^{2/3} t^{1/3}, \tag{118}$$

where I chose the integration constant such that the big bang occurs at $\tau = 0$.

7 Geometric Optics

Reading: *MTW 22.5, 22.6*

When the wavelength of electromagnetic waves is much shorter than the characteristic length over which the background is changing we can describe the propagation of the waves in terms of pointlike photons that move along null geodesics. This is called the **geometric optics approximation**.

Consider an observer in FRW spacetime at time τ_0. Because of isotropy the past lightcone of an observer at $\chi = 0$ is made of null rays at constant θ, φ coordinates, so they travel along

$$\chi(\tau) = \tau_0 - \tau. \tag{119}$$

The four-momentum of these photons is given by $k^\mu = \left(\frac{d\tau}{d\lambda}, \frac{d\chi}{d\lambda}, 0, 0\right)$, where λ is the affine parameter. From (119) follows $k^\chi = -k^\tau$. The χ component of the geodesic equation reads

$$\frac{d}{d\lambda} k_\chi = \frac{1}{2} k^\alpha k^\beta \partial_\chi g_{\alpha\beta} = 0, \tag{120}$$

where the last equality follows from the fact that $\partial_\chi g_{\alpha\beta} \neq 0$ only if $\alpha, \beta = \theta$ or φ, but $k^\alpha k^\beta \neq 0$ only if $\alpha, \beta = \tau$ or χ. Hence, k_χ is a constant and

$$k^\mu = \frac{c}{a^2}(1, -1, 0, 0), \tag{121}$$

for a constant c. This means that the frequency of the photon measured by a comoving observer at the time of emission $\omega_e = -u_\mu k^\mu$ redshifts by the time it arrives to the observer

$$\omega_0 = \omega_e \frac{a_e}{a_0}. \tag{122}$$

This motivates defining the **redshift** z as the time variable:

$$1 + z = \frac{a_0}{a}. \tag{123}$$

So for instance when we talk about a galaxy at redshift $z = 2$, we mean a galaxy along our past lightcone at the time when $a/a_0 = 1/3$. In our cosmological observations we can often use spectroscopy to identify atomic lines from which the original frequency ω_e and z of the emission point follows. If we could also determine the radial distance $a_0\chi$, or the circumference-defined radius $a_0 r(\chi)$, then the relation $a_0\chi(z)$ or $a_0 r(z)$ would allow us to fix the cosmological model, i.e. to determine $\Omega_m, \Omega_\Lambda, \Omega_r, \Omega_K$.[2] Apart from the parallax effect, two conceptually simple methods to

[2] Recall that a_0 relates comoving distances to physical distances we measure on Earth. We could set $a_0 = 1$ by allowing the spatial curvature k to be an arbitrary constant rather than $0, \pm 1$. Either way it is only $a_0\chi$ or $a_0 r$ that is relevant for fixing the cosmological model.

determine $r(\chi(z))$ are using standard rulers and standard candles.

Standard Rulers are cosmological sources with a known physical size ℓ. We can determine r by measuring the angular size $\Delta\varphi$ of such a ruler at redshift z. Using the fact that the φ coordinates of photons along our past lightcone are fixed, we find that (for $\Delta\varphi \ll 1$)

$$\ell = ar(\chi(z))\Delta\varphi = d_A(z)\Delta\varphi, \tag{124}$$

where the **angular diameter distance** is defined as

$$d_A(z) = \frac{a_0 r(\chi(z))}{1+z}. \tag{125}$$

So measuring $\Delta\phi$ and z of a standard ruler allows us to determine $a_0 r(\chi(z))$. A standard ruler in our universe is the BAO scale: it is an enhancement of the correlation function of galaxies at a comoving distance that has to do with the relatively well understood physics of the hot photon-baryon plasma in the early universe.

Standard Candles are sources with known proper luminosity L (erg/s). We can determine r by measuring their flux f (erg/cm^2/s) and redshift z:

$$f = \frac{L}{4\pi a_0^2 r^2(\chi(z))\ (1+z)^2}, \tag{126}$$

where one factor of $1/(1+z)$ arises from the redshift of the emitted photons and the other from the decrease in the arrival rate of photons compared to their emission rate. It is common to define the **luminosity distance** as

$$d_L(z) = a_0(1+z)r(\chi(z)), \tag{127}$$

in terms of which $f = L/4\pi d_L^2(z)$.

The above idea of following the evolution of a *swarm of photons* can be applied much more generally to any curve spacetime, as long as the geometric optics approximation holds. Using the terminology of *MTW 22.6*, suppose we paint red the photons in an infinitesimal volume of phase space $\Delta v_x \Delta v_k$ (measured by a locally inertial observer). Then at any later time, the region of the phase space they occupy is going to be deformed but its volume $\Delta v'_x \Delta v'_k$ (measured by any locally inertial observer) will be the same

$$\Delta v'_x \Delta v'_k = \Delta v_x \Delta v_k, \tag{128}$$

and always the same photons guard the boundaries of this region. This is the **Liouville Theorem** in curved spacetime. It has the following interesting consequence. Suppose the photons are emitted by a blackbody. Then (as we will see) the density of photons per phase space element is

$$f_e(\omega) = \frac{2}{(2\pi)^3\left(e^{\omega/T}-1\right)}, \tag{129}$$

where $\omega = |\boldsymbol{k}|$ is the frequency measured by the observer who lives on the surface of the source. If these photons arrive to a distant observer via a complicated journey that satisfies the assumption of geometric optics, but is otherwise arbitrary, then generically they get redshifted by a factor α ($= 1 + z$ in FRW). Crucially, α is the same for all photons that travel in a sufficiently narrow tube around the same path, regardless of their original frequency ω. By Liouville theorem the phase space density is the same, i.e.

$$f_o(\omega_o) = f_e(\alpha\omega_o). \tag{130}$$

Since $f_e(\omega)$ is just a function of ω/T, the observed photons also have a blackbody distribution with a redshifted temperature

$$T_o = \frac{T}{\alpha}. \tag{131}$$

For instance, this is what happens to the Cosmic Microwave Background (CMB) photons. To an excellent approximation they had a blackbody spectrum with $T \sim 3000\text{K}$ at the time when the universe became neutral, and today we observe them as a blackbody with $T_o \sim 3\text{K}$.

