Lectures on General Relativity
I. Introduction

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Abstract

General Relativity is introduced as a nonlinear theory of interacting massless spin-2 gravitons. Unlike most textbooks I start from relativistic quantum field theory in flat spacetime and attempt at constructing a relativistic version of Newtonian gravity. First, I discuss linearized spin-0 gravity and its phenomenology at some length. Motivated by the observational failure of spin-0 gravity I discuss linearized spin-2 gravity, and gravitational waves. Then, I discuss the inconsistency of linearized spin-2 gravity and introduce General Relativity as the unique nonlinear completion of the theory. Finally, I discuss geometry and connect with the common track.

Contents

1 Poincaré Transformations, Causal Structure, Group Theory

Reading: *Weinberg GR 2.1, 2.2*

Euclidean Geometry. Many concepts in Special Relativity are simple generalizations of more familiar concepts in Euclidean Geometry. Consider two points on a plane: $\boldsymbol{r}_A = (x_A, y_A)$, $\boldsymbol{r}_B = (x_B, y_B)$. The Euclidean distance squared of the two points is

$$I_{AB} = |\boldsymbol{r}_A - \boldsymbol{r}_B|^2 = (x_A - x_B)^2 + (y_A - y_B)^2. \tag{1.1}$$

One can ask what are the transformations of the 2d plane that leave this distance invariant. **Global Translations** clearly satisfy this property:

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a} \tag{1.2}$$

with $\boldsymbol{a}$ a constant, because it cancels from the difference $\boldsymbol{r}_A - \boldsymbol{r}_B$.

The only other transformations that preserves the Euclidean distance are **rotations**:

$$x' = x\cos\theta - y\sin\theta, \qquad y' = y\cos\theta + x\sin\theta, \tag{1.3}$$

where $\theta =$ constant.

It is very convenient during the whole course to adopt matrix notation. So the Euclidean vectors (and tensors) carry indices $i, j, \cdots$ which run over $1, 2, \cdots, d$ where d is the dimension. For instance, in a plane we denote

$$r^1 = x, \qquad r^2 = y \tag{1.4}$$

in terms of which the interval can be written as

$$I_{AB} = \sum_i (r_A - r_B)^i (r_A - r_B)^i = (r_A - r_B)^i (r_A - r_B)^j \delta_{ij} \tag{1.5}$$

and the **Kronecker delta** is defined as $\delta_{ij} = 1$ if $i = j$ and $= 0$ otherwise. Repeated indices are summed over: in two dimensional Euclidean space

$$(r')^i = R^i_j r^j = R^i_1 r^1 + R^i_2 r^2. \tag{1.6}$$

In this notation a 2d rotation is a 2×2 matrix:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.7}$$

with components R^i_j. Now we see that the characteristic property of rotations can be stated in terms of rotation matrices:

$$\delta_{ij} R^i_m R^j_n = \delta_{mn}. \tag{1.8}$$

Rotations form a group: There is a multiplication rule that combines two rotations into another rotation ${R_3}^i_j = {R_1}^i_k {R_2}^k_j$. There is a unit element $R^i_j = \delta^i_j$. And for every rotation there is an inverse rotation. The rotation group is a continuous group; we can continuously increase the rotation angle from 0. Continuous groups are characterized by their dimension. The group of $2d$, and $3d$ rotations are respectively 1- and 3-dimensional: there is only one type of rotation on a $2d$ plane and there are 3 in three dimensions.

1. How many independent rotations are there in d Euclidean dimensions? Hint: Consider an infinitesimal rotation

$$R^i_j = \delta^i_j + \epsilon^i_j, \qquad \epsilon^i_j \ll 1, \tag{1.9}$$

   and determine what condition equation (1.8) implies on components of ϵ^i_j. Then count the number of independent components.

Poincaré transformations. Special relativity puts time and space on a more equal footing. In order to have an invariant velocity c (the speed of light) one can find a direct generalization of Euclidean translations and rotations:

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu. \tag{1.10}$$

In 3+1 spacetime dimensions Greek indices run over $0, 1, 2, 3$, and as before the repeated indices are summed over. Translations are parametrized with the arbitrary constant vector a^μ. The **Lorentz transformation** matrix Λ^μ_ν is the generalization of Rotation such that the interval

$$I_{AB} \equiv -\eta_{\mu\nu}(x_A - x_B)^\mu (x_A - x_B)^\nu \tag{1.11}$$

between two spacetime events A and B is preserved. An **Event** is a point in space-time, identified by $x^\mu = (x^0, x^1, x^2, x^3)$. Setting $c = 1$, the Minkowski metric becomes $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Since it is sign-indefinite, I_{AB} can have either sign. We can introduce a causal structure: two events can be time-like separated ($I_{AB} > 0$), space-like separated ($I_{AB} < 0$), or null separated ($I_{AB} = 0$). (Mark them on a space-time diagram!)

2. Two events are separated by a time-like interval. Find a boost as a function of $x_A - x_B$ to the frame in which the two events happen at the same spatial point ${x'_A}^i = {x'_B}^i$. Is there a frame where the two events are simultaneous, ${x'_A}^0 = {x'_B}^0$?

   Reminder: A boost is fully determined in terms of a velocity 3-vector $\boldsymbol{v}$. (Bold symbols stand for 3-vectors $\boldsymbol{v} = (v^1, v^2, v^3)$.) A boost along x direction acts as follows (using the notation

$t = x^0$, $x = x^1$, $y = x^2$, $z = x^3$)

$$x' = \gamma(x - vt), \qquad t' = \gamma(t - vx), \qquad y' = y, \qquad z' = z, \tag{1.12}$$

where the boost factor γ is defined as

$$\gamma = \frac{1}{\sqrt{1 - v^2}}. \tag{1.13}$$

Those events that are time-like separated are called **causally connected** since they can influence one another in a Lorentz invariant theory. The space-like separated events are in turn causally disconnected. Moreover, depending on the sign of x^0_{AB} two causally connected events A and B have a time-order. If $x^0_{AB} > 0$ (< 0) A is to the future (past) of B. This order is preserved by proper Lorentz transformations. Therefore, in a Lorentz invariant theory boosted observers agree on the initial value formulation of physics.

3. Alice, Bob and Charlie meet for breakfast at the bar. Afterwards, Bob sees Alice and Charlie moving away from him along the same straight line in opposite directions, with constant velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_C$ respectively.

   (a) **Worldline** is the trajectory of a timelike observer on a spacetime diagram. It consists of a continuum of events and can be drawn once $x^i(t)$ is known. Draw wordlines of Alice, Bob, and Charlie, assuming that they all meet at $t = 0$ according to the bar's clock.

   (b) Any continuum of points in spacetime is called a **Curve**. The **proper length** of the curve is defined by breaking it into infinitesimal straight intervals and summing up the Poincaré invariant length $d\tau$ of the infinitesimal intervals, given by

   $$d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu. \tag{1.14}$$

   This is positive for the infinitesimal elements of a timelike curve, or a worldline. The proper length of a worldline is called **proper time**. Calculate τ_A, τ_B, τ_C as functions of t.

   (c) Write the components of Alice's **4-velocity** in Bob's frame, i.e. $u_A = dx^\mu_A/d\tau_A$. Draw u^μ_A on the spacetime diagram. What is $\eta_{\mu\nu} u^\mu_A u^\nu_A$? What is $\eta_{\mu\nu} u^\mu_A u^\nu_B$?

   (d) Find Charlie's velocity in Alice's frame.

4. * Repeat the same exercise as (1) for Lorentz transformations.

5. * $3d$ rotations form a subgroup of proper Lorentz transformations in $3 + 1d$ (i.e. the combination of two rotations is another rotation, and every rotation has an inverse rotation). Any proper Lorentz transformation (one with $\Lambda^0_0 \geq 1$ and $\det \Lambda = 1$) can be uniquely decomposed

in terms of a pure boost and a rotation:

$$\Lambda^{\mu}_{\nu} = B^{\mu}_{\alpha} R^{\alpha}_{\nu}, \tag{1.15}$$

where $R^0_0 = 1$, $R^0_i = R^i_0 = 0$ and R^i_j is a $3d$ rotation. A pure boost B that takes an object from rest to velocity $\boldsymbol{v}$ is given by

$$\begin{aligned} B^0_0(\boldsymbol{v}) =& \gamma = \frac{1}{\sqrt{1-\boldsymbol{v}^2}}, \\ B^i_0(\boldsymbol{v}) =& B^0_i(\boldsymbol{v}) = \gamma v^i, \\ B^i_j(\boldsymbol{v}) =& \delta_{ij} + v_i v_j \frac{\gamma - 1}{\boldsymbol{v}^2}. \end{aligned} \tag{1.16}$$

Determine v and the matrix R in terms of the components of Λ.

6. * Two observers move in opposite directions on a circle of radius R with constant angular velocities ω_1 and ω_2. When they first meet, they synchronize their clocks. When they meet again, whose clock will be delayed, and by how much?

2 Relativistic Dynamics, Constant Acceleration

Reading: *Weinberg GR 2.3, 2.4, 2.5*

The main principle of special relativity is that the laws of physics have to look the same in all inertial frames (or to all inertial observers), which are related by Poincaré transformations. Therefore, they have to relate Lorentz scalars to Lorentz scalars, Lorentz vectors to Lorentz vectors and so on. These relations are called **covariant**. Moreover, in the appropriate limit they have to reproduce the laws of non-relativistic dynamics.

In particular, **energy and momentum conservation** which hold in non-relativistic processes, take the covariant form

$$\sum_{\text{initial}} p_i^\mu = \sum_{\text{final}} p_f^\mu, \tag{2.1}$$

where the momentum 4-vector of a particle with rest-mass m is defined as

$$p^\mu = m u^\mu = m\frac{dx^\mu}{d\tau}. \tag{2.2}$$

Since dx^μ is a four-vector and $d\tau$ invariant under Lorentz transformations, (2.1) is covariant. Moreover, in the non-relativistic limit

$$p^i \to m v^i, \qquad p^0 \to m + \frac{1}{2} m v^2, \qquad \text{non-relativistic} \tag{2.3}$$

so (2.1) reduces to non-relativistic energy-momentum conservation. Note that the initial and final particles are always the same in non-relativistic processes because the kinetic energies are too small to have particle creation or annihilation.

Force four-vector is defined via

$$m\frac{d^2x^\mu}{d\tau^2} = f^\mu. \tag{2.4}$$

Since $d^2x^\mu/d\tau^2$ is a four-vector, the above equation is covariant, namely it holds in all inertial frames, if and only if f^μ is a four-vector.

Newtonian dynamics is extremely successful at low velocities. To maintain this, we demand that in the rest frame (RF)

$$m\frac{d^2x^i}{dt^2} = F_{\text{Newton}}, \qquad \text{Rest Frame.} \tag{2.5}$$

Next use $d\tau = dt/\gamma$ to write

$$\frac{d^2x^i}{dt^2} = \gamma^{-1}\frac{d}{d\tau}\left(\gamma^{-1}\frac{dx^i}{d\tau}\right) = \gamma^{-2}\frac{d^2x^i}{d\tau^2} - v^j\frac{dv^j}{d\tau}\frac{dx^i}{d\tau}. \tag{2.6}$$

In the rest frame $\gamma = 1$ and $v^j = 0$, therefore

$$f^i_{\text{RF}} = F^i_{\text{Newton}}. \tag{2.7}$$

1. What is f^0_{RF}? Hint: use the fact that $u^\mu = dx^\mu/d\tau$ has a unit norm $u_\mu u^\mu = -1$.

2. **Rindler Observer** Bob starts at rest at the origin in frame F. He then moves along the x-axis with constant acceleration a. By "acceleration" in special relativity I mean: go to the instantaneous rest frame of Bob, F', in which Bob's velocity is $v' = dx'/dt' = 0$, where x' and t' are the coordinates in frame F'. Then in frame F' Bob has acceleration $dv'/dt' = a$. Find the equation of motion of Bob $x(t)$ in frame F.

3. **Rindler Horizon** Alice stays at the origin in frame F and sends text-messages to Bob. After a time t the messages will never reach Bob. Calculate t.

   FYI: Rindler horizon provides an extremely useful and simple setup to understand lots of properties (both classical and quantum mechanical) of black hole horizons in general relativity. From Bob's point of view Alice falling behind the Rindler horizon is not much different from Alice jumping inside a black hole of radius $R = 1/(2a)$. Fortunately, unlike black holes there is no singularity hiding behind the Rindler horizon.

4. Calculate the redshift of the signals as a function of τ.

5. * Now suppose that Alice and Bob are initially at rest, respectively at $x = X_A$ and $x = X_B$ with $X_A > X_B > 0$. At $t = 0$ they launch their spaceships with constant proper accelerations $1/X_A$ and $1/X_B$. Bob is sending a message to Alice every one minute. What is the frequency at which Alice receives the messages?

3 Electromagnetism, Variation Principle

Reading: *Weinberg GR 2.6, 2.7*

1. **Axial Gauge.** Maxwell theory can be formulated in terms of $\boldsymbol{E}, \boldsymbol{B}$, or equivalently, in terms of $F_{\mu\nu}$. Because $\varepsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta} = 0$, one can introduce a **gauge field** A_μ and give an action formulation of electromagnetism

$$S[A_\mu] = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 + A_\mu J^\mu \right], \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.1}$$

   Show that varying this action with respect to A_μ gives $\partial_\nu F^{\nu\mu} = -J^\mu$.

   The formulation in terms of A_μ has gauge redundancy:

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \alpha \Longrightarrow F_{\mu\nu} \to \tilde{F}_{\mu\nu} = F_{\mu\nu}. \tag{3.2}$$

   Use the gauge redundancy (find α) to set $\tilde{A}_3 = 0$.

2. **Pure Gauge Configuration.** If $F_{\mu\nu} = 0$ everywhere, then the gauge field is a pure gauge $A_\mu = \partial_\mu \alpha$. Find α.

   Hint: It is useful to remember how in Newtonian physics potential energy is defined for a curl-free static force field $\boldsymbol{F}(\boldsymbol{x})$ – not to be confused with $F_{\mu\nu}$:

$$(\nabla \times \boldsymbol{F})^i = \varepsilon^{ijk}\partial_j F_k = 0 \Longrightarrow \boldsymbol{F} = -\nabla\phi. \tag{3.3}$$

   This definition only fixes the potential difference between two points:

$$\phi(\boldsymbol{x}) = \phi(\boldsymbol{0}) - \oint_{\boldsymbol{0}}^{\boldsymbol{x}} d\boldsymbol{x}' \cdot \boldsymbol{F}(\boldsymbol{x}'). \tag{3.4}$$

   The integral is defined along a path C connecting the two points $\boldsymbol{0}$ and $\boldsymbol{x}$. For $\phi(\boldsymbol{x})$ to be well-defined, the answer must be independent of the path. Equivalently, we must have

$$\oint d\boldsymbol{x} \cdot \boldsymbol{F}(\boldsymbol{x}) = 0, \qquad \text{for all closed loops.} \tag{3.5}$$

   This is related to $\nabla \times \boldsymbol{F}$ by the Stoke's theorem and is guaranteed by the condition $\nabla \times \boldsymbol{F} = 0$.

Aharonov-Bohm Effect. If the spacetime region with $F_{\mu\nu} = 0$ has a nontrivial topology the above construction can fail. Imagine there is a solenoid with a magnetic flux $\Phi \neq 0$. Then

$$\oint dx^\mu A_\mu = \Phi, \qquad \text{if the loop goes around the solenoid.} \tag{3.6}$$

Here A_μ is not a pure gauge configuration even outside of solenoid, even though $F_{\mu\nu} = 0$ outside. Of course this is not a surprise since the condition for A_μ being a pure gauge was that $F_{\mu\nu} = 0$ everywhere.

A more interesting situation arises if we cut and throw away the part of the spacetime that is occupied by the solenoid. Now we have a spacetime with a boundary. The original solution for A_μ is still a perfectly valid solution of Maxwell equations, and now $F_{\mu\nu} = 0$ everywhere since the solenoid is not part of the spacetime anymore. Yet clearly A_μ is not a pure gauge.

3. **Point-Particle Action** Find the charged particle equation of motion by varying the following action

$$S[X, A] = m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} - q \int d\sigma \frac{dX^\mu}{d\sigma} A_\mu(X). \tag{3.7}$$

4. **Charged Particle.** Adding the Maxwell action $S = -\frac{1}{4} \int d^4x F_{\mu\nu}^2$ and varying with respect to A_μ gives the Maxwell equations. What is the electric current corresponding to a point particle with charge q? Hint: To vary with respect to A_μ, the coupling of A_μ to the worldline in (3.7) has to be expressed in terms of an action defined as the integral of a Lagrangian density over the spacetime d^4x. This can be done using a Dirac delta function $\delta^4(x^\mu - X^\mu(\sigma))$. Use the delta function and the fact that for any f and g

$$\int d\sigma g(\sigma) \delta(f(\sigma)) = \sum_n \frac{g(\sigma_n)}{|df(\sigma_n)/d\sigma|}, \quad f(\sigma_n) = 0 \tag{3.8}$$

to perform the $d\sigma$ integral.

5. * **Wordline Reparametrization.** Check that the point particle action (3.7) is invariant under worldline reparametrization

$$\sigma \to \sigma' = f(\sigma) \tag{3.9}$$

where f is a differentiable and monotonic, but otherwise arbitrary function.