8 Thermodynamics

Reading: *Weinberg Cosmology 3.1, Baumann's lectures on Thermal history*

We have seen that our universe contains a thermal radiation component that becomes hotter at higher redshifts (recall that we use redshift z as a time variable; it runs backward). This hot radiation eventually ionizes the universe when it reaches the temperature $T \sim 1\text{eV}$, at $z \sim 1000$, and brings into (an approximate) thermal equilibrium electrons and baryons (baryons are protons and neutrons; neutrons are unstable unless they are captured in ions). At temperature $T \sim 1\text{MeV}$ electrons become relativistic, and neutrinos come into equilibrium with photons. [...]. Eventually at $T > 100\text{GeV}$ all standard model particles are thermalized and relativistic. The goal of this section is to review some basics of thermodynamics.

Thermodynamic quantities can be computed by the help of the thermal partition function

$$Z = \text{Tr} e^{-\beta(H - \sum_i \mu_i Q_i)} = \sum_{E_n} e^{-\beta(E_n - \sum_i \mu_i Q_{i,n})}, \tag{132}$$

where H is the Hamiltonian, β is the inverse temperature, Q_i are the maximum set of commuting conserved charges, and μ_i are chemical potentials associated to them. The sum is over all energy levels of the Hamiltonian, which can be chosen such that they are also eigenstates of the conserved charges since $[Q_i, H] = 0$ and $[Q_i, Q_j] = 0$.

We will focus on a weakly coupled system where energy levels are described in the Fock basis labeled by particle momenta and occupation numbers. The occupation number of a given momentum eigenstate $\boldsymbol{k}$ with energy

$$\varepsilon(k) = \sqrt{k^2 + m_a^2}, \qquad k \equiv \sqrt{|\boldsymbol{k}|^2} \tag{133}$$

is arbitrary if particle a is a boson but only 0 or 1 if it is a fermion. It is also common to introduce a chemical potential for every species of particles. So for particle a with charge $q_{i,a}$ under the symmetry Q_i, we have

$$\mu_a = \sum_i \mu_i q_{i,a}. \tag{134}$$

Antiparticles have opposite chemical potential $\mu_{\bar{a}} = -\mu_a$, and particles with no conserved charges (which are their own antiparticles) have $\mu = 0$.

Under the above approximation the partition function factorizes in terms of the product over the species $Z = \prod_a Z_a$, with

$$Z_a = \prod_{\boldsymbol{k}} \left[\sum_{n=0}^{n_{\max}} e^{-\beta n(\varepsilon(k) - \mu_a)} \right]^g, \tag{135}$$

where g is the number of internal degrees of freedom (such as polarizations) on which ε and μ does not depend and the upper bound of the sum $n_{\max}$ is 1 for fermions and ∞ for bosons. We can perform the sum over the geometric series in the latter case, and get the following simple result

$$\log(Z_a) = \pm gV \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \log\left(1 \pm e^{-\beta(\varepsilon(k)-\mu_a)}\right), \tag{136}$$

with the plus sign for fermions and the minus sign for bosons and we have approximated the sum over the momentum levels at finite volume by an integral

$$\sum_{\boldsymbol{k}} \to V \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}. \tag{137}$$

From the definition of Z, we can see that the thermal average of the number of particles of type a is

$$\langle N_a \rangle = \frac{1}{\beta}\partial_{\mu_a} \log(Z) = V \int d^3\boldsymbol{k} f_a(k), \tag{138}$$

where the phase space density of species a is given by

$$f_a(k) = \frac{g}{(2\pi)^3 \left(e^{\beta(\varepsilon(k)-\mu_a)} \pm 1\right)}, \tag{139}$$

again plus for fermions and minus for bosons. The overall factor of V in (138) results from the translation invariance. More relevant quantities for us are the densities, such as $n_a \equiv N_a/V$. Similar results can be obtained for the energy density and pressure

$$\rho = \sum_a \int d^3\boldsymbol{k} \varepsilon(k) f_a(k), \tag{140}$$

$$p = \frac{1}{3}\sum_a \int d^3\boldsymbol{k} k \varepsilon'(k) f_a(k), \tag{141}$$

where the sum runs over particles and antiparticles.

1. Derive the above expression for p by finding the rate of momentum transfer to an element of area at the wall of the box containing the thermal gas of particles. Hint: Show that $v = d\varepsilon(k)/dk$ for a relativistic dispersion relation (133).

2. Find the number density and the energy density of bosons and fermions in the relativistic limit $\varepsilon(k) = k$.

Finally, the total entropy S is defined in terms of the (expectation value of the) energy and total number of particles via

$$Z = e^{S-\beta(E-\sum_a \mu_a N_a)}. \tag{142}$$

From this expression and the above definitions we can derive the following identity for the entropy density $s = S/V$:

$$s = \beta(p + \rho - \sum_a \mu_a n_a). \tag{143}$$

3. The first law of thermodynamics follows from this expression if we multiply both sides by TV and vary, and assume

$$dp = sdT + \sum_a n_a d\mu_a. \tag{144}$$

   Verify this identity.

9 Thermal History

Reading: *Weinberg Cosmology 2, 3, Baumann's lectures on Thermal history*[3]

Strictly speaking the constituents of an expanding universe are never in thermal equilibrium. Thermal equilibrium is a stationary phase. But if the rate of interactions among some degrees of freedom is much faster than the expansion rate of the universe (as it is in a pot of boiling water), they will be in a state of quasi-equilibrium and the thermodynamic results can be applied. The two-body interactions are often the most relevant ones since it is much harder for three or more particles to meet simultaneously. At equilibrium, the average rate at which one particle A interacts with B particles is

$$\Gamma_A = n_B \langle \sigma v_{AB} \rangle \tag{145}$$

where n_B is the number density of B particles, v_{AB} is the relative velocity and the average is over its distribution. If $\Gamma_A \gg H$ then the assumption of equilibrium for this reaction is justified.