6. * Derive the equations of motion following from the action

$$S_{\text{Polyakov}} = \frac{1}{2} \int d\sigma e^{-1} l \dot{X}^\mu \dot{X}_\mu - e^2 m^2 \boldsymbol{r}, \qquad \dot{X}^\mu = \frac{dX^\mu}{d\sigma},$$

where e (which is called the **worldline metric**) is an additional dynamical variable. Are these equations equivalent to the equation of motion for a point-particle? Note that in this action (unlike the Nambu-Goto action), one can set $m = 0$. Does it reproduce what you expect for a massless particle? What is the transformation of e under reparametrizations of σ?

7. * In addition to the physical time translation invariance $X^0 \to X^0 + \epsilon$ the action for a point-like particle is obviously invariant also under the worldline time translation (a special case of worldline reparametrization),

$$X^\mu(\sigma) \to X^\mu(\sigma + \epsilon) \ .$$

Normally, by Noether theorem one expects to obtain the conserved "worldline energy", corresponding to this symmetry. Calculate this energy.

8. * **Differential Forms.** Maxwell theory can be elegantly formulated in terms of differential forms. A rank-r form $\omega_{\mu_1\cdots\mu_r}$ is a fully anti-symmetric covariant tensor. What is the highest rank form in d dimensions?

Exterior derivative is a way of constructing a higher rank form from a lower-rank form field $\omega_{\mu_1\cdots\mu_r}(x)$:

$$(d\omega)_{\mu_1\cdots\mu_{r+1}} = \partial_{[\mu_1}\omega_{\mu_2\cdots\mu_{r+1}]} \tag{3.10}$$

where [] denotes full antisymmetrization, e.g. $A_{[\mu\nu]} = (A_{\mu\nu} - A_{\nu\mu})$. $d\omega$ is called an **exact form**. A form Ω is called a **closed form** if $d\Omega = 0$. Show that all exact forms are closed: $dd\omega = 0$ for all ω.

Hodge Dual of a rank-r form is a rank-$(d-r)$ form:

$$(\star\omega)_{\mu_1\cdots\mu_{d-r}} = \varepsilon_{\mu_1\cdots\mu_{d-r}}{}^{\nu_1\cdots\nu_r}\ \omega_{\nu_1\ldots\nu_r}. \tag{3.11}$$

Find $\star\star\,\omega$.

Show that the Maxwell equations can be written as

$$d\star F = \star J, \qquad dF = 0. \tag{3.12}$$

9. * The identity $dF = 0$ implies that (locally) there exists a one-form field A such that $F = dA$ — in components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Prove this by giving an explicit integral formula for A_μ in terms of $F_{\mu\nu}$. Is your answer unique? (Hint: Recall how potential energy is defined for a curl-free static force field.)

10. * **Cluster Decomposition.** The first term in the point particle action (3.7) is the Lorentzian length of the particle trajectory $m\tau[X^\mu(\sigma)]$ (square brackets emphasize the fact that τ is a function of functions $X^\mu(\sigma)$, i.e. it is a *functional*). Suppose we replace this term by a nonlinear function of the length, such as $(\tau[X^\mu(\sigma)])^2$. Show that for a free particle the new action is extremized by a straight trajectory.

However, in the presence of a force (such as the EM force) the motion of the particle changes in a funny way. The acceleration will depend on the full history. This theory breaks cluster decomposition principle, namely the result of experiments performed at far separated points depend on one another.

4 Noether Theorem, Symmetries and Conservation Laws

Reading: *Weinberg QFT I, 7.3, 7.4*

Consider an action

$$S[\phi] = \int d^4x \mathcal{L} \tag{4.1}$$

where ϕ is a general notation for all dynamical variables. A symmetry transformation is a transformation

$$\phi \to \phi + \delta_s \phi \tag{4.2}$$

under which the variation of S is a boundary term:

$$\delta S[\phi + \delta_s \phi] = \int d^4x \partial_\mu \Lambda^\mu. \tag{4.3}$$

1. Consider the action of a free, massless scalar field

$$S = -\frac{1}{2} \int d^4x (\partial\phi)^2, \qquad (\partial\phi)^2 \equiv \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{4.4}$$

a It is invariant under a constant shift $\phi \to \phi + c$, with $\Lambda^\mu = 0$.

b Under a linear shift $\delta_s \phi = b_\mu x^\mu$, the action changes as

$$\delta S = -\int d^4x \partial_\mu (b^\mu \phi). \tag{4.5}$$

c Under a spacetime **translation** $x^\mu \to x^\mu + a^\mu$ the fields change as $\phi \to \phi + a^\mu \partial_\mu \phi$, and the action changes

$$\delta S = -\int d^4x \partial_\mu \phi a^\nu \partial_\mu \partial_\nu \phi = -\frac{1}{2} \int d^4x \partial_\mu (a^\mu (\partial\phi)^2). \tag{4.6}$$

2. Consider the worldline action $S_{PP} = \int d\sigma \sqrt{\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}}$. It is invariant under both translations and Lorentz transformations

$$X^\mu \to X^\mu + a^\mu + \omega^\mu_\nu X^\nu, \qquad \text{with} \quad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{4.7}$$

From the symmetries we can derive **conservation laws** as follows. Multiply the infinitesimal symmetry transformation by a function $\epsilon(x)$ with finite support around a point x_0:

$$\phi \to \phi + \epsilon(x) \delta_s \phi. \tag{4.8}$$

Because $\epsilon(x \to \infty) = 0$ the boundary term in the variation of the action vanishes. Moreover, the variation would be zero if $\partial_\mu \epsilon = 0$ so one obtains

$$\delta S[\phi + \epsilon(x)\delta_s \phi] = \int d^4x \partial_\mu \epsilon \ J^\mu = -\int d^4x \epsilon \partial_\mu J^\mu, \tag{4.9}$$

for some vector J^μ which depends on ϕ and its derivatives. In the last step I integrated by parts and used the asymptotic behavior $\epsilon(x \to \infty) = 0$. On-shell, the action is stationary. In particular, its variation under (4.8) must vanish. Since $\epsilon(x)$ is arbitrary, we must have

$$\partial_\mu J^\mu|_{\text{on-shell}} = 0. \tag{4.10}$$

Thus J^μ is a conserved current once the equations of motion are satisfied.

3. Consider again the free massless scalar field.

a Promoting the shift symmetry of the massless scalar field to a spacetime varying function[1]

$$\phi \to \phi + c(x) \tag{4.11}$$

leads to

$$\delta S = -\int d^4x \partial_\mu c(x) \partial^\mu \phi. \tag{4.12}$$

Therefore $J_c^\mu = -\partial^\mu \phi$. It is clearly conserved on-shell

$$\partial_\mu J_c^\mu = -\Box \phi = 0. \tag{4.13}$$

b * Find the conserved current J_b^μ corresponding to the linear shift, and verify its conservation.

c **Stress-Energy Tensor.** Consider spacetime dependent translations

$$\phi \to \phi + a^\mu(x)\partial_\mu \phi. \tag{4.14}$$

where $a^\mu(x \to \infty) = 0$. The change of the action after some partial integrations can be written as

$$\delta S = -\int d^4x \partial_\mu a^\nu(x)[\partial^\mu \phi \partial_\nu \phi - \delta^\mu_\nu \frac{1}{2}(\partial\phi)^2]. \tag{4.15}$$

There is one conserved current for each component of a^ν. They are collected in a single rank-2 tensor called stress-energy tensor T^μ_ν. $\partial_\mu T^\mu_\nu$ can be easily seen to be proportional to $\Box\phi$.

d * For a massive field show that

$$T^\mu_\nu = \partial^\mu \phi \partial_\nu \phi + \delta^\mu_\nu(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2) \tag{4.16}$$

[1]To avoid clutter, I often combine the arbitrary function $\epsilon(x)$ with the parameter of transformation – the constant c in this example – to define a single spacetime dependent parameter $c(x) = \epsilon(x)c$.

and that it is conserved on-shell.

4. The translation symmetry of the worldline action leads to a one-dimensional conservation law: Take
$$X^\mu \to X^\mu + a^\mu(\sigma) \tag{4.17}$$
where $a^\mu(\sigma)$ has finite support. The change in the action is
$$\delta S_{PP} = \int d\sigma \frac{da^\mu}{d\sigma} P_\mu, \qquad P_\mu = m\frac{dX_\mu}{d\tau}. \tag{4.18}$$
This implies that $dP_\mu/d\sigma = 0$.

From a conserved current one can derive a **conserved charge**. Let's define
$$Q(t) \equiv \int d^3\boldsymbol{x} J^0(t, \boldsymbol{x}). \tag{4.19}$$
Unless there is a nonzero flux at spatial infinity $Q(t)$ is time-independent, because
$$\frac{d}{dt}Q(t) = \int d^3\boldsymbol{x}\partial_0 J^0(t, \boldsymbol{x}) = \int d^3\boldsymbol{x}\partial_i J^i(t, \boldsymbol{x}) = \oint_{|\boldsymbol{x}|\to\infty} d\Sigma^i J^i. \tag{4.20}$$
In canonical formalism Q is the generator of symmetry transformation (see *Weinberg*).

5. * The above conservation of Q is a special case of a more general property. Show that $\partial_\mu J^\mu = 0$ is equivalent to
$$d \star J = 0, \tag{4.21}$$
where J stands for the 1-form J_μ. Integrating this equation over any spacetime region R gives
$$0 = \int_R d \star J = \int_{\partial R} \star J \tag{4.22}$$
where ∂R denotes the boundary of R and I used the Stoke's theorem.

6. * **Nambu-Goto Action.** Analog of the worldline action for a string (or more generally an n-dimensional brane, with point-particle and string corresponding, respectively, to $n = 0$ and $n = 1$) is the Nabmu-Goto action. For a string, $n = 1$, it is
$$S = -\ell_s^{-2} \int d^2\sigma \sqrt{-\det h_{\alpha\beta}}\ ,$$
ℓ_s^{-2} is the string tension, α, β are worldsheet indices which run over $0, 1$, and the *induced metric* of the worldsheet is defined as
$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \qquad \mu, \nu = 0, \cdots, d.$$

Check that the Nambu–Goto action is invariant under worldsheet reparametrizations,

$$X^\mu(\sigma^\alpha) \to X^\mu(f^\alpha(\sigma)) \; .$$

5 Representations of Poincaré Group and Spin

Reading: *Weinberg QFT vol I: 2.4, 2.5*

Observable quantities furnish representations of symmetries of nature. Under a symmetry transformation, some observables are invariant, like proper time under Lorentz transformations, others transform in a prescribed way, like energy and momentum which form a Lorentz vector. In quantum mechanics the wavefunction Ψ determines the outcome of observation and must transform accordingly $\Psi \to U\Psi$. Two wavefunctions that differ by a constant phase factor are identified in quantum mechanics,

$$\Psi \sim e^{i\phi}\Psi. \tag{5.1}$$

This implies that under composition of any two $g_1, g_2 \in G$, the symmetry group, U satisfies

$$U(g_1)U(g_2) = e^{i\phi(g_1,g_2)}U(g_1 g_2). \tag{5.2}$$

Hence U's form a representation of G. If the phase $\phi(g_1, g_2)$ cannot be set to zero by a redefinition of U this is called a projective representation of the group. The probabilities must remain invariant if all state vectors are transformed together. This implies that U is unitary and linear

$$(U\Phi, U\Psi) = (\Phi, \Psi), \qquad U(a\Phi + b\Psi) = aU\Phi + bU\Psi, \tag{5.3}$$

or else antiunitary and antilinear. We don't need to worry about the latter case; see *Weinberg 2.2* for further discussion.

We are specially interested in continuous groups (or **Lie groups**), like rotation, where the group elements can be parametrized by a set of real numbers $\{\theta_a\}$, with $\theta_a = 0$ corresponding to the identity element. The unitary operators can be expanded around $\mathbf{1}$:

$$U(\{\theta_a\}) = \mathbf{1} + i\theta_a t_a + \mathcal{O}(\theta^2), \tag{5.4}$$

where t_a are called the generators of the group.[2] For U to be unitary, t_a must be Hermitian: $t_a^\dagger = t_a$. Moreover, the composition rules of the symmetry group implies certain commutation relations on t_a. Perhaps the most familiar example is the commutation relations between generators of rotations, i.e. the angular momentum algebra:

$$[L_i, L_j] = i\epsilon^{ijk}L_k, \tag{5.5}$$

where t_a are replaced by the angular momentum operators L_i.

1. Consider a small rotation θ_1 along x^1 followed by a small rotation θ_2 along x^2, followed by

[2] As usual repeated indices are summed over.

$-\theta_1$, followed by $-\theta_2$. Work at first order in θ_1 and first order in θ_2 (this includes $\mathcal{O}(\theta_1\theta_2)$). Verify (5.5) by demanding that the combination of four U's reproduce the net rotation.

The commutation relation of the generators $\{t_a\}$ is called the **Lie algebra** of the group. Applying a similar argument to the unitary representation of Poincaré group,

$$U(\omega,\epsilon) = 1 + \frac{1}{2} i\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu + \cdots \tag{5.6}$$

gives the Poincaré algebra

$$\begin{aligned} i[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho} - \eta^{\mu\rho}J^{\nu\sigma}, & (5.7)\\ i[P^\mu, J^{\rho\sigma}] &= \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho & (5.8)\\ i[P^\mu, P^\nu] &= 0. & (5.9) \end{aligned}$$

The generators of spacetime translations are momentum operators P^μ. The generators of proper Lorentz transformations $J^{\mu\nu}$ comprise boosts $K^i = J^{i0}$ and angular-momenta $J_i = \epsilon_{ijk}J^{jk}$.

2. * Why are the conserved charges associated to a symmetry identified with the generators of those transformations? Hint: Repeat the derivation of the Noether theorem in the path integral formalism and with the insertion of a local operator $O(y)$. Derive the Ward identity:

$$i\partial_\mu T\{J^\mu(x)O(y)\} = \delta^4(x-y)\delta O(y), \tag{5.10}$$

where T denotes time-order product, and δO is the variation of O under the symmetry transformation. Integrate this equation over a thin slab of spacetime that contains y and extends to spatial infinity.

As in the case of rotations in non-relativistic quantum mechanics (QM), we need to find irreducible unitary representations of the symmetry group and their dimensionality to identify different particle species. In the same sense that there is no intrinsic difference between spin up and spin down electrons (they can be transformed into one another by a rotation), the irreducible representation of Poincaré group is a set of states that can be transformed into one another by a combination of translations, boosts, and rotations.

Of course, unless $P^\mu = 0$ (which corresponds to vacuum) these representations are infinite dimensional. Lorentz boosts span the entire mass-shell (or light cone) keeping P^2 fixed. As in non-relativistic quantum mechanics we can use the fact that momentum operators commute with the Hamiltonian $H = P^0$ to diagonalize them simultaneously:

$$P^\mu\Psi_{p,\sigma} = p^\mu\Psi_{p,\sigma} \tag{5.11}$$

with σ standing for all other quantum numbers, e.g. the analog of spin up and spin down states of electron in non-relativistic QM. If we have a state with multiple particles then p^μ is the center of mass momentum, and σ would characterize, among other things, the relative momenta of those particles. Hence it would be a continuous label. Our goal here is to classify single particle states and study their interactions perturbatively. So we ignore such a possibility, and take σ to be discrete.

Having diagonalized momentum operators, Lorentz transformations Λ^μ_ν have to be taken care of next. Apply a general Lorentz transformation Λ^μ_ν to $\Psi_{p,\sigma}$. The new state $U(\Lambda)\Psi_{p,\sigma}$ is still an eigenstate of P^μ with eigenvalue $p'^\mu = \Lambda^\mu_\nu p^\nu$, because using the Poincaré algebra (check)

$$P^\mu U(\Lambda)\Psi_{p,\sigma} = U(\Lambda)\Lambda^\mu_\nu P^\nu \Psi_{p,\sigma} = \Lambda^\mu_\nu p^\nu U(\Lambda)\Psi_{p,\sigma}. \tag{5.12}$$

Therefore, we can write

$$U(\Lambda)\Psi_{p,\sigma} = \sum_{\sigma'} D_{\sigma\sigma'}(\Lambda, p)\Psi_{\Lambda p,\sigma'}. \tag{5.13}$$

The matrix $D_{\sigma\sigma'}$ has a block-diagonal structure with each block corresponding to an irreducible representation. Note that in non-relativistic QM the problem of finding irreducible representations of the group of Galilean transformations has a relatively simple answer. Since Galilean boosts commute, any decomposition of Λ into a boost and a rotation gives a unique answer R^i_j for the rotation component. The matrix $D_{\sigma\sigma'}$ will depend only on R^i_j, which is an element of $SO(3)$ and whose irreducible representations are particles of definite spin.

In contrast, Lorentz boosts commute into rotations (**Thomas precession**) so we need to work harder to find $D_{\sigma\sigma'}$. However, if one is only interested in classifying spin degeneracy of massive particles (and not the explicit transformation law) the answer is the same as before. This is because at small velocities we have to recover Galilean transformations and the dimensions of irreducible representations are integer numbers, which cannot jump to a different value by continuously changing the velocity. The novel feature of the relativistic theory is the existence of massless representations.