In our universe, the existence of a small radiation component today, implies (as we will see) that at redshift $z \sim 1000$, the hydrogen atoms, which constitute the majority of baryon content of the universe, will be ionized. Moreover, the increase in the density of free electrons (which have a large cross-section with photons) brings the mixture of photons electrons and protons in thermal equilibrium. At earlier times the temperature of (now radiation dominated universe) increases so much that all SM particles become relativistic and come into equilibrium with photons. Indeed at such high temperatures we can estimate the rate of any 2-to-2 SM interaction by

$$\Gamma \sim g^4 T, \tag{146}$$

where g is a representative for the SM coupling constants. On the other hand, during radiation domination

$$H^2 \approx \frac{8\pi G}{3}\rho_r \propto \frac{T^4}{M_{\rm pl}^2} \tag{147}$$

where $M_{\rm pl} \equiv \sqrt{3/8\pi G} \sim 10^{18}$GeV, and the proportionality constant in the last formula is of order of the number of relativistic degrees of freedom, which would be about 100 for SM at $T \gg 100$GeV. Comparing (146) and (147) shows the assumption of equilibrium is well-justified when $T \sim 10^3$GeV. (One actually gets that for any $100\text{GeV} \ll T \ll 10^{13}\text{GeV}$ $\Gamma \gg H$, but we don't know enough about the energy range 10^3GeV-10^{13}GeV to trust this conclusion.)

Starting from this quasi-equilibrium initial state, we will review some major events that happen as the universe cools.

During the radiation dominated phase the energy density of the universe depends on which

[3] My presentation is approximately a shortened version of this reference, which can be found here `http://cosmology.amsterdam/education/cosmology/`.

particles are relativistic. For such (weakly coupled) particles at zero chemical potential the Bose-Einstein and Fermi-Dirac distributions give

$$\rho_a = \frac{\pi^2}{30} g_a T^4 \begin{cases} 1, & \text{bosons} \\ \\ \frac{7}{8}, & \text{fermions} \end{cases} \tag{148}$$

The total energy density is therefore

$$\rho \approx g_*(2\sigma_B T^4) \tag{149}$$

where σ_B is the Stefan-Boltzmann constant and the **effective number of relativistic degrees of freedom** is defined via

$$g_* = \sum_b g_b + \frac{7}{8}\sum_f g_f. \tag{150}$$

From the relation $p_a = \rho_a/3$, and $s_a = (p_a + \rho_a)/T$ it follows that $s_a = 4\rho_a/3T$ and hence

$$s = \frac{8}{3} g_{*,S}\sigma_B T^3, \tag{151}$$

where $g_{*,S}$ is the **effective number of relativistic degrees of freedom in entropy**. So far we assumed all relativistic degrees of freedom are in equilibrium and therefore have the same temperature. This implies $g_{*,S} = g_*$. However, as we will see below this condition no longer holds after neutrino decoupling.

When all SM particles are relativistic we have

$$g_* = 106.75. \tag{152}$$

As temperature drops with the expansion of the universe, heavy SM particles start to become non-relativistic and their number density (and energy density $\approx mn$) falls exponentially $\propto exp(-m/T)$. For instance, most of tops and anti-tops annihilate by the time $T = m_t/6 \approx 30\text{GeV}$ and g_* reduces by

$$\Delta g_* = -\frac{7}{8} \times 3 \times 4 = -\frac{21}{2}, \tag{153}$$

where the factor of 3 comes from the number of colors and 4 from spin degrees of freedom of t and $\bar{t}$. Note that even if a top particle does not find an anti-top to annihilate with, it will eventually decay into lighter SM particles. For a stable particle with only short-range interactions, the total number of particles can *freeze out* to a finite value after it becomes non-relativistic and dilute enough such that the number changing processes go out of equilibrium. This is one of the standard scenarios for explaining the origin of dark matter.

Evolution of temperature. Total entropy is conserved in thermal equilibrium. Hence the entropy density dilutes as $s \propto 1/a^3$. This implies that temperature does not simply drop as $1/a$

while the number of relativistic degrees of freedom is changing:

$$T \propto \frac{1}{g_{*,S}^{1/3} a}. \tag{154}$$

QCD phase transition happens at $T \sim 150$MeV. Right above this temperature up, down and strange quarks and gluons are relativistic, but below they confine into hadrons and mesons, among which only the three pions are relativistic, resulting in $\Delta g_* = -44.5$.

Above we have neglected chemical potentials. Recall that they are associated to conserved (or approximately conserved) charges, and they lead to an asymmetry between particle and anti-particle number. In the non-relativistic limit, they often translate into the conservation of the number of the lightest particle(s) carrying the charge. Since baryon number is conserved and nonzero in our universe the total number of protons and neutrons (which are almost degenerate and in thermal and chemical equilibrium until $T \sim 1$MeV) does not decay to zero. From the proton equilibrium density,

$$n_p \approx 2 \left(\frac{m_p T}{2\pi} \right)^{3/2} e^{(\mu_p - m_p)/T}, \tag{155}$$

we conclude that in order for n_p to fall as $1/a^3$

$$\frac{m_p - \mu_p}{T} = \frac{3}{2} \log \left(a^2 \frac{T}{m_p} \right) + \text{const.} \tag{156}$$

which implies $m_p - \mu_p \to 0$ approximately as $1/a$. This nonzero baryon number is obviously essential for our existence. Also, several cosmological observables are extremely sensitive to its exact value (e.g. the relative height of the peaks in the CMB power spectrum). However, its contribution to the energy density of the universe is negligible until $T \sim 1$eV.

Neutrino decoupling. At $T \sim 1$MeV, the SM particles we are left with are photons, electrons and positrons, baryons and neutrinos. Neutrinos are thermalized with the rest via

$$\begin{gathered} \nu + e^- \leftrightarrow \nu + e^- \\ \nu + \bar{\nu} \leftrightarrow e^+ + e^-. \end{gathered} \tag{157}$$

The rate of these processes can be estimated by using the Fermi theory. For the cross-section we have $\sigma \propto G_F^2$, and since all components are relativistic the only relevant dimensionful parameter to build the rate is T:

$$\Gamma \sim G_F^2 T^5. \tag{158}$$

During the radiation dominated era we have

$$T \sim g_*^{-1/4} (M_{\rm pl} H)^{1/2} \propto t^{-1/2}. \tag{159}$$

This implies that $\Gamma \propto t^{-5/2}$ drops faster than $H \sim 1/t$. Therefore, neutrinos decouple at some point. This is when

$$\Gamma \sim H \Rightarrow T \sim (G_F^2 M_{\rm pl})^{-1/3} \sim 1\text{MeV}. \tag{160}$$

Below this temperature neutrinos are effectively a gas of free relativistic particles, which as we have seen maintain their (fermionic) blackbody distribution, but with

$$T_\nu \propto \frac{1}{a}. \tag{161}$$

From the neutrino oscillation, we know that at least some of the neutrino species must be massive $\sum_i m_{\nu_i} > 60$ meV. Hence, today neutrinos will contribute to the matter content of the universe. The smallness of their mass leaves a signature in the structure formation in our universe, and cosmological observations (like galaxy surveys) are a promising avenue to determine $\sum_i m_{\nu_i}$.