Nevertheless, I treat both cases. The trick is to choose a reference momentum k^μ, and define arbitrary momentum states by applying a standard Lorentz transformation $L^\mu_\nu(p)$ to the reference state:

$$\Psi_{p,\sigma} \equiv N(p)U(L(p))\Psi_{k,\sigma}, \tag{5.14}$$

where $N(p)$ is a normalization factor. The action of an arbitrary Lorentz transformation $U(\Lambda)$ on $\Psi_{p,\sigma}$, which takes it to a superposition of states with momentum $\Lambda p = \Lambda L(p)k$, can be decomposed into a Lorentz transformation W^μ_ν which leaves k^μ invariant, followed by $L(\Lambda p)$. More explicitly, using (5.2),

$$U(\Lambda)\Psi_{p,\sigma} = N(p)U(L(\Lambda p))U(W(\Lambda, p))\Psi_{k,\sigma}, \qquad W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p). \tag{5.15}$$

The transformations W which leave k^μ invariant form a subgroup of Lorentz transformations which is called the **little group**. Apparently

$$U(W(\Lambda,p))\Psi_{k,\sigma} = \sum_{\sigma'} D_{\sigma\sigma'}(W)\Psi_{k,\sigma'} \tag{5.16}$$

and hence, finding representations of Lorentz group reduces to finding the representations of the little group. The action of $U(\Lambda)$ on any $\Psi_{p,\sigma}$ is then determined from its little group image W

$$U(\Lambda)\Psi_{p,\sigma} = \frac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D_{\sigma\sigma'}(W)\Psi_{\Lambda p,\sigma'}, \tag{5.17}$$

This is called the method of **induced representations**.

For **massive particles** $-p^2 = m^2 > 0$, it is natural to choose $k^\mu = (m, \mathbf{0})$. $L(p)$ can be taken to be a pure boost (defined in section 1) with parameter $v^i = p^i/p^0$. The little group is $SO(3)$ rotations whose irreducible representations are, as anticipated, states of integer and half integer spin.

For **massless particles** $p^2 = 0$, choose $k^\mu = (\kappa, \kappa, 0, 0)$ where $\kappa = 1eV$. $L(p)$ can be taken to be a boost along x^1 that takes k^μ to $(p^0, p^0, 0, 0)$ (find the boost parameter), followed by a rotation that takes $\hat{x}^1$ to $\hat{p}$. The little group is more interesting.

3. Find the algebra of the little group for a massless particle.

   Solution: To find the algebra we need to study small Lorentz transformation that leave k^μ invariant. Rotations in $2-3$ plane are definitely a subgroup. In addition to those, a boost along x^2 (or x^3) can be combined with a rotation in $1-2$ plane (respectively, $1-3$ plane) to leave k^μ invariant. Thus the generators are

   $$J = J^{23}, \quad A \equiv J^{02} - J^{12}, \quad B \equiv J^{03} - J^{13}. \tag{5.18}$$

   Using (5.7)-(5.9) we find

   $$[A,B] = 0, \qquad i[J,A] = B, \qquad i[J,B] = -A \tag{5.19}$$

   which is the algebra of the symmetry group of Euclidean plane (two translations and a rotation J).

As in the case of momentum operators, we can diagonalize A, B

$$A\Psi_{k,a,b} = a\Psi_{k,a,b}, \qquad B\Psi_{k,a,b} = b\Psi_{k,a,b}. \tag{5.20}$$

However, if a or b is nonzero they must be a continuous label because under a rotation by angle θ

$$a' = a\cos\theta - b\sin\theta, \qquad b' = a\sin\theta + b\cos\theta. \tag{5.21}$$

There is no compelling evidence for the existence of such continuous spin representations in nature.[3] We ignore them here and set $a = b = 0$. Thus we are left with the representations of $SO(2)$: Rotations around the momentum vector whose eigenvalues are called **helicity**. Since a rotation by 2π has to give ± 1, the helicity of massless representations are either integer or half integer. The latter has to do with the fact that the Lorentz group is doubly connected (*Weinberg 2.7*) and therefore it admits projective representations (with half-integer spin and helicity). A rotation by 2π corresponds to a closed path in the group manifold that cannot be shrunk to zero. While a rotation by 4π can always be shrunk to zero.

So far there is no reason for helicities to come in pairs $h = \pm\sigma$ when $\sigma \neq 0$. However, they have to because, firstly, parity changes the sign of h, and secondly, even if parity is not respected by interactions of a particle, causality of interactions that involve any massless particle requires the existence of the opposite helicity state antiparticle (*Weinberg 5.9*).[4] The gauge symmetries of the theories of photons and gravitons are also the consequence of the same requirement. A Lorentz vector A_μ has too many components to describe two helicities of the photon. Gauge symmetry is the result of the redundancy of this description. This is the general strategy to construct a relativistic quantum theory: to package the creation and annhilation operators for the irreducible representations of Poincaré into fields and to write local lorentz-invariant Lagrangians for these fields.

4. * Find the number of degrees of freedom (polarizations) of dilaton (massless scalar), photon (massless spin 1), and graviton (massless spin 2) in $d = 5$ spacetime dimensions. (Are there photons or gravitons in $d = 2$ or $d = 3$?)

[3]Nevertheless, they have been revisited recently by Schuster and Toro (1404.0675).

[4]The argument goes as follows. To construct a causal interacting theory, we need to combine creation and annihilation operators in local fields that transform under (non-unitary) representations of Loretnz group. To construct such fields, the annihilation operator for any massless state of helicity $\sigma \neq 0$ has to be combined with the creation operator of a massless field with helicity $-\sigma$.

6 A Relativistic Theory of Gravity

Reading: *Weinberg GR: 2.8*

Newtonian gravity is very successful in explaining a wide range of phenomena. But it cannot fit in a relativistic framework since it involves action at a distance. There cannot be any instantaneous interaction in relativity because simultaneity is not Lorentz invariant, even though it is a good approximate notion for slowly moving objects. Incidentally, Newtonian gravity looks extremely similar to electrostatics with mass playing an analogous role as the electric charge. The Newtonian potential φ is related to mass density ρ via Poisson equation

$$\nabla^2\varphi = 4\pi G\rho, \tag{6.1}$$

and massive objects moving in this field experience a force $-m\nabla\varphi$. There is a gravitational analog of the Coulomb $1/r^2$ force between massive objects.

By the end of 19th century it was known that electrostatics is just a special limit of the Maxwell theory, a relativistic theory which involves magnetic forces and electromagnetic waves. And today we understand it as the unique theory of interacting massless spin-1 particles called photons. Today, we also know that Newtonian gravity is a special limit of general relativity, the unique theory of interacting massless spin-2 particles, or **gravitons**. In the next few lectures, I will take a non-historical approach to reach this conclusion. Our guiding principles will be special relativity and quantum mechanics. In this framework long-range interactions can only result from the exchange of local degrees of freedom, i.e. particles.[5] However unlike electromagnetism, gravity cannot be described by the exchange of spin-1 photons. Like electric charges repel each other while all (positive) masses attract gravitationally.

It is perhaps more natural to try a massless spin-0 field with a relativistic equation of motion

$$(-\partial_t^2 + \nabla^2)\varphi = -4\pi G T, \tag{6.2}$$

where the source T has a non-relativistic limit $-T \to \rho$. At small velocities, we can neglect ∂_t compared to ∇ and recover (6.1).

1. * Show that a massive force carrier ($m_\varphi \neq 0$) with non-relativistic limit

$$(\nabla^2 - m_\varphi^2)\varphi = 4\pi G\rho \tag{6.3}$$

leads to a short-range potential $-GMe^{-m_\varphi r}/r$ for a point source $\rho = M\delta^3(\boldsymbol{x})$.

So far we specified how matter sources φ. How does φ affect the motion of matter? What kind

[5]To be more precise I should say small nonuniform motions of a particle can affect and be affected by another particle via the exchange of local degrees of freedom.

of force does the exchange of φ induce among massive point sources? In field theory the answer is unambiguous. One can use the perturbative QFT machinery to calculate the scattering amplitude due to the exchange of φ. I leave this to the exercises, and take an alternative approach. Let's try to couple φ to the worldline of a point particle. To lowest order in φ and its derivatives (and of course respecting worldline reparametrization invariance), this is

$$S_{pp} = -m \int d\sigma \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\sigma} \frac{dX^\nu}{d\sigma}} (1 + \lambda\varphi(X)). \tag{6.4}$$

This changes the massive particle equation of motion to

$$m \frac{d}{d\tau} \left((1 + \lambda\varphi) \frac{dX_\mu}{d\tau} \right) = -m\lambda \partial_\mu \varphi. \tag{6.5}$$

In the rest frame and to lowest order in φ, we obtain the following equation of motion

$$m\ddot{X}^i = -\lambda m \nabla_i \varphi + \mathcal{O}(\varphi^2) \tag{6.6}$$

Thus if φ is to be identified with the Newtonian potential, we must set $\lambda = 1$.

The worldline coupling (6.4) also leads to a source for φ. Introducing a kinetic term for φ and extending the worldline coupling to a spacetime action gives

$$\int d^4x \left[-\frac{1}{2} \kappa (\partial\varphi)^2 - m\varphi \int d\tau \delta^4(x - X(\tau)) \right]. \tag{6.7}$$

2. The sign before the worldline action (6.4) is important for our result. Show that it must be negative?

Varying (6.7) with respect to φ gives

$$\kappa \Box \varphi = \frac{m}{\dot{X}^0} \delta^3(x^i - X^i(t)). \tag{6.8}$$

For a non-relativistic particle $\dot{X}^0 \simeq 1$ and we recover (6.1) by setting $\kappa = 1/4\pi G$. Note that the r.h.s. is indeed the trace of the stress-energy tensor for a point-particle,

$$T^{\mu\nu} = \frac{p^\mu p^\nu}{p^0} \delta^3(x^i - X^i(t)). \tag{6.9}$$

So we take this as the definition of relativistic scalar gravity at linear level. Namely, given a matter action $S_m[\psi]$, with ψ representing all matter fields, we introduce the coupling $S = \int \varphi T^\mu_\mu$, where $T_{\mu\nu}$ is the total stress-energy tensor of $S_m[\psi]$.[6] Such a universally coupled scalar field is often called **dilaton**. And the theory we just developed is the linear version of the theory of gravity that Nordström proposed in 1913 (`https://en.wikipedia.org/wiki/Nordstrom's_theory_of_`

[6]Note that at lowest order in perturbation theory this is just a cubic coupling between matter fields and φ. For

gravitation).

Our scalar gravity model by construction passes all non-relativistic tests. Hence, to confirm or to reject it we need to examine its intrinsically relativistic predictions. The most obvious examples are gravitational red-shift, bending of light, precession of the perihelion of Mercury, and gravitational wave emission from close binaries.

Gravitational Redshift. Good clocks are small and have rapid internal dynamics. What does happen if they are immersed in an external field that is almost constant compared to their size and period? There is no universal answer to this question. The external field might change some fundamental feature based on which the clock operates. (For instance, an external magnetic field shifts the energy levels of an atomic clock.) However, we expect some universality to arise in a gravitational field. If dilaton φ is coupled, as above, by replacing $X^\mu \to X^\mu(1+\varphi)$ everywhere in the matter action, then all that happens to good clocks is that their natural time-scales get stretched, or their frequency redshifted:

$$\omega = (1+\varphi)\omega_0, \tag{6.11}$$

where ω_0 is the frequency away from gravitational sources, i.e. when $\varphi = 0$. This phenomenon has been measured on Earth; precise atomic clocks at different altitudes work at different rates https://en.wikipedia.org/wiki/Pound-Rebka_experiment.

Light Unbent. Let me next show that ultra-relativistic particles decouple from scalar gravity. Consider the motion of a particle in a static gravitational field (say of the Sun)

$$m\frac{d}{d\tau}\left((1+\varphi)\frac{dX_\mu}{d\tau}\right) = -m\partial_\mu\varphi. \tag{6.12}$$

The $\mu = 0$ component of this equation gives us a constant of motion

$$E = m(1+\varphi)\frac{dX^0}{d\tau}. \tag{6.13}$$

We can use this equation to express derivatives with respect to τ in terms of those with respect to time $t = X^0$:

$$\frac{d^2X^i}{dt^2} = -\frac{m^2}{E^2}\partial_i\varphi + \mathcal{O}(\varphi^2). \tag{6.14}$$

Now taking the limit $m \to 0$ while keeping the energy of the particle E fixed makes the r.h.s. negligible. Therefore, light rays move on straight lines. Nowadays we have very accurate measurements that light rays are indeed bent by about 1.7 arcseconds by the Sun. So the simplest relativistic

instance, a massive scalar field ψ in d dimensions is coupled via the vertex

$$S = \int d^dx\, \varphi\left[\frac{2-d}{2}(\partial\psi)^2 - \frac{d}{2}m^2\psi^2\right]. \tag{6.10}$$

model of gravity is experimentally ruled out. Nevertheless, it is very instructive to pursue other predictions of scalar gravity, because it is a simple setup which shares lots of conceptual similarities with spin-2 gravity. (Or imagine we couldn't measure light.)

3. * Couple the Polyakov action to φ by changing $\eta_{\mu\nu} \to (1+2\varphi)\eta_{\mu\nu}$. Show that the same conclusion about massless particles can be directly derived from the Polyakov action with m set to zero.

4. * Calculate T^μ_μ for the Maxwell theory. Does it agree with your expectation? Repeat the same exercise for a massless scalar field.

5. * Calculate the differential cross-section for scattering in attractive $1/r$ potential, with strength C. Use Born approximation.

6. * Calculate the differential cross-section for scattering of two non-relativistic particles of mass m, M $(m \ll M)$, due to the tree-level exchange of a dilaton coupled to T^μ_μ. Use the rest frame of M. Is there a choice of C in the previous problem that gives the same result?

7 Dilaton Waves

Reading: *Landau-Lifshitz vol2 sections 62, 66, 67*

One of the most exciting features of the relativistic theory of gravity is the emergence of gravitational waves, very much analogous to the emergence of electromagnetic waves once electrostatics is completed into the relativistic Maxwell theory. The motions of matter sources cause the dilaton field φ to locally fluctuate. These fluctuations are governed by a wave equation $(\partial_t^2 - \nabla^2)\varphi = 0$, so they propagate at the speed of light, carrying away energy and information. The goal of this lecture is to learn how to calculate the dilaton field of moving masses, and the radiation of dilaton waves by compact systems.

Retarded Potential. The standard way to deal with the problem is to find the retarded Green's function $G(x, x')$ which is the solution to

$$\Box_x G_R(x, x') = \delta^4(x - x') \tag{7.1}$$

with the boundary condition $G(x, x') = 0$ if $x^0 < x'^0$. The answer is

$$G_R(x, x') = -\frac{\delta(t - t' - |\boldsymbol{r} - \boldsymbol{r}'|)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}\theta(t - t'). \tag{7.2}$$

For aesthetic reasons I use $\boldsymbol{r}$ to denote spatial coordinates: $x^\mu = (t, \boldsymbol{r})$. The step function θ $[\theta(t - t' \geq 0) = 1$ and $\theta(t - t' < 0) = 0]$ ensures that G_R vanishes for $t < t'$. Note, however, that G_R vanishes everywhere outside of the future light-cone of x', as expected from Lorentz invariance of the system. (In this particular case G_R is non-vanishing just on the future light-cone.) Had it been nonzero anywhere outside of the future light-cone, at that point x and x' would have been spacelike separated and by a Lorentz transformation one could have made G_R nonzero at some $t < t'$.

1. Derive (7.2).

   Solution: Since there is no explicit x dependence G is a function only of $x - x'$. It is convenient to use Fourier basis:

   $$G_R(x - x') = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} G_{\boldsymbol{k}}(t - t')e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}. \tag{7.3}$$

   We get

   $$(-\partial_t^2 - k^2)G_{\boldsymbol{k}}(t - t') = \delta(t - t'), \qquad G_{\boldsymbol{k}}(t - t' < 0) = 0, \tag{7.4}$$

   where $k = \sqrt{\boldsymbol{k}^2}$. There is a unique solution:

   $$G_{\boldsymbol{k}}(t - t') = -\frac{1}{k}\theta(t - t')\sin k(t - t'). \tag{7.5}$$

Next, Fourier transform this expression back to real space (at $t' = 0$, $\boldsymbol{r'} = 0$, $t > 0$; $t', \boldsymbol{r'}$ can be easily restored at the end by using translational invariance):

$$
\begin{aligned}
G_R(t, \boldsymbol{r}) &= \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} G_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot(\boldsymbol{r})} \\
&= -\frac{1}{2\pi^2 r} \int_0^\infty dk \sin kt \ \sin kr \\
&= -\frac{1}{8\pi^2 r} \int_{-\infty}^\infty dk [\cos k(t-r) - \cos k(t+r)] \\
&= -\frac{1}{8\pi^2 r} \mathrm{Re} \int_{-\infty}^\infty \mathrm{dk} [\mathrm{e}^{\mathrm{ik(t-r)}} - \mathrm{e}^{\mathrm{ik(t+r)}}] \\
&= -\frac{1}{4\pi r} [\delta(t-r) - \delta(t+r)].
\end{aligned}
\tag{7.6}
$$

The argument of the second delta function is positive definite, so it can be ignored. Restoring t' and $\boldsymbol{r'}$ gives (7.2).