Returning to the hot universe, we know since neutrinos were originally in thermal equilibrium with photons, T_ν and T (photon temperature) redshift in the same way until **electron-positron annihilation**. The number of relativistic degrees of freedom in the sector that is in equilibrium with photons starts from 11/2 (= 2 for photons $+\frac{7}{8} \times 4$ for e^+e^-) at $T > m_e$ and drops to 2 when $T \ll m_e$. It follows from (154) that afterwards

$$\frac{T}{T_\nu} = \left(\frac{11}{4}\right)^{1/3}. \tag{162}$$

Below this temperature, we can continue to use (149) and (151) with

$$\begin{aligned} g_* &\approx 2 + \frac{7}{8} 3 \times 2 \times \left(\frac{T_\nu}{T}\right)^4 \\ g_{*,S} &\approx 2 + \frac{7}{8} 3 \times 2 \times \left(\frac{T_\nu}{T}\right)^3, \end{aligned} \tag{163}$$

which are no longer equal. The approximation sign in these equations is because the neutrinos are not 100% decoupled during e^+e^- annihilation. This is usually taken into account (for ρ) by introducing an **effective number of neutrino species** $N_{\rm eff} = 3.046$.

1. Calculate Ω_ν as a function of $\sum_i m_{\nu_i}$, given that the CMB temperature today is 2.73K and $H_0 \approx 70$km/s/Mpc.

Big Bang Nucleosynthesis (BBN). The difference between the neutron and proton mass is

$$Q = m_n - m_p \approx 1.3\text{MeV}. \tag{164}$$

Since $n_p + n_n$ is conserved, in thermal equilibrium they are in detailed balance

$$\frac{n_n}{n_p} = e^{-Q/T}. \tag{165}$$

If the neutrons do not get bound inside nuclei like He^4 they will decay and all baryon number will be in the form of protons, which would then combine with electrons to form Hydrogen atoms. (Other elements can be produced also during supernova explosions but there is good evidence that heavier elements existed before supernovas.) An important nucleus in the process of nucleosynthesis is deuterium (made of one proton and one neutron) whose binding energy is 2.22 MeV. Once deuterium forms below this temperature it goes through a chain of reactions that produces He^3, He^4, and Li^7.[4]

Recombination. The photon-baryon-electron plasma remains in a quasi-equilibrium state until much lower temperature when there are not enough high energy photons to ionize hydrogen atoms. Photons and electrons are tightly coupled because of the Thompson scattering $\gamma + e^- \to \gamma + e^-$, and electrons and protons because of the Coulomb scattering $e^- + p \to e^- + p$. When most of free electrons combine into neutral hydrogen atoms H, the rate for Thompson scattering significantly drops and photons decouple. This thermal gas of free photons is called the **Cosmic Microwave Background (CMB)**

To determine the recombination and photon decoupling temperature, we should analyze the reaction

$$e^- + p^+ \leftrightarrow H + \gamma, \tag{166}$$

which implies $\mu_e + \mu_b = \mu_H$. The binding energy of hydrogen is

$$B \equiv m_e + m_p - m_H = 13.6\text{eV}. \tag{167}$$

By the time n_H becomes appreciable, e^-, p^+ and H are non-relativistic and satisfy

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{(\mu_i - m_i)/T}, \tag{168}$$

where $g_e = g_p = g_H/2 = 2$ (for H there are two spin states of p^+ times two of e^-). We can eliminate μ_i by looking at the combination

$$\frac{n_H}{n_e n_p} \approx \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B/T}, \tag{169}$$

where in the prefactor we approximated $m_H \approx m_p$. Because of the neutrality of the universe $n_e = n_p$, and this is equal to n_H/n_e^2. It is useful to express everything in term of X_e the fraction of free electrons out of all electrons. This is approximately $X_e \approx n_e/n_b$ because the majority of

[4] Weinberg presents a very nice account of BBN in his popular science book "The First Three Minutes".

baryons at this stage are protons (either free, or in hydrogen atoms). Under the same approximation $n_H \approx n_b(1 - X_e)$. So we obtain the so-called Saha equation

$$\frac{1 - X_e}{X_e^2} \approx \frac{2\zeta(3)}{\pi^2} \left(\frac{2\pi T}{m_e}\right)^{3/2} \eta_b e^{B/T}, \tag{170}$$

where we defined the baryon-to-photon ratio

$$\eta_b \equiv \frac{n_b}{n_\gamma} \tag{171}$$

which below e^+e^- annihilation is a constant that is known empirically to be $\mathcal{O}(10^{-10})$. We also used

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3. \tag{172}$$

Because η_b is exponentially small in our universe, we see that for X_e to deviate significantly from 1, that is, for the majority of electrons and protons to combine into hydrogen atoms, T has to be well below B. In practice, $T_{\text{rec}} \approx 0.3\text{eV} \ll 13.6$ eV. Intuitively, this follows from the fact that until arriving at such a low temperature there are enough ionizing photons in the tail of the blackbody spectrum to keep protons and electrons unbound.

10 Cosmological Perturbations

Reference: *Weinberg Cosmology 6.2, 6.3, 6.4*

Our universe is not exactly homogenous and isotropic. The late universe is extremely inhomogenous at short distances, but as we go to cosmological scales, or to earlier times the inhomogeneities become smaller and can be described as perturbtaions on FRW. Observation of these perturbations at various scales can teach us about (1) the initial state of our universe, and since we have to evolve them to the observer, also about (2) the cosmological parameters of the FRW background, and (3) microscopic physics.

Fortunately, the initial perturbations turn out to be extremely small and hence linear perturbation theory gives a very accurate description at large scales and early times. So we start with a toy model of a massless scalar field on flat FRW

$$S = \frac{1}{2}\int d\tau d^3\boldsymbol{x}\ a^2[\phi'^2 - |\nabla\phi|^2], \tag{173}$$

where τ is the conformal time, a the scale factor and prime denotes $d/d\tau$. Since the equation of motion is linear and $\boldsymbol{x}$-independent, it is diagonal in the momentum basis:

$$\phi''_{\boldsymbol{k}} + 2\mathcal{H}\phi'_{\boldsymbol{k}} + k^2\phi_{\boldsymbol{k}} = 0, \tag{174}$$

where $\mathcal{H} \equiv a'/a = \dot{a} = aH$ is the comoving Hubble parameter, $k^2 = \sum_i k_i^2$ and

$$\phi(\tau, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\phi_{\boldsymbol{k}}(\tau)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{175}$$

$\boldsymbol{k}$ is called the comoving momentum. Note that it is a constant vector. At large k (geometric optics limit), it can be thought of as the conserved momentum of the ϕ particles $k_i = g_{i\mu}k^\mu$.

Equation (174) has two extreme regimes:

1. **Superhorizon** $k \ll \mathcal{H}$: the solution is a superposition of a "growing mode" and a "decaying mode"
$$\phi_{\boldsymbol{k}}(\tau) = c_g + c_d\int^\tau \frac{d\tau'}{a^2}. \tag{176}$$

2. **Subhorizon** $k \gg \mathcal{H}$: the solution is wavelike with an adiabatically changing amplitude,
$$\phi_{\boldsymbol{k}}(\tau) = \frac{1}{a}\left(c_+e^{ik\tau} + c_-e^{-ik\tau}\right). \tag{177}$$

Consider an FRW cosmology with a single energy component with $p = w\rho$. The Friedmann equation in this case gives $H^2 = H_0^2 a^{-3(1+w)}$, from which we learn

$$\frac{1}{\mathcal{H}} \propto a^{\frac{1}{2}(1+3w)}. \tag{178}$$

$\mathcal{H}^{-1}$ is called the **comoving horizon**. It grows if $w > -1/3$. Of course this conclusion also holds if a cosmology contains multiple components, but it is dominated by one or a subset of the components with $w_i > -1/3$. This is the case in our universe during its hot phase, and for most of the subsequent evolution until $z \sim 0.2$, where Λ starts dominating. Hence, the perturbations of our hypothetical field ϕ start in the superhorizon regime at early enough time, and (unless k is extremely small) they will enter the subhorizon regime. In order for the perturbations to remain finite as $\tau \to 0$, we must have $c_d = 0$. This will fix the ratio c_+/c_- in the subhorizon regime.

A useful way to think about the superhorizon regime is the **separate universe** picture. In this regime the gradients are much smaller than the curvature length. Therefore, one can think of points separated by $r \sim 1/k$ as different FRW cosmologies that evolve independently with different homogeneous initial conditions for ϕ and ϕ'. Any initial ϕ' quickly redshifts, and one is left with a time-independent configuration. When $k > \mathcal{H}$, the gradients become relevant and $\phi_{\boldsymbol{k}}$ starts oscillating.

Adiabatic Perturbations

What is the initial condition for perturbations in our universe? Let's consider the simplest option. Imagine at some early time all constituents of the universe are in thermal equilibrium and there is no conserved charge. Then T fully determines the state of an FRW cosmology. Moreover, T is just a time variable. Different values of T correspond to different times in the evolution of the same cosmology. If instead of having $T =$ constant on an early time slice, we let it fluctuate at superhorizon scales, then, as long as these temperature fluctuations are superhorizon, *different patches of the universe go through the same history but with a relative time-shift.* This is called adiabatic initial condition. We can generalize it to include conserved charges, or other energy contents, by ensuring the basic property of having identical (but shifted with respect to one another) histories at superhorizon scales.

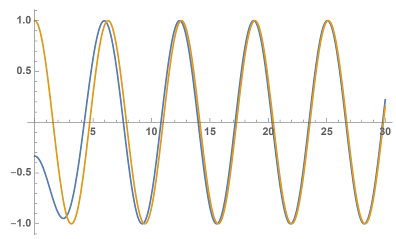
During radiation domination, it is easy to solve for the evolution of adiabatic fluctuations. At subhorizon scales these are just the sound waves satisfying

$$\delta_{\boldsymbol{k}}'' + c_s^2 k^2 \delta_{\boldsymbol{k}} \approx 0, \tag{179}$$

where $\delta_{\boldsymbol{k}} = \delta\rho_{\boldsymbol{k}}/\bar{\rho}$, and $c_s^2 = 1/3$. To match with the superhorizon initial condition, one has to include metric fluctuations and linearize Einstein equations and $\nabla_\mu T^\mu_\nu = 0$ to find

$$\delta_{\boldsymbol{k}} = A_{\boldsymbol{k}} \frac{\omega\tau(\omega^2\tau^2+2)\cos(\omega\tau) - 2\sin(\omega\tau)}{\omega^3\tau^3}, \tag{180}$$

where $\omega = c_s k$ and $A_{\boldsymbol{k}}$ is the unknown amplitude. After the modes cross the *sound horizon*, $\omega\tau > 1$, the solution approaches $A_{\boldsymbol{k}}\cos(\omega\tau)$ which is just the solution to (179). The solution (180) and its asymptotic limit both with $A_{\boldsymbol{k}} = 1$ are plotted respectively in blue and orange

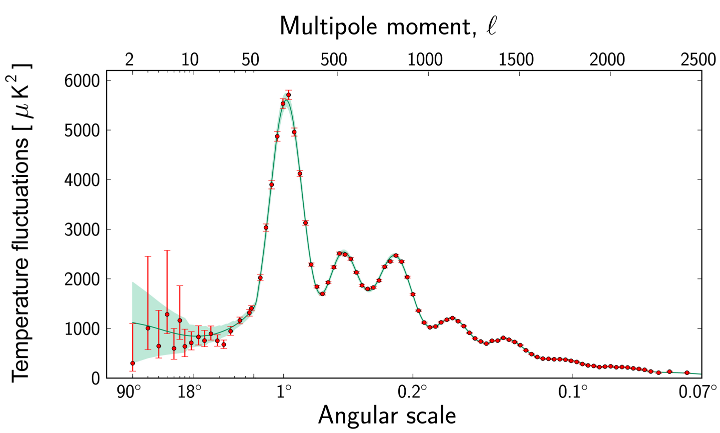


We have good evidence that the perturbations in our universe are predominantly adiabatic, and their amplitude $A_{\boldsymbol{k}}$ is an approximately scale-invariant Gaussian random variable,

$$\langle A_{\boldsymbol{k}} A_{\boldsymbol{k}'} \rangle = \frac{\Delta}{k^{4-n_s}} (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}'), \tag{181}$$

with $1 - n_s \ll 1$, $\Delta \sim 10^{-9}$.