For a generic source $T(x) \equiv T^\mu_\mu(x)$, the solution to the equation

$$\Box\varphi = -4\pi G T(x), \tag{7.7}$$

with outgoing boundary condition (i.e. no incoming φ wave as $t \to -\infty$) can be easily checked to be $-4\pi G \int d^4x' G_R(x, x') T(x')$. This is the retarded potential

$$\varphi(t, \boldsymbol{r}) = G \int d^3\boldsymbol{r'} \frac{T(t - |\boldsymbol{r} - \boldsymbol{r'}|, \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|}. \tag{7.8}$$

The retardation $t - |\boldsymbol{r} - \boldsymbol{r'}|$ makes it manifest that information propagates at the speed of light. However in spite of being conceptually illuminating, this exact solution is too complicated to be useful practically. To proceed, imagine we have a compact system of masses with typical size R (by which I mean either a bound system of size R or a system made of non-relativistic masses which do not move significantly while being observed), and study the large $r \gg R$ behavior of the solution. I don't make any assumption about the time-dependence of the source. In general it can be Fourier transformed in time

$$T(t, \boldsymbol{r'}) = \int \frac{d\omega}{2\pi} e^{-i\omega t} T(\omega, \boldsymbol{r'}), \tag{7.9}$$

where reality of $T(t, \boldsymbol{r'})$ implies $T(\omega, \boldsymbol{r'}) = T^*(-\omega, \boldsymbol{r'})$. Since the equation is linear and there is no explicit time-dependence, different frequencies are decoupled. Thus, let's focus on one of the frequencies

$$T_\omega = T(\omega, \boldsymbol{r'}) e^{-i\omega t} \tag{7.10}$$

and find the corresponding φ_ω solution (note that T_ω and φ_ω have dimensions of T/ω and φ/ω, respectively)

$$\varphi_\omega(t,\boldsymbol{r}) = G\int d^3\boldsymbol{r}'\frac{T(\omega,\boldsymbol{r}')e^{-i\omega(t-|\boldsymbol{r}-\boldsymbol{r}'|)}}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{7.11}$$

Because of the linearity of (7.8), the full solution will be the superposition

$$\varphi(t,\boldsymbol{r}) = \int\frac{d\omega}{2\pi}\varphi_\omega(t,\boldsymbol{r}). \tag{7.12}$$

We are interested in the solution at $r \gg R$ so that

$$|\boldsymbol{r}-\boldsymbol{r}'| = r - \hat{r}\cdot\boldsymbol{r}' + \mathcal{O}(r'^2/r), \tag{7.13}$$

can be truncated. In two extreme regimes, when $\omega r \ll 1$ and when $\omega r \gg 1$, further approximations can be made. In the **Near Zone** $r \ll 1/\omega$, which also implies $\omega r' \ll r'/r$, the corrections to the exponent in (7.11) can be neglected compared to the r'/r corrections coming from the denominator. The solution is then approximately

$$\varphi_\omega(t,\boldsymbol{r}) = Ge^{-i\omega(t-r)}\int d^3\boldsymbol{r}'\frac{T(\omega,\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{7.14}$$

Expanding the denominator in powers of r'/r gives a multipole expansion very much analogous to what is encountered in electrostatics. The l^{th} multipole has the $\hat{r}$ dependence of the l^{th} spherical harmonic Y_{l0} and is suppressed by $(r'/r)^l$ with respect to the monopole solution, which goes as $1/r$. We recover the full non-relativistic answer if we can further neglect powers of ωr compared to R/r. We then have an approximately instantaneous reaction of φ to the variations of the source.

The opposite regime $r \gg 1/\omega$ is called the **Wave Zone**. The solution is now approximately of the form

$$\varphi_\omega(t,\boldsymbol{r}) \approx -G\frac{A(\omega,\hat{r})e^{-i\omega(t-r)}}{r}, \tag{7.15}$$

where

$$A(\omega,\hat{r}) = \int d^3\boldsymbol{r}'T(\omega,\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'}. \tag{7.16}$$

Thus φ_ω is an outgoing wave. We are not interested in r'/r corrections in this regime because, as we will see, the energy flux of the free waves is $\mathcal{O}(\varphi^2)$. Therefore, it is only the leading $1/r$ term that is responsible for taking away energy to infinity. Imagine we are detecting the waves coming from a faraway source, with characteristic frequency ω, and say at distance r_0 from the Earth center so that $\omega r_0 \gg 1$. Near the Earth we can write $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{x}$, and use $|\boldsymbol{x}| \sim R_{\mathrm{E}} \ll r_0$ to write the solution as

$$\varphi_\omega(t,\boldsymbol{x}) \approx Ne^{-i\omega(t-\hat{\boldsymbol{r}}_0\cdot\boldsymbol{x})}, \tag{7.17}$$

with N a normalization constant (including the phase $e^{i\omega r_0}$). That is, the derivatives are dominated by derivatives of the phase. This is a **Plane Wave** with wave-vector $\boldsymbol{k} = \omega\hat{\boldsymbol{r}}_0$. Plane waves are the simplest solutions of the free field equations, and are usually used as a basis for the asymptotic states and to quantize the theory. For instance, deriving a formula for the energy flux of the radiation becomes extremely easy since the problem has essentially become $1+1$ dimensional.

2. Derive a formula for the **Energy Flux** (erg/sec.cm^2) in terms of stress-energy tensor. What does it give for a plane dilaton wave?

3. * Derive the Poynting flux for the Maxwell field, using $E_i = F_{0i}$ and $B_i = \varepsilon_{ijk}F_{jk}$.

4. * Transverse vibrations of a non-relativistic string with mass density ρ and tension T are described by the action

$$S = \int dtdx \left[\frac{1}{2}\rho(\partial_t u)^2 - \frac{1}{2}T(\partial_x u)^2\right]. \tag{7.18}$$

Derive the stress-energy tensor. Find the sound speed c_s. Find the energy flux for a right-moving wave $u(t,x) = u_0 \cos(\omega(t - x/c_s))$.

Let's now return to the original problem with a general time-dependent source (7.9). We are often interested in one of the following cases.

- Either the system is (approximately) periodic with period Δt, in which case we would like to know the average **Radiated Power (Luminosity)** (erg/sec). In this case, the Fourier integral in (7.9) becomes a discrete sum

$$T(t,\boldsymbol{r}') = \sum_{n\in\mathbb{Z}} T(\omega_n,\boldsymbol{r}')e^{-i\omega_n t}, \qquad \omega_n \equiv 2\pi n/\Delta t, \tag{7.19}$$

with a similar discrete sum replacing (7.12):

$$\varphi(t,\boldsymbol{r}) = \sum_{n\in\mathbb{Z}} \varphi_{\omega_n}(t,\boldsymbol{r}). \tag{7.20}$$

Given an expression for the energy flux, the average luminosity (erg/sec) is obtained by integrating the flux (erg/sec.cm^2) over a sphere of constant radius r, and averaging it over one period

$$\langle L\rangle = -\frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} dt \int d\hat{r}\; r^2\; T^r_0. \tag{7.21}$$

Substituting (7.20) and (7.15) in (**??**), we obtain

$$\langle L\rangle = \frac{G}{4\pi\Delta t}\int_{t_0}^{t_0+\Delta t} dt \int d\hat{r} \sum_{n,n'\in\mathbb{Z}} (-\omega_n\omega_{n'})A(\omega_n,\hat{r})A(\omega_{n'},\hat{r})e^{-i(\omega_n+\omega_{n'})(t-r)}. \tag{7.22}$$

The integral over one period collapses the double sum over n into the diagonal because

$$\int_{t_0}^{t_0+\Delta t} dte^{-i(\omega_n+\omega_{n'})t} = \Delta t\ \delta_{n,-n'}. \tag{7.23}$$

So we finally get

$$\langle L\rangle = \frac{G}{4\pi}\int d\hat{r}\sum_{n\in\mathbb{Z}}\omega_n^2|A(\omega_n,\hat{r})|^2, \tag{7.24}$$

where I used the reality condition $A(\omega,\hat{r}) = A^*(-\omega,\hat{r})$.

- Or else, there is a process like a scattering event or a merger with a finite temporal extent, during which dilaton waves are emitted. Here one often asks what is the total amount of energy released in gravitational radiation. In this case the Fourier transform is continuous and we have

$$E_{\rm tot} = -\int_{-\infty}^{\infty} dt\int d\hat{r}\ r^2\ T_0^r. \tag{7.25}$$

Repeating similar steps as above gives

$$E_{\rm tot} = \frac{G}{4\pi}\int d\hat{r}\int_{-\infty}^{\infty} dt\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}(-\omega\omega')A(\omega,\hat{r})A(\omega',\hat{r})e^{-i(\omega+\omega')(t-r)}. \tag{7.26}$$

Using the definition of Dirac delta

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} dte^{-i\omega t}, \tag{7.27}$$

we obtain

$$E_{\rm tot} = \frac{G}{4\pi}\int d\hat{r}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2|A(\omega,\hat{r})|^2. \tag{7.28}$$

Multipole Expansion. Let me summarize what we learned so far. For a time-dependent source of characteristic frequency ω and characteristic size R, the solution looks like an outgoing wave for $r \gg 1/\omega$. To study radiation (for instance the amount of energy that escapes from the system to infinity) we only care about the leading $1/r$ behavior of solution. Higher order corrections in R/r can be ignored. So the solution is of the form

$$\varphi(t,\boldsymbol{r}) = \frac{\hat{\varphi}(t-r,\hat{r})}{r} + \mathcal{O}(1/r^2), \tag{7.29}$$

where I used $|\boldsymbol{r}-\boldsymbol{r}'| = r - \hat{r}\cdot\boldsymbol{r}' + \mathcal{O}(1/r)$ and defined

$$\hat{\varphi}(t-r,\hat{r}) \equiv G\int d^3\boldsymbol{r}'T(t-r+\hat{r}\cdot\boldsymbol{r}',\boldsymbol{r}'). \tag{7.30}$$

So depending on the structure of the source, the radiation will be anisotropic (depends on $\hat{r}$) because of the $\hat{r}\cdot\boldsymbol{r}'$. It is convenient to expand the $\hat{r}$ dependence of $\hat{\varphi}$ in terms of an orthogonal basis on the sphere, i.e. spherical harmonics:

$$\varphi(t, \boldsymbol{r}) = \frac{G}{r} \sum_{l,m} a_{lm}(t-r) Y_{lm}(\hat{r}). \tag{7.31}$$

$l = 0, 1, 2, \dots$ are called monopole, dipole, quadrupole and so on. Due to the orthonormality of Y_{lm} basis, the total emitted power which is obtained by integrating the flux over the sphere is

$$L = G \sum_{l,m} \dot{a}_{lm}^2(t). \tag{7.32}$$

So far we treated $T = T_\mu^\mu$ as a generic time-dependent source. Next, we restrict to an important class of compact systems, namely **Non-Relativistic** systems with typical velocity $v = \omega R \ll 1$. In this regime we can Taylor expand $T(t - r + \hat{r} \cdot \boldsymbol{r}', \boldsymbol{r}')$ in powers of $\hat{r} \cdot \boldsymbol{r}'$:

$$T(t - r + \hat{r} \cdot \boldsymbol{r}', \boldsymbol{r}') = \sum_n \frac{1}{n!} \partial_t^n T(t - r, \boldsymbol{r}')(\hat{r} \cdot \boldsymbol{r}')^n. \tag{7.33}$$

The n^{th} order term in the sum is suppressed by v^n, so for a given precision the sum can be truncated. On the other hand the n^{th} order term can only contribute to $l \leq n$ multipoles. Therefore, **the multipole expansion is an expansion in powers of velocity**. To second order, we get (check)

$$\varphi(t, \boldsymbol{r}) = \frac{G}{r} \left[m(t-r) + \hat{r}_i \dot{d}_i(t-r) + \frac{1}{18}(3\hat{r}_i\hat{r}_j - \delta_{ij})\ddot{q}_{ij}(t-r) + \cdots \right] \tag{7.34}$$

where over-dot denotes d/dt and the monopole, dipole, and quadrupole are respectively defined as

$$\begin{aligned} m(t) &= \int d^3\boldsymbol{r}' T(t, \boldsymbol{r}') + \cdots, \\ d_i(t) &= \int d^3\boldsymbol{r}' r_i' T(t, \boldsymbol{r}') + \cdots, \\ q_{ij}(t) &= \int d^3\boldsymbol{r}' (3r_i' r_j' - r'^2 \delta_{ij}) T(t, \boldsymbol{r}') + \cdots. \end{aligned} \tag{7.35}$$

Note that we have neglected relative corrections of order v^2 to each moment (this is the meaning of ellipses). Given that multipole expansion is itself an expansion in powers of v, one can use, say, the quadrupole term without worrying about the corrections to the monopole only if monopole emission vanishes because of a symmetry reason.

5. * Relate a_{2m} to q_{ij}.

Phenomenology. The discussion of non-relativistic limit is important firstly because many astrophysical sources (such as binary systems) are non-relativistic. Secondly, the dynamics of these sources is governed by the gravitational attraction, which implies that $v^2 \sim \varphi$. Therefore, a reliable calculation of radiation in the relativistic regime (when $v^2 = \mathcal{O}(1)$) requires a nonlinear definition of the theory of gravity. We have not defined such a theory yet, and in any case the problem is not tractable analytically.

Let us now consider a concrete example by calculating the power-loss of a binary system composed of two neutron stars in a circular orbit with frequency ω. It is indeed the case that both monopole and dipole radiation are zero by symmetry. The monopole term is spherically symmetric, and the spherical average of the source term in the center of mass frame is time-independent. In this frame no vector can be associated to the system either, hence $d_i = 0$.

6. * Show that in the center of mass frame the monopole and dipole moment of any solid object is time-independent. As a result the monopole of a binary system must be proportional to the eccentricity of the orbit. Is the dipole moment always zero?

Thus the leading emission is quadrupole emission. Integrating the expression for the flux over the sphere and using

$$\int d\hat{r}(3\hat{r}_i\hat{r}_j - \delta_{ij})(3\hat{r}_k\hat{r}_l - \delta_{kl}) = \frac{4\pi}{5}(3\delta_{ik}\delta_{jl} + 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}) \tag{7.36}$$

we get for the energy loss of the orbit

$$P_{\text{loss}} = -\frac{G}{270}\left(\frac{d^3 q_{ij}}{dt^3}\right)^2. \tag{7.37}$$

Assuming equal masses M at separation R in $x-y$ plane, the time-dependent part of q_{ij} is (exercise)

$$q_{xy} = \frac{3}{4}MR^2\sin(2\omega t), \quad q_{xx} = \frac{3}{4}MR^2(1+\cos(2\omega t)), \quad q_{yy} = \frac{3}{4}MR^2(1-\cos(2\omega t)). \tag{7.38}$$

Using the Kepler's law $\omega^2 = r_g/R^3$ where $r_g = 2GM$, we obtain

$$P_{\text{loss}} = -\frac{r_g^5}{15GR^5}. \tag{7.39}$$

This loss of power causes the orbit to shrink with a rate (check)

$$\dot{R} = -\frac{8r_g^3}{15R^3}. \tag{7.40}$$

This concrete prediction of scalar gravity has actually been tested observationally, and has failed miserably. Astronomers have measured the orbit evolution of a few such binaries in the Universe extremely precisely and over many years (`https://en.wikipedia.org/wiki/Hulse-Taylor_binary`). The result is in perfect agreement with general relativity, which predicts a power loss that is by a factor of 6 larger than (7.37).

8 Spin-2 Gravity

Reading: `https://video.ias.edu/pitp-2011-arkani-hamed1`

We have seen that the scalar theory of gravity, though theoretically sound, is not phenomenologically adequate. We argued before that gravitational attraction cannot be described by a spin-1 particle. So the next candidate for graviton is a massless spin-2 particle, which has two degrees of freedom with helicity ± 2. In a relativistic theory, such a particle is described by a symmetric tensor $h_{\mu\nu}$. Compare this with photons which are described by a vector field A_μ. In Maxwell theory, the description in terms of A_μ is redundant since in $d = 4$ there are 4 components in A_μ while there are only two photon polarizations. However, the theory has a gauge symmetry $A_\mu \to A_\mu + \partial_\mu \alpha$ and this eliminates the extra degrees of freedom. To preserve this symmetry the electric current J^μ which sources A_μ has to be conserved. The gauge symmetry and the need for such a conserved source are both manifest in the A_μ equation of motion

$$\partial_\nu F^{\nu\mu} = -J^\mu. \tag{8.1}$$

Acting by ∂_μ on both sides the l.h.s. vanishes identically, because of antisymmetry of $F^{\mu\nu}$, so $\partial_\mu J^\mu = 0$.

There are 10 independent components in $h_{\mu\nu}$, exceeding the number of graviton polarizations by 8. And there is a bigger gauge symmetry called **Linearized Diffeomorphisms** (or "linear diffs" for short)

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{8.2}$$

with the gauge parameter now being an arbitrary x-dependent 4-vector ξ^μ. The equation of motion for $h_{\mu\nu}$ has to be a tensor equation, and to respect the gauge symmetry, the source term must be a conserved tensor. The natural candidate is the stress-energy tensor $T^{\mu\nu}$. With these requirements the $h_{\mu\nu}$ equation has to be

$$\Box h_{\mu\nu} - \partial_\mu \partial_\sigma h^\sigma_\nu - \partial_\nu \partial_\sigma h^\sigma_\mu + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\sigma \partial_\rho h^{\sigma\rho} = c T_{\mu\nu}. \tag{8.3}$$

The proportionality coefficient c depends on the normalization of $h_{\mu\nu}$. We will fix it by demanding that, first, the gravitational coupling is

$$S_{\text{coupl.}} = \frac{1}{2} \int d^4x h^{\mu\nu} T^{(0)}_{\mu\nu}, \tag{8.4}$$

and second, Newtonian gravity is recovered in the non-relativistic limit (as we shall see). The super-script (0) on $T_{\mu\nu}$ indicates that it is calculated to zeroth order in $h_{\mu\nu}$, as appropriate for a linearized theory of gravity. To simplify the notation I will drop (0), but it is implicitly assumed until we get to the discussion of nonlinear gravity.

1. Derive the homogeneous part of (8.3).

   Solution: First, use (8.2) to set $\partial^\mu \tilde{h}_{\mu\nu} = 0$:

$$\Box\xi_\nu + \partial^\mu\partial_\nu\xi_\mu = -\partial^\mu h_{\mu\nu}. \tag{8.5}$$

   There are residual diffs which preserve this condition, satisfying

$$\Box\tilde{\xi}_\nu + \partial^\mu\partial_\nu\tilde{\xi}_\mu = 0. \tag{8.6}$$

   The most general form of the equation of motion in this gauge is

$$\Box\tilde{h}_{\mu\nu} + a\partial_\mu\partial_\nu\tilde{h}^\alpha_\alpha + b\eta_{\mu\nu}\Box\tilde{h}^\alpha_\alpha = cT_{\mu\nu}, \tag{8.7}$$

   with a, b, c to be determined. Next, demand that this equation is invariant under the residual diffs (8.6). This forces $a = 1$. Then, take the divergence ∂^μ of both sides. The l.h.s. vanishes identically if and only if $b = -1$. We will fix c later.

   Finally, we need to write a diff invariant equation that in $\partial^\mu\tilde{h}_{\mu\nu} = 0$ reduces to the above equation. This means that the full equation contains $\partial^\mu h_{\mu\nu}$. With $a = -b = 1$ it is easy to see that the l.h.s. of (8.3) is the only diff invariant generalization of (8.7).

As in the case of scalar gravity once the coupling of graviton to matter fields is determined, we can read off the gravitational force either by looking at scattering of particles due to the exchange of gravitons or by coupling $h_{\mu\nu}$ to the worldline action. The second route is more straightforward, so let us follow it here. Recall that for a point-particle

$$T^{\mu\nu} = m\int d\tau \dot{X}^\mu\dot{X}^\nu\delta^4(x^\mu - X^\mu(\tau)). \tag{8.8}$$

(As before over-dot means $d/d\tau$.) Using (8.4), we obtain the following worldline action

$$\begin{aligned} S_{pp} =& -m\int d\sigma\sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\sigma}\frac{dX^\nu}{d\sigma}}\left(1 - \frac{\frac{dX^\mu}{d\sigma}\frac{dX^\nu}{d\sigma}}{2(d\tau/d\sigma)^2}h_{\mu\nu}(X)\right) \\ =& -m\int d\sigma\sqrt{-(\eta_{\mu\nu} + h_{\mu\nu}(X))\frac{dX^\mu}{d\sigma}\frac{dX^\nu}{d\sigma}} + \mathcal{O}(h^2_{\mu\nu}). \end{aligned} \tag{8.9}$$

Varying with respect to X^μ, gives the particle equation of motion in the gravitational field as

$$m\frac{d}{d\tau}\left(\frac{\eta_{\mu\nu} + h_{\mu\nu}}{\sqrt{1 - h_{\alpha\beta}\dot{X}^\alpha\dot{X}^\beta}}\frac{dX^\nu}{d\tau}\right) = \frac{1}{2}m\partial_\mu h_{\alpha\beta}\dot{X}^\alpha\dot{X}^\beta. \tag{8.10}$$

To simplify this equation, perform a spatial diff ξ^i to set $h_{0i} = 0$.[7] Then, in the rest frame where

[7] This is a short way of saying that perform a diff with $\partial_0\xi_i = h_{0i}$ so that $\tilde{h}_{0i} = 0$. And then forget about the old

$\dot{X}^i = 0$, we find $\ddot{X}^i = \mathcal{O}(\partial_i h_{\mu\nu})$. Therefore for $\mu = i$, we can ignore $h_{\mu\nu}$ on the l.h.s., to get

$$m\ddot{X}^i = \frac{1}{2} m \partial_i h_{00} + \mathcal{O}(h_{\mu\nu}^2). \tag{8.11}$$

This implies that in the non-relativistic regime h_{00} is related to the Newtonian potential by

$$h_{00} = -2\varphi, \qquad \varphi \text{ the Newtonian potential, and } h_{0i} = 0. \tag{8.12}$$

2. Given the above relation derive the proportionality coefficient of $T_{\mu\nu}$ in (8.3).

   Solution: In the Newtonian limit the dominant component of the stress-energy tensor is $T^0_0 = \rho$, and the time-derivatives are small. Using these facts together with the trace of (8.3) one can obtain an equation for h_{00}. The trace of (8.3) implies

   $$\Box h - \partial_\sigma \partial_\rho h^{\sigma\rho} = \frac{c}{2-d} T^\mu_\mu. \tag{8.13}$$

   Combining this with the 00 component of (8.3), and using the approximation $\partial_0 \approx 0$, $T^\mu_\mu \approx -T_{00} = -\rho$ give

   $$\nabla^2 h_{00} = c \frac{3-d}{2-d} \rho. \tag{8.14}$$

   Substituting (8.12), taking $d = 4$, and comparing with the Poisson equation, we obtain

   $$c = -16\pi G. \tag{8.15}$$

The last thing to do in this lecture is to find an action for $h_{\mu\nu}$. This allows us to quantize gravity and to determine the energy-momentum content in gravitational waves. Writing a quadratic action that gives (8.3) is extremely simple. Introduce the second order differential operator $\mathcal{E}^{\alpha\beta}_{\mu\nu}$ such that $\mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta}$ gives the l.h.s. of (8.3). Explicitly, this is of the form

$$\mathcal{E}^{\alpha\beta}_{\mu\nu} = \delta^{\alpha\beta}_{\mu\nu} \Box - \delta^\alpha_\nu \partial_\mu \partial^\beta + \cdots \tag{8.16}$$

Finding the rest of the terms is left to you as an exercise. Then the quadratic action for $h_{\mu\nu}$ is

$$S_g = \frac{1}{64\pi G} \int d^4x\, h^{\mu\nu} \mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} \tag{8.17}$$

plus the source term (8.4).

3. Derive the equation of motion from the action $S = \int d^4x \left[\frac{1}{2}\phi\Box\phi + \phi J\right]$. Write the action in the first order form (where there is at most one derivative per field).

$h_{\mu\nu}$ and drop the tilde from the new one. In the future I may just say "choose the gauge $h_{0i} = 0$".

The gravitational action S_g can be simplified a bit, and be written in the following first order form

$$S_g = -\frac{1}{64\pi G}\int d^4x\left[(\partial_\alpha h_{\mu\nu})^2 - 2(\partial_\mu h_{\mu\nu})^2 + 2\partial_\mu h_{\mu\nu}\partial_\nu h^\alpha_\alpha - (\partial_\alpha h^\mu_\mu)^2\right]. \tag{8.18}$$

4. Consider a transverse-traceless plane wave propagating in x direction:

$$h_{yy} = -h_{zz} = h_+(t,x), \qquad h_{yz} = h_{zy} = h_\times(t,x), \tag{8.19}$$

and all other components of $h_{\mu\nu}$ equal to zero. The energy-momentum tensor of a freely propagating wave is in its plane of motion. Derive an effective $1+1$ dimensional action. Find the stress-energy tensor, and using that the energy flux carried by plane gravitational waves.

Solution: The only term in (8.18) that survives is the first term. Let's use initial Latin indices for the two dimensional space (t,x), and suppose the transverse plane is compact with area A

$$S_g = -\frac{A}{32\pi G}\int d^2x\left[(\partial_a h_+)^2 + (\partial_a h_\times)^2\right], \tag{8.20}$$

This is identical to the action of a free scalar field in $1+1d$. Dividing by A we get the $3+1d$ stress-energy tensor

$$t_{ab} = \frac{1}{16\pi G}[\partial_a h_+ \partial_b h_+ - \frac{1}{2}\eta_{ab}(\partial_c h_+)^2], \tag{8.21}$$

and a similar expression for $h_\times$. I use lower-case t_{ab} for the stress-energy tensor of gravitons to distinguish it from that of matter fields which appears in the source term (8.4). The energy flux is

$$-t^1_0 = -\frac{1}{16\pi G}\dot{h}_+ h'_+ = \frac{1}{16\pi G}\dot{h}^2_+, \tag{8.22}$$

where I used $h'_+ \equiv \partial_x h_+ = -\dot{h}_+$.

5. * By Weinberg-Witten theorem a theory with a massless spin-2 particle cannot have a non-trivial (ordinarily) conserved stress-energy tensor. Why does our derivation of the energy flux in the gravitational waves make sense despite this fact?

9 Phenomenology of Spin-2 Gravity

Reading: *Landau-Lifshitz vol2 sections 107, 110*

We formulated spin-2 gravity in the last lecture. Now we should understand what it predicts.

Redshift. As discussed in the context of scalar gravity, to study the performance of good clocks (that is small clocks with fast internal dynamics), the gravitational field can be treated as a constant matrix $h_{\mu\nu}$. As before we set $h_{0i} = 0$ by an appropriate diff so as to make connection with dilaton gravity and Newtonian gravity. Looking back at the worldline action (8.9), we see that the only effect of such a constant field is to replace

$$\eta_{\mu\nu} \to g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}, \tag{9.1}$$

with zero $0i$ components. Hence, the internal dynamics of clocks would be the same if we measured time and distances differently:

$$\tilde{X}^0 = \sqrt{1 - h_{00}}X^0, \qquad \tilde{X}^i = \sqrt{\lambda_i}R^i_j X^j, \quad \text{no sum over } i \tag{9.2}$$

where R^i_j is a rotation matrix that diagonalizes g_{ij}, and λ_i are its eigen-values:

$$g_{ij} = \sum_m \lambda_m R^m_i R^m_j. \tag{9.3}$$

In particular, the frequency of the clocks that are at rest redshifts as

$$\omega = (1 - \frac{1}{2}h_{00})\omega_0 = (1 + \varphi)\omega_0, \tag{9.4}$$

which is the same as in dilaton gravity. In fact, the dilaton gravity could have been formulated in a similar fashion by taking

$$g_{\mu\nu} = (1 + 2\varphi)\eta_{\mu\nu}. \tag{9.5}$$

Note that in the $\tilde{X}^\mu$ system of measurement there is the usual (special relativistic) rules of translating the rate of moving clocks to one another. Thus, it is natural to modify the definition of proper time of an observer who moves inside gravitational field to be what her clock measures:

$$d\tilde{\tau}^2 = -\eta_{\mu\nu}d\tilde{X}^\mu d\tilde{X}^\nu = -g_{\mu\nu}dX^\mu dX^\nu. \tag{9.6}$$

Good clocks always work at the same rate in terms of $\tilde{\tau}$. Soon we will get used to the new definition and drop the tilde. In terms of $\tilde{\tau}$ the particle equation of motion (8.10) takes the form

$$m\frac{d}{d\tilde{\tau}}\left(g_{\mu\nu}\frac{dX^\nu}{d\tilde{\tau}}\right) = \frac{1}{2}m\partial_\mu g_{\alpha\beta}\frac{dX^\alpha}{d\tilde{\tau}}\frac{dX^\beta}{d\tilde{\tau}}. \tag{9.7}$$

This equation is called the **Geodesic Equation**, describing the shortest lines that can be drawn on a manifold with metric $g_{\mu\nu}$.[8] Of course, so far we derived it assuming small $h_{\mu\nu}$, ignoring $\mathcal{O}(h_{\mu\nu}^2)$ corrections, and continue to do so in the rest of this lecture. Later I will return to the nonlinear version.

As we saw in the context of scalar gravity, the motion of test particles in the **static field** of astrophysical objects is an important testing ground for the theory of gravity. The frequency of signals propagating linearly on such a background remains conserved.[9] Hence we can set up experiments to easily measure gravitational redshift. Moreover, as in the case of dilaton gravity there is a conserved energy for particle propagating on this background. The zeroth component of the geodesic equation implies that the following quantity is a constant of motion

$$mg_{0\mu}\frac{dX^\mu}{d\tilde{\tau}} = E. \tag{9.8}$$

The calculation of bending of light in spin-2 gravity is left to you as an exercise.

Gravitational Waves. There is a magical gauge, sometimes called radiation gauge,

$$\partial^\mu \psi_{\mu\nu} \equiv \partial^\mu (h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha_\alpha) = 0. \tag{9.9}$$

In this gauge the equation of motion for $h_{\mu\nu}$ becomes

$$\Box \psi_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{9.10}$$

Everything we said about dilaton waves can now be repeated for $\psi_{\mu\nu}$ almost unchanged. Far from time-dependent sources there will be gravitational radiation:

$$\psi_{\mu\nu} = \frac{4G}{r}\int d^3\boldsymbol{r}' T_{\mu\nu}(t - |\boldsymbol{r}-\boldsymbol{r}'|, \boldsymbol{r}'). \tag{9.11}$$

They look like plane waves with $\boldsymbol{k} = \omega\hat{r}$. They carry energy and momentum, and they can be measured by monitoring the motion of test masses and laser beams.

However, there are still too many components in $\psi_{\mu\nu}$. We can further simplify it by using the residual diffs that preserve (9.9), namely diffs that satisfy

$$\Box \xi_\mu = 0. \tag{9.12}$$

In particular, for a plane wave of momentum k^μ, (i.e. $h_{\mu\nu}(x) = \hat{h}_{\mu\nu}e^{ik_\mu x^\mu}$ with $\hat{h}_{\mu\nu}$ =const.) we

[8] To be honest they are the longest for Lorentzian signature. Any time-like trajectory can be deformed into a zigzagging one made of null segments which has zero proper time. The universal description is stationary, or optimum.

[9] This is saying that we can do a Fourier transform in t and different Fourier components $e^{i\omega t}$ decouple.

should use $\xi_\mu = \hat{\xi}_\mu e^{ik_\mu x^\mu}$, which automatically satisfies (9.12), because $k_\mu k^\mu = 0$. Take

$$\hat{\xi}_\mu = ak_\mu + b\bar{k}_\mu + \sum_{s=1,2} c_s \epsilon_\mu^{(s)}, \tag{9.13}$$

with a, b, c_s arbitrary complex numbers,

$$\bar{k}^\mu \equiv (k^0, -\boldsymbol{k}) = \omega(1, -\hat{k}) \tag{9.14}$$

and $\epsilon_\mu^{(s)}$ two orthonormal polarization vectors that span the transverse plane:

$$\epsilon_\mu^{(s)} \epsilon^{(s')\mu} = \delta^{ss'}, \qquad k^\mu \epsilon_\mu^{(s)} = \bar{k}^\mu \epsilon_\mu^{(s)} = 0. \tag{9.15}$$

The change in $\hat{h}_{\mu\nu}$ is

$$\begin{aligned} \delta\hat{h}_{\mu\nu} =& ik_\mu \hat{\xi}_\nu + ik_\nu \hat{\xi}_\mu \\ =& 2iak_\mu k_\nu + ib(k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu) + \sum_s ic_s (k_\mu \epsilon_\nu^{(s)} + k_\nu \epsilon_\mu^{(s)}). \end{aligned} \tag{9.16}$$

We can make $\hat{h}_{\mu\nu}$ purely spatial ($h_{0\mu} = 0$), transverse ($k^\mu h_{\mu\nu} = 0$), and traceless ($h_\alpha^\alpha = 0$) by choosing

$$a = -\frac{\bar{k}^\mu \bar{k}^\nu \hat{h}_{\mu\nu}}{8i\omega^4}, \qquad b = \frac{\hat{h}_\mu^\mu}{4i\omega^2}, \qquad c_s = \frac{\bar{k}^\mu {\epsilon^{(s)}}^\nu \hat{h}_{\mu\nu}}{2i\omega^2}. \tag{9.17}$$

Therefore, in this gauge $\psi_{\mu\nu} = h_{\mu\nu}$ and for a wave propagating along x^1, the nonzero components are

$$h_+ = \frac{1}{2}(h_{22} - h_{33}), \qquad h_\times = h_{23}. \tag{9.18}$$

We have already derived the stress-energy tensor of these modes. Solving them in terms of $T_{\mu\nu}$ in the non-relativistic limit and deriving the quadrupole formula is nicely explained in LL §110. However, for completeness I'll summarize it here.