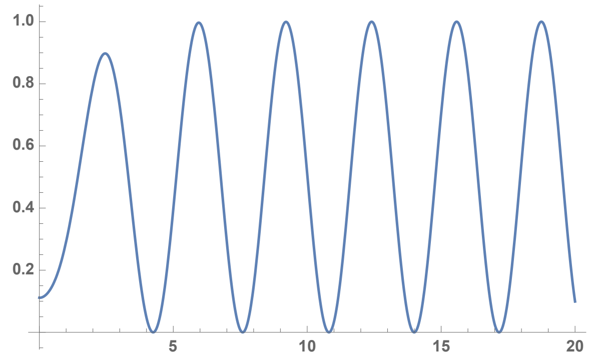
One important manifestation of these perturbations is in the CMB temperature anisotropy, i.e the variation of the CMB temperature as we look in different directions in the sky. This arises from fluctuations at different locations on the last scattering surface and along the trajectory of photons from there to us along our past lightcone. Their statistical average as a function of angular scale looks like



(182)

The solid line is the prediction of the ΛCDM cosmological model, with an adiabatic initial condition.

Let's now imagine taking a snapshot at time τ_* of all k modes in (180), square them, correlate them and multiply by k^3 to cancel the $1/k^3$ in (181). Since the result only depends on $k\tau$ (up to $1-n_s$ corrections), the k dependence of the result would be the same as the τ dependence of $(\delta_{\boldsymbol{k}}(\tau)/A_{\boldsymbol{k}})^2$ at fixed $\boldsymbol{k}$:



(183)

We see that it bears some resemblance with (182). Of course (183) is too simplistic because (i) we only looked at perturbations at a fixed time rather than different angles along our past lightcone, (ii) we neglected $\Omega_m \neq \Omega_b \neq 0$, (iii) $\delta T/T$ is not exactly $\delta\rho/\rho$, (iv) there is a nonzero damping of the sound waves, ...

It is the sensitivity of the map (182) to all these details and the smallness of the error-bars that make CMB such a great probe of the cosmological model. For instance, keeping Ω_m fixed but changing Ω_b would change the relative height of the peaks, making CMB one of the best sources of evidence for the existence of dark matter.

Finally, two examples of less minimal initial conditions are (i) fluctuations in the chemical potentials, and (ii) fluctuations of a light scalar field, both at fixed T.

Structure Formation

Most large scale structures (galaxies, clusters of galaxies, etc) were formed during the matter domination. They are the result of the **Jeans instability**, i.e. the dominance of gravitational attraction over the pressure for a large enough body of mass. Since we are dealing with non-relativistic matter the basic idea can be understood in Newtonian gravity. Consider a ball of gas with radius R, density ρ, and $\frac{\partial p}{\partial \rho} = c_s^2 \ll 1$. The Euler equation for the gas velocity reads

$$\dot{\boldsymbol{v}} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p - \rho\nabla\Phi \tag{184}$$

where Φ is the Newtonian potential. We can estimate

$$|\nabla p| \sim \frac{c_s^2\rho}{R}, \qquad |\rho\nabla\Phi| \sim \rho\frac{GM}{R^2} \sim G\rho^2 R. \tag{185}$$

The maximum size at fixed ρ, where it is possible to have a pressure supported distribution can then be estimated to be

$$R_J = \frac{c_s}{\sqrt{G\rho}}. \tag{186}$$

For $R \gg R_J$ the matter distribution will collapse.

A flat matter dominated FRW never collapses. The Hubble expansion gives just enough kinetic energy to the elements to escape the gravitational well. However, in the presence of perturbations over-dense regions will collapse. Using $G\rho \sim H^2$, we find

$$R_J \sim \frac{c_s}{H} \ll \frac{1}{H}, \tag{187}$$

which implies that there is a large range of comoving scales where perturbations are subhorizon but "super-Jeans". They grow in time and form the structures like the ones we observe.

11 Inflation

Reference: *Weinberg Cosmology 4, Mukhanov Cosmology 5*

Inflation is a period of accelerated expansion that is hypothesized to occur before the hot phase. Originally, it was invented to explain several puzzles in the hot big bang cosmology, two of which will be reviewed below. However, it turned out to be the most compelling scenario to explain the origin of nearly scale-invariant adiabatic fluctuations that we observe today.

Flatness problem. Recall that we defined Ω_K, as

$$\Omega_K = -\frac{k}{H^2 a^2}, \tag{188}$$

which measures the fractional deviation of the total energy density from the critical energy density $1 - \rho/\rho_{\rm cr}$. Today we only have constrains on Ω_K:

$$|\Omega_K^0| < 10^{-3}. \tag{189}$$

It is useful to think of Ω_K as a function of time and ask how closeness to the critical density evolves in various phases of FRW cosmology. During matter domination, we have $\Omega_K \propto a$. So at matter-radiation equality ($z \simeq 3400$)

$$|\Omega_K^{\rm eq}| < 10^{-6}. \tag{190}$$

During the radiation era, $\Omega_K \propto a^2$. So at Grand Unification temperature $T_{\rm GUT} \sim 10^{16}$GeV, we can estimate

$$|\Omega_K^{\rm GUT}| < g_*^{-2/3} 10^{-56}, \tag{191}$$

where g_* is the number of relativistic degrees of freedom at $T_{\rm GUT}$. This is unknown, but it doesn't change the fact that $|\Omega_K^{\rm GUT}|$ is so incredibly small.[5] It is puzzling why the initial energy density was so close to the critical one, though it is of course a logical possibility to have exactly $k = 0$.