Let's fix some $\hat{k} = \hat{r} = \hat{x}^1$. Observe that the above gauge transformation to spacelike-transverse-traceless $h_{\mu\nu}$ does not change h_{23} and $h_{22} - h_{33}$. Moreover, it is always true that $\psi_{23} = h_{23}$, $\psi_{22} - \psi_{33} = h_{22} - h_{33}$. Therefore, we can immediately extract h_+ and $h_\times$ from the spatial components of (9.11) which have T_{ij} as a source integrated over $d^3\boldsymbol{r}'$. We can use stress-energy conservation to simplify any localized source (to simplify the notation I temporarily drop the prime on r):

$$\begin{aligned}
\int d^3\boldsymbol{r} T_{ij} &= \int d^3\boldsymbol{r}[\partial_k(r^j T_i^k) + r^j \partial_0 T_i^0] \\
&= -\int d^3\boldsymbol{r} r^j \partial_0[\partial_k(r^i T_0^k) + r^i \partial_0 T_0^0] + \text{B.T.} \\
&= -\int d^3\boldsymbol{r}[r^i \partial_0 T_j^0 + r^i r^j \partial_0^2 T_0^0] + \text{B.T.} \\
&= -\int d^3\boldsymbol{r}[T_j^i + r^i r^j \partial_0^2 T_0^0] + \text{B.T.}
\end{aligned} \tag{9.19}$$

where B.T. stands for boundary terms, which vanish for localized sources. This equation implies that

$$\int d^3\boldsymbol{r} T_j^i = -\frac{1}{2}\partial_0^2 \int d^3\boldsymbol{r} r^i r^j T_0^0, \tag{9.20}$$

For non-relativistic sources, $T_{00} \approx \rho$ and we get from (9.11)

$$h_{23} = \frac{2G}{3r}\ddot{q}_{23}, \qquad h_{22} - h_{33} = \frac{2G}{3r}(\ddot{q}_{22} - \ddot{q}_{33}). \tag{9.21}$$

I replaced the second inertial moment with the quadrupole moment (restoring the prime)

$$q_{ij} = \int d^3\boldsymbol{r}'(3r'_i r'_j - \delta_{ij} r'^2) T_{00}, \tag{9.22}$$

by removing the trace. The additional term is proportional to δ_{ij} and doesn't affect (9.21). In the last lecture we derived the expression for the flux:

$$F = \frac{1}{16\pi G}(\dot{h}_+^2 + \dot{h}_\times^2). \tag{9.23}$$

To find the total power, we need to integrate it over a sphere of large radius r. For this purpose it is useful to define the plus and cross polarization vector for a general $\hat{k} = \hat{r}$:

$$\epsilon_{ij}^{(s)}\epsilon_{ij}^{(s')} = \frac{1}{2}\delta^{ss'}, \qquad \hat{r}_i \epsilon_{ij}^{(s)} = 0, \qquad \epsilon_{ii}^{(s)} = 0, \qquad s = 1, 2. \tag{9.24}$$

In terms of these polarization vectors the expression for the flux in an arbitrary direction becomes

$$F = \frac{G}{36\pi r^2}\sum_s (\epsilon_{ij}^{(s)} \frac{d^3}{dt^3} q_{ij})^2. \tag{9.25}$$

Since the polarization vectors give a basis for transverse, traceless matrices the above sum over s

can be expressed purely in terms of $\hat{r}_i$ and δ_{ij}:

$$\sum_s \epsilon_{ij}^{(s)}\epsilon_{mn}^{(s)} = \frac{1}{4}[(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}) - \delta_{ij}\delta_{mn} + (\hat{r}_i\hat{r}_j\delta_{mn}+\hat{r}_m\hat{r}_n\delta_{ij}) \\ - (\hat{r}_i\hat{r}_m\delta_{jn}+\hat{r}_i\hat{r}_n\delta_{jm}+\hat{r}_j\hat{r}_m\delta_{in}+\hat{r}_j\hat{r}_n\delta_{im}) + \hat{r}_i\hat{r}_j\hat{r}_m\hat{r}_n]. \tag{9.26}$$

The integral of this expression over $\hat{r}$ can then be expressed just in terms of δ_{ij}:

$$\int d\hat{r} \sum_s \epsilon_{ij}^{(s)}\epsilon_{mn}^{(s)} = \frac{2\pi}{15}(3\delta_{im}\delta_{jn}+3\delta_{in}\delta_{jm}-2\delta_{ij}\delta_{mn}). \tag{9.27}$$

Using $q_{ii}=0$ we obtain

$$L = \frac{G}{45}\left(\frac{d^3 q_{ij}}{dt^3}\right)^2. \tag{9.28}$$

This is the famous **Quadrupole Emission** formula.

1. Derive (9.26) and (9.27).

   Solution: Write the polarization sum on the l.h.s. of (9.26) as a linear combination of 5 tensor structures that are fixed by symmetries. Determine the five unknowns by imposing (9.24). To derive (9.27) use

   $$\int d\hat{r}\hat{r}_i\hat{r}_j = \frac{4\pi}{3}\delta_{ij}, \tag{9.29}$$

   and

   $$\int d\hat{r}\hat{r}_i\hat{r}_j\hat{r}_m\hat{r}_n = \frac{4\pi}{15}(\delta_{ij}\delta_{mn}+\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}). \tag{9.30}$$

Gravitational waves can be detected by monitoring the distance between two free flying masses. If one of the masses is equipped with a laser and an accurate clock, and the other with a good mirror, the distance between the masses can be measured by timing how long it takes for a pulse of laser light to make the round-trip journey. This is essentially how the mainstream detectors such as LIGO work today. In 2016, LIGO announced the first direct detection of gravitational waves due to the merger of two black holes of roughly 30 solar mass each (`https://www.ligo.caltech.edu/detection`). Such a merger has different phases. The early phase and the longest is called **Inspiral**, during which the two objects are separate and they orbit one another. Our linearized theory gives a good description for the emission during most of this period. Towards the end of the inspiral phase and during the **Merger** and **Ringdown** phase the system is fully relativistic. During this short final period the full nonlinear formulation of general relativity is necessary to make predictions.

2. * A body of mass $m \ll M$ orbits a body of mass M. Is the eccentricity growing or decreasing? Assume non-relativistic velocity. (Use LL book for the energy and momentum loss rates.)

3. * The goal of this problem is to see why massless spin-2 particles are described by a symmetric tensor $h_{\mu\nu}$. Consider a helicity +2 single-particle state with momentum $k^\mu = (\omega, \omega, 0, 0)$:

$$|\boldsymbol{k}, +\rangle = a^\dagger(\boldsymbol{k}, +)|0\rangle. \tag{9.31}$$

By definition, under a rotation along $\hat{k} = \hat{x}$ the state transforms as

$$U_x(\theta)|\boldsymbol{k}, +\rangle = e^{2i\theta}|\boldsymbol{k}, +\rangle. \tag{9.32}$$

The field $h_{\mu\nu}(x)$ has to be constructed out of annihilation and creation operators multiplied by appropriate polarization vectors and mode-functions. In particular, this expansion includes

$$h_{\mu\nu}(x) \supset \varepsilon^*_{\mu\nu}(\boldsymbol{k}, +)a^\dagger(\boldsymbol{k}, +)e^{-ik\cdot x}. \tag{9.33}$$

Apply the same little group rotation to $h_{\mu\nu}$:

$$U_x(\theta)h_{\mu\nu}U_x^\dagger(\theta), \tag{9.34}$$

and focus on (9.33), the term whose momentum remains invariant. This can be calculated in two ways. One is to use (9.32). The other is to transform $\varepsilon^*_{\mu\nu}$.

a Show that choosing $\varepsilon_{\mu\nu}(\boldsymbol{k}, +) = \varepsilon^+_{\mu\nu}(\boldsymbol{k}) + i\varepsilon^\times_{\mu\nu}(\boldsymbol{k})$ gives the correct result.

b Imagine we wanted $h_{\mu\nu}$ to be an antisymmetric 2-form. Could you choose an appropriate $\varepsilon_{\mu\nu}$?

c Causality requires that the commutator of fields (and their Hermitian conjugates) vanish outside of the light-cone. This forces each creation operator to be paired with an annihilation operator. Show that to construct a causal massless spin-2 field we need both helicity +2 and helicity −2 states.

4. * Two objects of mass M have a head-on collision at event $(0, \vec{0})$. In the distant past, $t \to -\infty$, the masses started at $x \to \pm\infty$ with zero velocity.

   (a) Using Newtonian theory, show that $x(t) = \pm(9GMt^2/8)^{1/3}$.

   (b) For what separation is the Newtonian approximation reasonable?

   (c) Calculate, at leading order in $v/c \ll 1$, the gravitational waves emitted by the two objects at $(x, y, z) = (0, R, 0)$, with $R \gg x$ (i.e. calculate $h_{\mu\nu}(t)$ in the gauge $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0$).

5. * Consider a thin metal rod of mass M and length ℓ spinning at frequency ω around a symmetrical perpendicular axis.

a Show that the time-dependent part of the reduced quadrupole moment,

$$J_{ij} \equiv I_{ij} - \frac{1}{3}\delta_{ij} I,$$

where I_{ij} is the quadrupole moment tensor of the bar, is given by

$$\begin{aligned} J_{xx} &= \frac{\ell^2}{12} m \cos^2(\omega t), \\ J_{yy} &= \frac{\ell^2}{12} m \sin^2(\omega t), \\ J_{xy} &= \frac{\ell^2}{12} m \sin(\omega t)\cos(\omega t). \end{aligned} \tag{9.35}$$

b Use this result to compute the gravitational radiation luminosity emitted by the rod. What is its power assuming that $M = 10^3$ g, $\ell = 100$ cm, and $\omega = 1$ kHz. How long does it take for the rod to lose a significant part of its kinetic energy in GWs?

c Estimate the amplitude of the gravitational waves h at a distance of 1 km.

10 Nonlinear Gravity: I. General Covariance

Reading: *Weinberg GR 3.2, 3.3, Landau-Lifshitz vol2 sections 85, 86, 94*

Let us momentarily return to scalar gravity, which at linear order was defined by

$$\Box\varphi = -4\pi G T^{(0)}, \qquad T = T^{\mu}_{\mu}. \tag{10.1}$$

I explicitly added the label (0) to emphasize that the r.h.s. is evaluated to zeroth order in φ. Clearly, the total stress-energy tensor of the theory contains φ. First, because there is energy stored in the gravitational field of massive objects. For instance, in a gravitationally bound system this energy is negative twice the kinetic energy. Secondly, in the relativistic theory there are dilaton waves, which carry energy and momentum even in the absence of matter. So to have a nonlinear theory of scalar gravity one has to specify how the r.h.s. of (10.1) is generalized. However, from a fundamental viewpoint there is no unique answer. We could just postulate that (10.1) is the full description, or we could try to add higher order terms by adding higher order interactions to the action. Of course, different choices would lead to different predictions. Given that scalar gravity fails phenomenologically already at linear order, we don't need to pursue this further.

The situation is very different in the case of spin-2 gravity,

$$\mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = -16\pi G T^{(0)}_{\mu\nu}, \tag{10.2}$$

because the l.h.s. vanishes identically when acted on by ∂^{μ}. This implies that for the equation to be consistent divergence of the r.h.s. must vanish on-shell. This holds to zeroth order in $h_{\mu\nu}$ but no further than that! Again there is energy in the gravitational field of massive objects, and more importantly there are gravitational waves. Matter can source gravitational waves and lose energy, and can absorb them. Therefore, at nonlinear order in $h_{\mu\nu}$ the r.h.s. must look like

$$-16\pi G[T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + t^{(2)}_{\mu\nu} + \cdots] \tag{10.3}$$

where $T^{(n)}_{\mu\nu}$ incorporates the effect of coupling between matter and $h_{\mu\nu}$ at order $h^{n+1}_{\mu\nu}$, and $t^{(n)}_{\mu\nu}$ correspond to purely gravitational contributions such as the stress-energy tensor of the GWs derived in the last lecture. Solving this problem iteratively is nontrivial because once we add a term of order $h_{\mu\nu}$ to the r.h.s., we have to modify the action at $\mathcal{O}(h^2_{\mu\nu})$. This in turn contributes to stress-energy tensor at $\mathcal{O}(h^2_{\mu\nu})$ and hence requires a new term of $\mathcal{O}(h^3_{\mu\nu})$ to be added to the action and so on. There are several arguments at various levels of rigor showing that **General Relativity** is the unique answer to this problem.[10] Below I will give only some hints of why this is the case.

[10] See e.g. "Self-Interaction and Gauge Invariance" by Deser, and "Spin-2 Fields and General Covariance" by Wald.

1. * a) Consider a theory of two decoupled fields ψ and χ,

$$S[\psi,\chi] = S_1[\psi] + S_2[\chi]. \tag{10.4}$$

Show that there are two conserved stress-energy tensors $T_1^{\mu\nu}$ and $T_2^{\mu\nu}$, one for each action. (As a concrete example you can take ψ, χ to be free scalar fields and derive T_1 and T_2 explicitly.)

b) Add a small coupling between the two fields:

$$S \to S + S_{\text{int}}, \qquad S_{\text{int}} = \epsilon \int d^4x \chi^2 \psi^2. \tag{10.5}$$

Show that $T_1^{\mu\nu}$ and $T_2^{\mu\nu}$ are no longer conserved but $\partial_\mu T_1^{\mu\nu} = \mathcal{O}(\epsilon)$ and $\partial_\mu T_2^{\mu\nu} = \mathcal{O}(\epsilon)$. Rather there is only one conserved tensor

$$T^{\mu\nu} = T_1^{\mu\nu} + T_2^{\mu\nu} + T_{\psi\chi}^{\mu\nu}, \tag{10.6}$$

where $T_{\psi\chi}^{\mu\nu}$ is the stress-energy tensor derived from S_{int}.

The situation with gravity is similar. As long as gravitons and matter fields are decoupled, the stress-energy tensors for the matter fields denoted by $T_{\mu\nu}^{(0)}$ is conserved. One can also talk about the conserved energy and momentum of free gravitons obtained by integrating $t_{\mu 0}^{(2)}$ over the space. Once we add the coupling $h^{\mu\nu}T_{\mu\nu}^{(0)}$, neither $T_{\mu\nu}^{(0)}$ nor $t_{\mu\nu}^{(2)}$ are conserved. In particular $\partial^\mu T_{\mu\nu}^{(0)} = \mathcal{O}(h_{\mu\nu})$.

Let's look back at the Maxwell theory for some inspiration. Recall that the l.h.s. of the Maxwell equation $\partial_\nu F^{\nu\mu} = -J^\mu$ is also identically conserved, and hence the r.h.s. is the electric current which is conserved on-shell. We saw that this follows naturally once we realize that the description in terms of A_μ is redundant and the action must be invariant under $U(1)$ gauge transformations

$$\delta A_\mu = \partial_\mu \alpha(x). \tag{10.7}$$

We learned from the Noether theorem that if the action of a set of fields collectively denoted by ψ is invariant under a global symmetry with parameter a, when the parameter of the symmetry transformation is promoted to a function of the spacetime the action changes as

$$\delta S_\psi = -\int d^4x \ \partial_\mu a J^\mu. \tag{10.8}$$

Therefore, to ensure (10.7) we identified $a(x) = \alpha(x)$ and coupled A_μ to the Noether current associated to the global symmetry of the charged fields ψ

$$S_{A\psi} = \int d^4x A_\mu J^\mu. \tag{10.9}$$

This procedure of promoting a global symmetry to a local one (i.e. one with an arbitrary spacetime dependent parameter $a(x)$) by coupling the fields to a gauge field is called **Gauging the Symmetry**. The sourced Maxwell theory is so simple because photons and hence the A_μ field are electrically neutral. Therefore, we do not need to modify the homogeneous part of the action. But the conserved electric current J^μ will generically change after coupling charged particles to photons. For instance the source term in the Maxwell equation coupled to a charged scalar field is

$$J_\mu = -iq(\Phi^\dagger D_\mu \Phi - (D_\mu \Phi)^\dagger \Phi), \tag{10.10}$$

where $D_\mu = \partial_\mu + iqA_\mu$.

From this point of view, what we have done for spin-2 gravity is to gauge spacetime translations $x^\mu \to x^\mu - a^\mu$. When the parameters of translations become spacetime dependent and identified with the gauge parameters that transform $h_{\mu\nu}$ (i.e. $a^\mu(x) = \xi^\mu(x)$) we obtain a gauge symmetry that has several names: **General Covariance, Reparametrization Invariance, Diffeomorphism**, etc. The difference with the Maxwell theory is that graviton is charged under spacetime translations, namely, it carries energy and momentum. From the perspective of the Noether theorem this is saying that the $h_{\mu\nu}(x)$ field also transforms under translations. Therefore, the conserved Noether current $T_{\mu\nu}$, whose zeroth component used to be the density of energy and momentum, has to be modified once matter fields are coupled to $h_{\mu\nu}$. As mentioned above one way to construct the full theory is to iterate, starting from the linear theory.

Another way is to be smart like Einstein was and guess the full answer: Our goal is to write a nonlinear action for $h_{\mu\nu}$ and matter fields ψ such that it is invariant under

$$x^\mu \to \tilde{x}^\mu = x^\mu - \xi^\mu(x), \tag{10.11}$$

and such that for $h_{\mu\nu} \ll 1$ it reproduces our linearized theory.[11] To do this first divide the most general action for $h_{\mu\nu}, \psi$ into two pieces:

$$S[h_{\mu\nu}, \psi] = S_g[h_{\mu\nu}] + S_m[\psi, h_{\mu\nu}]. \tag{10.12}$$

The requirement of general covariance is quite strong. It almost uniquely fixes both actions. Let's focus on S_m first. Recall that at $h_{\mu\nu} = 0$, S_m is the volume integral of a Lagrangian which is a local function of fields and their derivatives and which is a **scalar** under Lorentz transformations. This is achieved by forming invariant products of vectors and derivatives using the Minkowski metric. For general coordinate transformations (10.11) this construction is insufficient:

$$\eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \neq \eta_{\mu\nu}. \tag{10.13}$$

[11]Note the distinction between the linearized diffs under which $\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}$ and full diffeomorphisms. Once ξ^μ is identified as the change of x^μ we have to include $\mathcal{O}(\xi h_{\mu\nu})$ terms such as $\xi^\lambda \partial_\lambda h_{\mu\nu}$ in the transformation law of $h_{\mu\nu}$.

We therefore replace all $\eta_{\mu\nu}$'s with a new metric tensor $g_{\mu\nu}(x)$ (made of $h_{\mu\nu}$) such that

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{10.14}$$

is invariant. (Note that the invariant interval ds^2 is commonly defined like this regardless of metric signature, so in mostly plus signature $d\tau^2 = -ds^2$.)

2. * Show that at linear order

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{10.15}$$

At $\mathcal{O}(h_{\mu\nu}^2)$ the relation depends on what our (so far unspecified) prescription is for the transformation of $h_{\mu\nu}$ at $\mathcal{O}(h_{\mu\nu}\xi^\alpha)$. For instance, we can take (10.15) as the full nonlinear definition of $h_{\mu\nu}$. Note that in any case $h_{\mu\nu}$ is not a tensor,

$$\tilde{h}_{\mu\nu}(\tilde{x}) \neq h_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} \tag{10.16}$$

because $\eta_{\mu\nu}$ is not. Indeed, if a tensor is zero at a point x in one coordinate system, it has to be zero in all others. The inhomogeneous transformation $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \cdots$ breaks this rule because we can start from $h_{\mu\nu} = 0$ and make it nonzero by almost any $\xi^\mu(x)$. Therefore, $h_{\mu\nu}$ **and** $\eta_{\mu\nu}$ **have no place in the covariant action unless they come together.**

3. * Show that under an infinitesimal diffeomorphism $g_{\mu\nu}$ changes as

$$\delta g_{\mu\nu} = g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\nu\alpha}\partial_\mu \xi^\alpha + \xi^\alpha \partial_\alpha g_{\mu\nu}. \tag{10.17}$$

The simplest example of a covariant matter action is perhaps the worldline action

$$S_{\rm pp} = -m\int d\sigma \sqrt{-g_{\mu\nu}(X)\frac{dX^\mu}{d\sigma}\frac{dX^\nu}{d\sigma}}. \tag{10.18}$$

This is what we encountered at linear order, but now it is a fully nonlinear generally covariant theory. More generally, we still need to make two additional changes to have a complete recipe for S_m.

Covariant Derivative. In Cartesian coordinates, derivatives of tensors are higher rank tensors. This is not the case in general coordinate systems; example:

$$\partial_\mu \tilde{A}_\nu = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\partial_\alpha\left(\frac{\partial x^\beta}{\partial \tilde{x}^\nu}A_\beta\right) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\partial_\alpha A_\beta + \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}A_\alpha. \tag{10.19}$$

The reason is clear. It is in general not meaningful to subtract components of vector fields at different spacetime points. For instance, in spherical coordinates A_θ at two different values of θ point in two different directions. However, one can introduce a covariant derivative ∇_μ to construct proper higher rank tensors from lower order ones. The trick is to combine (10.19) (and its analogs) with another non-covariant object made of derivatives of $g_{\mu\nu}$ such that the last term in the transformation cancels

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha \tag{10.20}$$

where $\Gamma^\alpha_{\mu\nu}$ are called Christoffel symbols. They don't form a tensor:

$$\tilde{\Gamma}^\sigma_{\mu\nu} = \frac{\partial \tilde{x}^\sigma}{\partial x^\gamma} \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \Gamma^\gamma_{\alpha\beta} + \frac{\partial x^\sigma}{\partial \tilde{x}^\alpha} \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}. \tag{10.21}$$

More generally, the covariant derivative of a tensor contains one Christoffel symbol per index:

$$\nabla_\mu T^{\beta_1 \cdots}_{\alpha_1 \cdots} = \partial_\mu T^{\beta_1 \cdots}_{\alpha_1 \cdots} + \Gamma^{\beta_1}_{\mu\nu} T^{\nu \cdots}_{\alpha_1 \cdots} + \cdots - \Gamma^\nu_{\mu\alpha_1} T^{\beta_1 \cdots}_{\nu \cdots} - \cdots \tag{10.22}$$

Notation. The ordinary derivative and covariant derivative are sometimes denoted by comma and semicolon, respectively:

$$A_{\mu,\nu} \equiv \partial_\nu A_\mu, \qquad A_{\mu;\nu} \equiv \nabla_\nu A_\mu. \tag{10.23}$$

4. * Show that the covariant derivative satisfies the Leibnitz rule

$$\nabla_\mu (T_{\alpha\beta} S^{\beta\gamma}) = \nabla_\mu T_{\alpha\beta} S^{\beta\gamma} + T_{\alpha\beta} \nabla_\mu S^{\beta\gamma}. \tag{10.24}$$

5. * Show that

$$\Gamma^\sigma_{\mu\nu} = \frac{g^{\sigma\rho}}{2} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{10.25}$$

has the same transformation property as (10.21). This is the unique choice that is symmetric in $\mu \leftrightarrow \nu$ and ensures

$$\nabla_\mu g_{\alpha\beta} = 0. \tag{10.26}$$

What is $\nabla_\mu g^{\alpha\beta}$?

From a purely geometrical point of view there is no unique covariant derivative. To any $\Gamma^\sigma_{\mu\nu}$ that satisfies (10.21) we can add an arbitrary tensor field $C^\sigma_{\mu\nu}$ and thereby get a new covariant derivative. (It is also true that any two choices Γ and Γ' differ by a tensor field.) In particular, the antisymmetric part

$$T^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu} \tag{10.27}$$

is a tensor called the **Torsion Tensor**. Geometrically, the vanishing of torsion tensor is motivated because it guarantees that

$$T^\sigma_{\mu\nu} = 0 \Rightarrow \nabla_\mu \nabla_\nu f(x) = \nabla_\nu \nabla_\mu f(x). \tag{10.28}$$

And the condition (10.26) ensures that (as we will see) the inner product is preserved under parallel transport. Thus apart from being special, there is no strict mathematical rule to enforce (10.25).

From our physical point of view, the situation is completely different. We started from a theory of matter coupled to gravitons, described by $h_{\mu\nu}$. Our goal is to add coupling between matter and $h_{\mu\nu}$ to construct a covariant theory. The only way to construct covariant derivatives using just $h_{\mu\nu}$ (so that $\nabla_\mu \to \partial_\mu$ when $h_{\mu\nu} = 0$) is to choose Christoffels as in (10.25), so we should stick to it.

6. * Rewrite the geodesic equation in terms of Christoffels. Note that they play the role of the gravitational field.

7. * Show that the variation of metric under linear diffs can be written as

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{10.29}$$

   Hint: it might be easier to calculate $\delta g^{\mu\nu}$ first.

So we replace ordinary derivatives with covariant derivatives. This procedure should be familiar from gauge theories like electromagnetism. In order to gauge a global symmetry we have to replace

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu. \tag{10.30}$$

The gauge field A_μ and the Christoffel symbols are sometimes called **Connections**, since they allow us to compare vectors and tensors at different spacetime points.[12]

Volume Element. The last step to build a covariant action is to replace the measure

$$d^4x \to d^4x\sqrt{-g}, \qquad g \equiv \det g_{\mu\nu}. \tag{10.31}$$

By taking the determinant of both sides of

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x), \tag{10.32}$$

one easily sees that the new measure is invariant, and hence the action is invariant if a scalar is integrated using this measure over the whole spacetime. It is convenient to absorb $\sqrt{-g}$ inside the Lagrangian density to make it a scalar density (rather than a scalar).

8. * Is the Levi-Civita symbol (the fully antisymmetric rank 4 object with $\varepsilon_{0123} = 1$) a tensor? If not, can you combine it with $\sqrt{-g}$ to construct a tensor?

[12] A more sophisticated way of saying this is the following. In gauge theories there is an independent copy of the symmetry group at each spacetime point. In order to compare fields that transform nontrivially under the symmetry group (e.g. vectors and tensors in GR) at two different points we need a connection that lives on the link between those two points. It transforms under the symmetry group at the end of the link and inverse of the group at the beginning of the link.

9. * Show that for any vector V^μ

$$\sqrt{-g}\nabla_\mu V^\mu = \partial_\mu(\sqrt{-g}V^\mu.) \tag{10.33}$$

Therefore, such a term in the Lagrangian reduces to a boundary contribution to the action.

To summarize, we learned that in order to couple matter to spin-2 gravity we have to write a generally covariant theory, in which the matter fields no longer see the flat Minkowski metric but rather $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and the coupling to the spin-2 field $h_{\mu\nu}$ is only through this metric. The theory is invariant under reparameterization of coordinates (10.11). For any point x_0^μ there exists a set of ξ^μ such that in the new coordinate system

$$g_{\mu\nu}(x_0) = \eta_{\mu\nu}, \qquad \partial_\sigma g_{\mu\nu}(x_0) = 0. \tag{10.34}$$

Moreover, different ξ^μ in this set are related by Poincaré transformations (see Weinberg 3.2 for a proof). In these coordinate systems the geodesic equation at $X^\mu(\tau_0) = x_0^\mu$ becomes

$$\left.\frac{d^2X^\mu}{d\tau^2}\right|_{\tau_0} = 0, \tag{10.35}$$

where τ denotes the proper time as measured by $g_{\mu\nu}$ (i.e. $\tilde{\tau}$ of the previous lecture). For this reason they are called **Local Inertial Frames**. The existence of local inertial frames for any x_0 is the Einstein's famous **Equivalence Principle**.

Even though GR textbooks often attribute a central role to the Equivalence Principle and general covariance, they do not lead uniquely to Einstein's theory of gravity. First of all, general covariance is an almost empty statement. Any theory could be formulated in a covariant way by following the above steps. It is only when we regard $g_{\mu\nu}$ (or equivalently $h_{\mu\nu}$) as a dynamical variable and introduce the action $S_g[h_{\mu\nu}]$ that we obtain a theory of spin-2 gravity. Equivalence Principle, on the other hand, is not empty as it relates gravity to geometry. However, it doesn't single out spin-2 gravity. Scalar gravity can be completed at nonlinear order into a covariant theory, with the metric given by

$$g_{\mu\nu} = e^{2\varphi}\eta_{\mu\nu}. \tag{10.36}$$

Then the Equivalence Principle holds in the sense that at any point x_0 there is a set of inertial frames in which the laws of physics are those of special relativity. I won't spend much time talking about these matters of principle, because they sound mysterious and deeper than what they actually are. Almost any GR text book devotes at least a section (often a chapter) on Equivalence Principle, which you can refer to for further discussion.

10. * Take $x_0 = 0$, and expand

$$\xi^\mu = A^\mu_\nu x^\nu + B^\mu_{\alpha\beta}x^\alpha x^\beta + \mathcal{O}(|x|^3). \tag{10.37}$$

Show that the condition $\tilde{g}_{\mu\nu}(0) = \eta_{\mu\nu}$ fixes A^μ_ν up to a Lorentz transformation, and the condition $\partial_\sigma \tilde{g}_{\mu\nu}(0) = 0$ uniquely fixes $B^\mu_{\alpha\beta}$. This proves the existence of local inertial frames.

Stress-Energy Tensor. Once we have a covariant formulation of the matter theory, we can give an alternative and more efficient method of deriving the stress-energy tensor: By general covariance

$$0 = \delta_\xi S[\psi, g_{\mu\nu}] = \int d^4x \left(\delta_\xi g^{\mu\nu} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} + \delta_\xi \psi \frac{\delta \mathcal{L}}{\delta \psi} \right). \tag{10.38}$$

We can use (10.29) and the general relation

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} \tag{10.39}$$

which holds between any tensor field and its inverse (satisfying $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$) to write the first term as

$$\begin{aligned} \delta_\xi g^{\mu\nu} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} =& (-\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \\ =& \sqrt{-g} \xi^\mu \nabla^\nu \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} + \text{total derivative} \end{aligned} \tag{10.40}$$

where in the second line we used the symmetry of $g^{\mu\nu}$, the Leibniz rule and (10.33). The total derivative term can be neglected because $\xi^\mu(x)$ is an arbitrary function of x which we take to be zero at infinity. So we have

$$\int d^4x \left(\sqrt{-g} \xi^\mu \nabla^\nu \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} + \delta_\xi \psi \frac{\delta \mathcal{L}}{\delta \psi} \right) = 0. \tag{10.41}$$

The second term is proportional to the equations of motion for matter fields, therefore it must vanish on-shell. Therefore, since ξ^μ are arbitrary

$$\nabla^\nu \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} \bigg|_{\text{on-shell}} = 0. \tag{10.42}$$

We can identify this covariantly conserved tensor as the covariant stress-energy tensor of the matter action:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}}. \tag{10.43}$$

This gives a very efficient prescription to derive a symmetric stress-energy tensor even in the absence of gravity. We can momentarily covariantize the theory by following the above steps, vary with respect to the $g^{\mu\nu}$ and then set it back to $\eta^{\mu\nu}$. The result is an ordinarily conserved tensor.

When $h_{\mu\nu}$ is finite (10.43) has to be regarded as the sum $\sum_{n=0}^\infty T^{(n)}_{\mu\nu}$ in (10.3) of all order corrections to the matter stress-energy tensor due to the gravitational interactions.

11. * Show that for a symmetric rank-2 tensor

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^\mu_\nu) - \frac{1}{2}\partial_\nu g_{\alpha\beta}T^{\alpha\beta} \tag{10.44}$$

11 Nonlinear Gravity: II. Einstein-Hilbert Action

Reading: *Landau-Lifshitz vol2 sections 95, 96*

Next we turn attention to $S_g[h_{\mu\nu}]$. We learned that the action must be generally covariant and hence only a function of $g_{\mu\nu}$, its inverse $g^{\mu\nu}$ and covariant derivatives ∇_μ. Therefore, it is a fully geometrical object. Moreover, at quadratic order in $h_{\mu\nu}$ it has to reproduce $S_g^{(2)}[h_{\mu\nu}]$ that was constructed in the previous lecture. And it has to be a spacetime integral of a Lagrangian which is a scalar density, perhaps up to total derivative terms. It is a simple exercise in differential geometry to show that there is only one such object, the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \tag{11.1}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is called Ricci scalar, the trace of Ricci tensor $R_{\mu\nu}$. The Ricci tensor itself is defined in terms of the trace of the Riemann tensor

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{11.2}$$

which is given by the following expression in terms of the metric and its derivatives

$$R^\mu_{\nu\alpha\beta} = \partial_\alpha\Gamma^\mu_{\nu\beta} - \partial_\beta\Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\mu_{\beta\lambda}\Gamma^\lambda_{\nu\alpha}. \tag{11.3}$$

For the moment, don't worry about the geometric meaning of these quantities. We will explore them in more detail later. Think of the Einstein-Hilbert action as just a messy nonlinear action for $h_{\mu\nu}$, with two derivatives in each term. It is unique because everything can be packed into a covariant expression.

To derive the equation of motion for $h_{\mu\nu}$ we can alternatively vary the action with respect to $g_{\mu\nu}$, or $g^{\mu\nu}$. The variation of S_{EH} gives (for a derivation see LL §95, or wait until I get a chance to talk about first order formalism)

$$\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_{\text{EH}}}{\delta g^{\mu\nu}} = \frac{1}{8\pi G}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R). \tag{11.4}$$

The tensor on the r.h.s. is called the **Einstein Tensor** $G_{\mu\nu}$. At linear order in $h_{\mu\nu}$ it reproduces the l.h.s. of the unique linear spin-2 field equation:

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}\mathcal{E}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta}. \tag{11.5}$$

The higher order terms should be thought of as the sum of purely gravitational contributions to the stress-energy tensor on the r.h.s. of (10.3), taken to the l.h.s. of the equation. Namely

$$G_{\mu\nu} - G^{(1)}_{\mu\nu} = -8\pi G \sum_{n=2}^{\infty} t^{(n)}_{\mu\nu}. \tag{11.6}$$

The same argument which lead to covariant conservation of $T_{\mu\nu}$ defined as (10.43) leads to

$$\nabla^\mu G_{\mu\nu} \equiv 0, \tag{11.7}$$

now as an identity (rather than an on-shell statement), because there is no field other than $g_{\mu\nu}$ in S_{EH}. This is called the **Bianchi Identity**. Thus the full equation of motion for $g^{\mu\nu}$ which is the **Einstein Equation**

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{11.8}$$

is covariantly conserved on-shell.

1. Show that the Einstein equation is a solution to the problem we encountered at the beginning of the lecture. Namely, show that if we move the nonlinear pieces of $G_{\mu\nu}$ to the r.h.s. the result is ordinarily conserved

$$\partial_\mu \tau^\mu_\nu \equiv \partial_\mu \left[T^\mu_\nu - \frac{1}{8\pi G}(G^\mu_\nu - G^{(1)\mu}{}_\nu)\right] = 0, \qquad \text{on-shell.} \tag{11.9}$$

Conserved Energy and Momentum. We Discovered that the stress-energy tensor is covariantly (and not ordinarily) conserved. Can we define a conserved energy-momentum 4-vector P_μ by integrating T^0_μ over a spatial surface, say

$$\int d^3x\ T^0_\mu, \tag{11.10}$$

as we did in Cartesian coordinates? Not surprisingly, the answer is no. Technically because in non-Cartesian coordinates the component μ at different spacetime points means different things, so we cannot simply add them up. (Remember that we needed a connection to subtract them at different points.) However, there is a more physical reason. The gravitational field also carries energy and momentum, even in the absence of matter. Any expression for energy has to incorporate that. $T_{\mu\nu}$ does not, because it vanishes if we set all matter fields to zero:

$$\psi = 0 \Longrightarrow T_{\mu\nu} = 0. \tag{11.11}$$

Furthermore, unlike non-gravitational physics there cannot exist any covariant expression for the local density of energy (or momentum). We learned that at any spacetime point we can set $h_{\mu\nu}$ and its first derivative to zero by choosing an inertial frame (what we cannot do is to set them to zero globally). So a would-be stress-energy tensor of the gravitational field must vanish at that point.