Horizon problem. Perhaps more puzzling is the fact that the CMB temperature is so close to being isotropic (with typical anisotropies $\delta T/T \sim 10^{-5}$). In hot big bang cosmology, the last scattering surface (LSS), by which we mean the sphere obtained by the intersection of our past lightcone and the photon decoupling hypersurface, can be divided into many causally disconnected patches. This is evident from the comoving radius at t_1 of the past lightcone of an observer who lives at $t_2 > t_1$

$$\chi(t_2, t_1) = \int_{t_1}^{t_2} \frac{dt}{a} = \int_{a_1}^{a_2} \frac{da}{H a^2}. \tag{192}$$

[5] Since inflation was invented when Grand Unification was popular, one often encounters such references to the GUT scale. Switching to other UV scales would not qualitatively change the conclusion.

We find $\chi(t_0, t_{\rm LSS}) \sim 1/H_0$, while

$$\chi(t_{\rm LSS}, t_{\rm GUT}) \sim \frac{1}{a_{\rm LSS} H_{\rm LSS}} \ll \frac{1}{H_0}. \tag{193}$$

Again, the horizon problem is not a contradiction. It is a question about naturalness of some initial condition.

To see how to solve flatness and homogeneity problems, let us examine the underlying reason behind them in a single-component FRW. In the former case, it is the growth of the comoving horizon in matter and radiation cosmology that leads to the smallness of $|\Omega_K|$ at early times:

$$|\Omega_K| = \frac{k}{\mathcal{H}^2} \propto a^{1+3w}. \tag{194}$$

If the hot phase was preceded by a long enough period dominated by a component with $w < -1/3$ then during that period $|\Omega_K|$ would decrease. Hence the initial condition for that phase could be $|\Omega_K| = \mathcal{O}(1)$. On the other hand, if the integral (192) diverges if the lower bound is taken $a_1 = 0$ then the past lightcones of all points intersect far enough in the past and the horizon problem would have been solved. For $p = w\rho$, we have

$$\chi(a_2, 0) = \frac{1}{H_0}\int_0^{a_2} da\ a^{\frac{1}{2}(3w-1)} = \begin{cases} \text{finite} & w > -\frac{1}{3} \\ \infty & w \le -\frac{1}{3}. \end{cases} \tag{195}$$

Hence the period with $w < -1/3$ could also solve the horizon problem. During this period the comoving horizon shrinks

$$\dot{\mathcal{H}} > 0 \Rightarrow \ddot{a} > 0 \tag{196}$$

so as alluded to above we need a period of accelerated expansion. This condition can be written as $-\dot{H}/H^2 < 1$. If $\dot{H} < 0$, it is called inflation.

The other possibility $\dot{H} > 0$, can occur, for instance, if we live in a bouncing cosmology that has a period of contraction in the past. During the bounce $\dot{H} > 0$. However, given that $|\Omega_K| \ll 1$ when the universe was small such a bouncing cosmology requires violating the Null Energy Condition (NEC), which is believed to hold in sensible classical theories.

1. Show that $\dot{H} > 0$ implies $\rho + p < 0$ in flat FRW.

Slow-roll Inflation

Inflation can be realized by a scalar field rolling down a potential, with action

$$S = \int dt d^3\boldsymbol{x}\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - V(\phi)\right]. \tag{197}$$

The stress-energy tensor is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right], \tag{198}$$

which on FRW background, and assuming homogeneity becomes diagonal $T_{\mu\nu} = \mathrm{diag}(\rho, pa^2, pa^2, pa^2)$ with

$$\rho = \frac{1}{2}\dot{\phi}^2 + V, \qquad p = \frac{1}{2}\dot{\phi}^2 - V. \tag{199}$$

Suppose V has a minimum at ϕ_0, with $V'(\phi_0) = 0$ and $V(\phi_0) > 0$. If ϕ is stuck at this minimum we have $w = p/\rho = -1$, the minimum allowed value by NEC. It is equivalent to having CC $\Lambda = V(\phi_0)$. However, this model has no clock to end the inflation, so it cannot be connected to the subsequent phases of our cosmology.

On the other hand, by choosing the potential to be sufficiently flat (to be quantified below) we get very close to $w = -1$, and still have a sign-definite $\dot{\phi} \neq 0$, so that $\phi(t)$ plays the role of the clock. So let us assume

$$\dot{\phi}^2 \ll V \Rightarrow w = -1 + \frac{\dot{\phi}^2}{V}. \tag{200}$$

Under this assumption, the expansion rate is

$$H^2 \approx \frac{V}{3M_{\rm pl}^2}, \qquad M_{\rm pl}^2 = \frac{1}{8\pi G}. \tag{201}$$

The conditions for this *slow-roll* solution can be derived from the ϕ equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{202}$$

where prime on V means $d/d\phi$. To maintain (200), and have a nonzero, sign-definite $\dot{\phi}$ for a long period, we need to have

$$|\ddot{\phi}| \ll H|\dot{\phi}| \tag{203}$$

This fixes

$$\dot{\phi} \approx -\frac{V'}{3H}. \tag{204}$$

Therefore, (200) holds if

$$\epsilon \equiv \frac{M_{\rm pl}^2 V'^2}{2V^2} \ll 1. \tag{205}$$

On the other hand (204) and (201) imply

$$\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{M_{\rm pl}^2 V''}{V} + \frac{M_{\rm pl}^2 V'^2}{2V^2}. \tag{206}$$

Therefore, for (203) to hold, we also need

$$\eta \equiv \frac{M_{\rm pl}^2 V''}{V} \ll 1. \tag{207}$$

ϵ and η are called slow-roll parameters. We also see that $-\dot{H}/H^2 \approx \epsilon \ll 1$. Hence H is approximately constant and the scale factor grows exponentially $a \propto \exp(Ht)$.

2. Show that the slow-roll solution (204) is an attractor. Namely, show that any initial condition $(\phi_i, \dot{\phi}_i)$ will converge to (204) with a characteristic time-scale $1/H$.

Since $\dot{\phi} \neq 0$, we can imagine a model in which after some time ϕ reaches a region of $V(\phi)$ where the slow-roll conditions no longer hold. An appropriate coupling between ϕ and standard model degrees of freedom (e.g. $\phi^2|H|^2$) could then result in the transfer of the energy density in ϕ into SM and initiating the hot phase. This is called **reheating**. The "re" in reheating comes from the idea that perhaps the universe was in a hot (but inhomogenous) phase before inflation began.

3. Suppose $V = \frac{1}{2}m^2\phi^2$. Find the field values for which slow-roll conditions hold. Near the bottom of the potential $m \gg H$ and the conditions don't hold. Show that the approximate solution for the homogenous field is

$$\phi(t) \approx \frac{A}{a^{3/2}} \cos(\omega t + \alpha), \tag{208}$$

where A and α are constants. How does the energy density decay with a?

In order for inflation to solve the flatness problem, we would need an initial $\Omega_K^i = \mathcal{O}(1)$ to decrease by about 56 orders of magnitude by the end of inflation. Using $\Omega_K = -k/\mathcal{H}^2 = -k/a^2H^2$ and the fact that H is approximately constant, we need $(a_i/a_r)^2 \sim 10^{-56}$ or

$$N_e \equiv \log \frac{a_r}{a_i} \approx 60., \tag{209}$$

where N_e is called the **number of e-folds**.

The flatness problem is similarly solved, by bringing the entire observable universe in causal contact:

$$\chi(t_r, t_i) = \int_{t_i}^{t_r} \frac{dt}{a} \approx \frac{1}{\mathcal{H}_i}, \tag{210}$$

which can be made larger than $1/H_0$ by taking inflation long enough.

4. Alice and Bob are comoving observers at physical distance $L < 1/H$, at $t = 0$. Bob sends a message to Alice every minute. At what frequency does Alice receive messages? When was

the last message that she receives sent by Bob? Suppose inflation is much longer than Alice's lifetime and neglect ϵ corrections.

Commentary: Bob is said to fall behind Alice's cosmological horizon. Much like when he falls behind a black hole horizon, no communication is possible afterward.

Origin of Perturbations

Because of the accelerated expansion, inflation rapidly dilutes any preexisting classical perturbations. In other words, since the comoving horizon expands momentum modes go from being subhorizon to superhorizon during inflation. It is natural to expect that whatever initial condition led to inflation, it did not have fluctuations at arbitrarily large momenta. In terms of the physical frequencies, this is saying that we expect an observer who lives at time t_i sees $\phi_{\boldsymbol{k}}$ modes with frequencies above some $\omega_{\max}$ not to be excited:

$$\frac{k_{\text{excited}}}{a_i} < \omega_{\max}. \tag{211}$$

For small enough a_i (equivalent to long enough inflation), $1/k_{\text{excited}}$ is a comoving distance much longer than the entire observable universe.

This (arguably justified) assumption would then fix the initial condition for the shorter wavelength modes. They are in Minkowski vacuum except their physical wavelength is being slowly stretched by the expansion of the universe. This initial state fully determines what happens when the modes become superhorizon. In particular, the one-point functions vanish, e.g. $\langle\phi_{\boldsymbol{k}}\rangle = 0$, and the statistical nature of the fluctuations results from their quantum mechanical origin. *Cosmological perturbations originated from the zero-point fluctuations of the inflaton field ϕ during inflation.*

A full derivation of the spectrum of these fluctuations is a beautiful calculation, but beyond the scope of these lectures. It requires taking into account both metric and ϕ fluctuations. Here I will explain some of the features of the result, and estimate its size. Let us focus on the two-point correlation function

$$\langle\phi_{\boldsymbol{k}}(t)\phi_{\boldsymbol{k}'}(t)\rangle = P(t,k)(2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k}'), \tag{212}$$

where I used the translation invariance and isotropy of the background. If t is such that $k/aH \ll 1$, that is, k is superhorizon, then we expect the fluctuations to be frozen up to slow-roll corrections. Because ϵ, η determine how the potential deviates from CC. So in this limit $P(t,k) \to P(k)$. Neglecting the slow-roll corrections the background metric looks like

$$ds^2 = -dt^2 + e^{2Ht}d\boldsymbol{x}^2. \tag{213}$$

This means that a rescaling of x can be compensated by a shift in time. But at superhorizon scales, there is no time-dependence. As a result the correlation function $\langle\phi(t,\boldsymbol{x})\phi(t,0)\rangle$ should be

$\boldsymbol{x}$-independent, up to slow-roll corrections. This implies that

$$P(k) = \frac{A}{k^{3+\mathcal{O}(\epsilon,\eta)}}, \tag{214}$$

Finally, by dimensional analysis

$$A \propto H^2. \tag{215}$$

Intuitively, this says that from the point of view of an observer who lives during inflation the field ϕ fluctuates with a typical amplitude H. In fact, this observer has a horizon and measures a finite temperature $T = H/2\pi$, and the fluctuations we are talking about are analogous to the Hawking radiation from a black hole.

Finally, we can use the fact that ϕ is effectively the clock that determines when inflation ends. We have seen that this clock is fluctuating over the space, so inflation would end at different times at different points. We can translate this into the metric by performing a compensating time shift $\Delta t = -\Delta\phi/\dot{\phi}$ to obtain

$$ds^2 \approx -dt^2 + e^{2Ht+2\zeta} d\boldsymbol{x}^2, \tag{216}$$

where

$$\zeta = -\frac{H\Delta\phi}{\dot{\phi}} = \mathcal{O}(H/\sqrt{\epsilon}M_{\mathrm{pl}}). \tag{217}$$

These fluctuations are indeed adiabatic because they correspond to identical histories but shifted with respect to one another at different locations.

5. Consider the average field over a ball of radius R

$$\bar{\phi}(t) = \frac{3}{4\pi^2 R^3} \int_{|x|<R} d^3\boldsymbol{x}\, \phi(t, \boldsymbol{x}). \tag{218}$$

   Use (79) to show that in Minkowski vacuum

$$\left\langle \bar{\phi}(t)^2 \right\rangle = \frac{9}{16\pi^2 R^2}. \tag{219}$$

   Next consider the spectrum (214) for $k > k_i$ and $P(k) = 0$ for $k < k_i$ and estimate $\left\langle \bar{\phi}^2 \right\rangle$ as a function of k_i and R.