But if a tensor vanishes in some frame it vanishes in all other frames. By repeating this argument at every spacetime point, one arrives at the conclusion that if an energy-momentum tensor exists for $h_{\mu\nu}$ field, it has to be zero everywhere.

The best we can hope for is to find a non-covariant expression for energy-momentum density such that once integrated over a spatial slice it gives a meaningful answer P_μ for the total energy-momentum of the spacetime. This is a nontrivial problem. Mathematicians and Physicists have been working on it up until today.[13] The answer exists provided that we can nail the asymptotics (behavior at large values of r) to some fixed spacetime, such as Minkowski. That is, if $h_{\mu\nu}$ and consequently matter fields which source $h_{\mu\nu}$ fall off sufficiently rapidly as we take $r \to \infty$ with t being kept fixed. Then there exist **Pseudo Tensors of Energy and Momentum**, one example of them given by τ^μ_ν in (11.9). Although τ^μ_ν is not unique, the following integral gives a unique total 4-momentum

$$P_\mu = \int d^3x \tau^0_\mu \tag{11.12}$$

provided it is taken over an entire time-slice that as $r \to \infty$ matches a constant t slice of the fixed asymptotic Minkowski spacetime. Here, I am skipping lots of details about the fall-off rate of deviations from Minkowski, as well as the proof of why under Lorentz transformations of the asymptotic coordinates, P_μ behaves like a 4-vector. They can be found in LL §96. To learn even more about them, you should wait until we acquire more geometric tools.

However, the definition I just gave has a very nice property that everyone can appreciate immediately. On the equations of motion, we have

$$\tau^\mu_\nu|_{\text{on-shell}} = \frac{1}{8\pi G} G^{(1)\mu}{}_\nu = -\frac{1}{16\pi G}\partial_\alpha H^{\alpha\mu}_\nu, \tag{11.13}$$

where I used the relation (11.5) between the linearized Einstein tensor and the second order differential expression $\mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta}$, which can be written as the total derivative of an anti-symmetric rank-3 object:

$$H^{\alpha\mu}_\nu = \partial^\alpha h^\mu_\nu + \delta^\mu_\nu \partial^\beta h^\alpha_\beta + \delta^\alpha_\nu \partial^\mu h^\beta_\beta - \{\alpha \leftrightarrow \mu\}. \tag{11.14}$$

Inserting the $\mu = 0$ component of (11.13) in (11.12), and using the antisymmetry of $H^{\alpha\mu}_\nu$, we obtain a beautiful result:

$$P_\mu = -\frac{1}{16\pi G}\int d^3x\ \partial_i H^{i0}_\mu = -\frac{1}{16\pi G}\oint_{r\to\infty} dS_i\ H^{i0}_\mu. \tag{11.15}$$

Just like the Gauss's law which relates the total electric charge within a region of space to the flux of the electric field through the boundary, the energy-momentum of the entire spacetime is related to the flux of H^{i0}_μ at infinity. There is no longer any reference to the interior of the spacetime. This expression can be used even if there is a hole inside the spacetime (in fact there are holes in GR, with a singularity inside them). Since asymptotically the spacetime is Minkowski, $h_{\mu\nu}$ has to approach zero. Hence, we can return to the linearized theory and treat $h_{\mu\nu}$ as a tensor field which

[13] For an example see `https://arxiv.org/abs/hep-th/9902121`.

lives on Minkowski spacetime. It is no longer a surprise that P_μ is a 4-vector in asymptotia.

2. * Check that the asymptotic expression for energy $\propto \int d\Omega r^2 H_0^{0r}$ is indeed invariant under linearized diffs.

12 Manifolds

Reading: *Wald 2.1, 2.2, 2.3*

In the last lectures, I argued that to study spin-2 gravity at nonlinear level one has to go beyond the flat Minkowski spacetime and consider spacetime metric $g_{\mu\nu}$ as a dynamical degree of freedom to be determined by the distribution of matter. In the next few lectures we develop some useful mathematical tools that are relevant to this problem.[14]

Manifolds. *Manifold is a topological space with a collection of open subsets O_α each of which equipped with a one-to-one and onto map $\psi_\alpha : O_\alpha \to U_\alpha \in \mathbb{R}^d$. When $O_\alpha \cap O_\beta \neq \emptyset$ then $\psi_\alpha \circ \psi_\beta^{-1}$: $\psi_\beta(O_\alpha \cap O_\beta) \to \mathbb{R}^d$ is a smooth (C^∞) function.*

If this is too abstract to digest, think of manifold as the surface of an apple. There are little ants who are doing geometry on this surface. They draw lines and circles and they measure angles. Or imagine ancient civilizations trying to produce a map of their territories. Each map (or chart) covers a finite patch of the Earth which includes one state and parts of the neighboring states just to make clear where the borders are. No single one of these charts covers the entire Earth but their collection does (or could). Moreover, when the charts of two neighboring states are compared they should agree on common regions.

As a more concrete example consider a two dimensional sphere S^2, embedded in three dimensions:

$$X^2 + Y^2 + Z^2 = 1. \tag{12.1}$$

Every point on S^2 is uniquely identified with a point in $\mathbb{R}^3$ satisfying the above condition. We can cover S^2 with six charts $f_{i\pm}$, each of which covering one hemisphere:

$$f_{z+} : \{(X, Y, Z) \in S^2 | Z > 0\} \to \mathbb{R}^2, \qquad f_{z+}(X, Y, Z) \equiv (x^1, x^2) = (X, Y). \tag{12.2}$$

Note that we followed the common practice of denoting embedding coordinates with upper-case symbols and the internal coordinates with lower-case ones. The other five $f_{i\pm}$ are defined in an analogous way to f_{z+}. One can easily check that $f_{i\pm} \circ f_{j\pm}^{-1}$ (where $\circ$ means composition of two maps) are smooth functions when their domains overlap (check).

It is not possible to cover the entire S^2 with just one chart, but we can cover it with two charts using **Stereography**. One chart is obtained by projecting every point $p \in S^2$ on the

[14] These are very standard topics, which can be found in many textbooks. Depending on your depth of interest in math and your patience you can choose different paths. Wald is very mathematical and concise. Carroll is a more pedagogical version of Wald, expanding it by a factor of almost two. Weinberg and Landau-Lifshitz are very minimal on geometrical aspects. LL is as usual very compact but it is sufficient (Zel'dovich learned GR from LL). Nevertheless, if you choose to read a more mathematical book be aware that sometimes they are too orthodox in avoiding coordinate systems. Don't be afraid of $\mu, \nu, \cdots$, it is often extremely efficient and illuminating to use coordinate systems and tensor components. Great physicists like Weinberg and Landau used them all the time so there is no reason to be ashamed of doing so.

plane at $Z = -1$ by extrapolating the line that connects p to $(0, 0, 1)$. This chart doesn't cover $(0, 0, 1) \in S^2$. The other chart can be chosen to be projection on $Z = 1$ plane from the bottom point $(0, 0, -1)$.

Vectors. In flat space, like $\mathbb{R}^3$, once we choose an origin, every two points a and b define a finite displacement vector

$$\Delta \boldsymbol{x}_{ab} = \boldsymbol{x}_a - \boldsymbol{x}_b \tag{12.3}$$

which corresponds to another point on $\mathbb{R}^3$. In general, this is impossible in curved space. For instance, there is no sense in which two points on a sphere define a third point. However, the notion of **Infinitesimal Displacement** continues to exist. So one defines a **Tangent Space** V_p as a d-dimensional vector space at every point $p \in M$. Think of tangent planes on the surface of an apple. At different points these planes are not parallel to one another. Hence, there is no simple way of comparing vectors that belong to the tangent space of two different points $V_p, V_{p'}$.

Vectors can be defined as a list of d real numbers and a transformation law under coordinate transformations. Imagine one of the ancient civilizations had systematically misaligned compasses. Their description of the direction of rivers and passes near the border would differ from their neighbors', but at every point in the overlap of two charts there is a single rotation that relates all directions.

However, there is a more abstract way of defining a vector $v \in V_p$. It is a linear map from the set $\mathcal{F}$ to $\mathbb{R}$ that satisfies the Leibniz rule, where $\mathcal{F}$ is the set of all smooth functions $f : M \to \mathbb{R}$. Namely, for $f, g \in \mathcal{F}$ and $a, b \in \mathbb{R}$

$$v(af + bg) = av(f) + bv(g), \qquad v(fg) = gv(f) + fv(g). \tag{12.4}$$

Amazingly, this is equivalent to the previous definition.

1. Show that if f is a constant function $f(p) = c, \forall p \in M$ then $v(f) = 0$.

2. Show that partial derivatives form a basis for vectors. In a coordinate system, partial derivative at point p (with $x(p) = a$) with respect to μ coordinate is defined as

$$X_\mu : \mathcal{F} \to \mathbb{R}, \qquad \forall f \in \mathcal{F} \Rightarrow X_\mu(f) = \partial_\mu f(p) = \frac{\partial f}{\partial x^\mu}(p), \tag{12.5}$$

The vector transformation law for v^μ under a coordinate transformation

$$x^\mu \to \tilde{x}^\mu(x), \tag{12.6}$$

where $\tilde{x}^\mu(x)$ is a smooth invertible map, follows from the transformation rules for partial derivatives:

$$\frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\nu}. \tag{12.7}$$

Curves. A curve is a function from real numbers (or an interval $I \subset \mathbb{R}$) to the manifold $C : \mathbb{R} \to M$. At every point p that a curve passes through, it defines a **Tangent Vector** $T_p \in V_p$. Formally, it can be defined by its action on smooth functions: For any smooth function on the manifold $f \in \mathcal{F}$, we can combine $f \circ C : \mathbb{R} \to \mathbb{R}$ to get a real function $f \circ C(t \in \mathbb{R})$ (definition: $f \circ g(t) \equiv f(g(t))$). Then

$$T_p(f) = \frac{d}{dt} f \circ C(t_p), \qquad C(t_p) = p. \tag{12.8}$$

Less formally and using a coordinate system, a curve is a set of d functions $x^\mu(t)$. At any point along the curve

$$T_p^\mu = \frac{dx^\mu}{dt}(t_p). \tag{12.9}$$

3. Let's go back to the S^2 example. We can consider a Longitude which is characterized much easier in spherical coordinates (recall that $X^2 + Y^2 + Z^2 = 1$)

$$\theta = \sin^{-1}\sqrt{X^2+Y^2}, \qquad \phi = \tan^{-1}(Y/X). \tag{12.10}$$

Longitude is a constant ϕ curve, $L_\phi : [0, \pi] \to S^2$, described in the f_{z+} chart as

$$x^1(t) = X(t) = \cos\phi \sin t, \qquad x^2(t) = Y(t) = \sin\phi \sin t. \tag{12.11}$$

The tangent vector is then

$$T^\mu(x(t)) = \frac{dx^\mu}{dt} = (\cos\phi\cos t, \sin\phi\cos t) = (\cos\phi\cos\theta, \sin\phi\cos\theta). \tag{12.12}$$

We could have considered a curve that runs twice faster $L'_\phi : [0, \pi/2] \to S^2$:

$$x^1(t) = X(t) = \cos\phi \sin 2t, \qquad x^2(t) = Y(t) = \sin\phi \sin 2t. \tag{12.13}$$

Then we would have $T'(p) = 2T(p)$.

Let's also describe a longitude $L_{\phi=0}$ with $T^\mu = (\cos\theta, 0)$ in $f_{x,+}$ chart. Using tildes for the coordinates in this chart

$$\tilde{x}^1(t) = Z(t) = \cos t, \qquad \tilde{x}^2(t) = Y(t) = 0. \tag{12.14}$$

The tangent is now $\tilde{T}^\mu = (-\sin\theta, 0)$. Note that the naive Euclidean norm $|T| = \sqrt{(T^1)^2 + (T^2)^2}$ is not preserved: $|T| \neq |\tilde{T}|$. Without a *metric* there is no invariant notion of length on a manifold.

Vector Fields. A vector field (or a tangent field) is an assignment of a tangent vector $v_p \in V_p$ to every point $p \in M$. Acting on a smooth function from the manifold to real numbers, $f : M \to \mathbb{R}$, a vector field produces another function $v(f) : M \to \mathbb{R}$. The field is called smooth, if for any f,

$v(f)$ is a smooth function. A vector field generates a set of **Integral Curves** on the manifold by setting

$$T(p) = v_p, \tag{12.15}$$

very much like the electric field lines.

4. As an example consider the field that generates **Latitudes** on the sphere, in f_{z+} chart

$$S^{\mu}_{\theta,\phi} = (-\sin\theta\sin\phi, \sin\theta\cos\phi). \tag{12.16}$$

   The integral curves are lines of constant θ, i.e. latitudes $l_\theta : [0, 2\pi) \to S^2$ defined as

$$x^1(s) = X(s) = \sin\theta\cos s, \qquad x^2(s) = Y(s) = \sin\theta\sin s. \tag{12.17}$$

Commutator, Lie Derivative. Given two vector fields v, w one can define another field

$$[v, w](f) = v(w(f)) - w(v(f)), \qquad \forall f \in \mathcal{F}. \tag{12.18}$$

This is called the commutator of v and w, or the Lie derivative of w with respect to v, $L_v w$. In components

$$[v, w]^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu. \tag{12.19}$$

d mutually commuting and linearly independent vector fields form a coordinate basis. Their integral curves can be used to chart the manifold.

5. Take vector fields T, S corresponding to the tangent vectors of longitudes and latitudes with uniform parametrizatio. We have

$$T = T^\mu \partial_\mu = \partial_\theta, \qquad S = S^\mu \partial_\mu = \partial_\phi, \tag{12.20}$$

   and as a result

$$T^\nu \partial_\nu S^\mu = S^\nu \partial_\nu T^\mu = \cos\theta(-\sin\phi, \cos\phi). \tag{12.21}$$

   Therefore, $[T, S] = 0$. The corresponding coordinate system is (θ, ϕ) coordinates.

   The vanishing of the commutator ensures that if we start from a point p and assign to it $(t = 0, s = 0)$ and move along T field for curve parameter t, i.e. for

$$\Delta x^\mu = \int_0^t dt' T^\mu(x(t')), \tag{12.22}$$

   and then move along S for curve parameter s, we end up at the same point as if we first moved along S for s and then along T for t. Hence, we can unambiguously label points on the manifold (at least in the vicinity of p) by the pair (t, s).

For an example of non-commuting fields consider again longitudes and latitudes, but this time parameterize latitudes non-uniformly. For instance:

$$x^1(s) = X(s) = \sin\theta\cos(s(1+\cos\theta)), \qquad x^2(s) = Y(s) = \sin\theta\sin(s(1+\cos\theta)). \quad (12.23)$$

The tangent vector S changes to $S' = (1+\cos\theta)\partial_\phi$ and the commutator

$$[T, S'] = -\sin^2\theta(-\sin\phi, \cos\phi) \neq 0. \quad (12.24)$$

It is easy to see that the assignment of (t, s) would be path dependent and hence T, S' do not form a coordinate basis.

Tensors. Once the tangent space is defined the construction of tensors is an easy task. Of course as before we could have defined them in terms of their transformation properties. Alternatively, we can define the **Cotangent Space** V_p^* at any point $p \in M$ as the collection of **Covectors**. A covector $\omega \in V_p^*$ is a linear map from the tangent space at p to real numbers: $\omega : V \to \mathbb{R}$. We can choose a basis for them:

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu \quad (12.25)$$

and expand

$$\omega = \omega_\mu dx^\mu \Longrightarrow \omega(v) = \omega_\mu v^\mu. \quad (12.26)$$

The standard transformation property of covariant vectors for ω_μ immediately follows from the transformation property of vector components.

A tensor of rank (k, q) is then defined as a multilinear map $T : \underbrace{V \times \cdots \times V}_{q \text{ times}} \times \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \to \mathbb{R}$, were multilinear means separately linear in every argument. Tensors can be expanded in the coordinate basis ∂_μ, dx^μ and the components satisfy the well-known tensor transformation rules. For instance for a rank $(2, 1)$ tensor

$$T = T^{\mu\nu}_\sigma \partial_\mu \partial_\nu dx^\sigma. \quad (12.27)$$

The upper indices that are summed together with ∂_μ basis are called **Contravariant**, and those that sum with dx^μ are called **Covariant**. **Tensor Fields** are defined analogously to vector fields.

Metric Tensor. In order to talk about almost any local geometric concept on manifolds, such as straight lines, angles, area one needs to have a notion of distance. That is, we need to associate a length to an infinitesimal displacement (a vector) and an angle between two different displacements. This is a symmetric map from two vectors in V_p (for all $p \in M$) to real numbers, which is a symmetric covariant rank-2 tensor field, called the metric g. It has to be non-singular, i.e. if $g(v, w) = 0$ for all $w \in V_p$ then $v = 0$. Therefore, the metric is invertible. The components of the inverse metric g^{-1} are denoted by the same symbol g:

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma. \quad (12.28)$$

Metric gives a natural one-to-one map between elements of tangent space and cotangent space: For every $v \in V_p$ fill one of the slots of the metric with v, i.e. $g(v, .)$. This is an element of V_p^*, with components given by the familiar formula

$$g(v, .)_\mu = g_{\mu\nu} v^\nu. \tag{12.29}$$

Similarly, covariant and contravariant components of tensors can be mapped together.

Embedded Manifolds. Let's have a closer look at embedding of manifolds in higher dimensional spaces. In particular, let's focus on embedding in flat Euclidean (or Minkowski) space of dimension $d+1$. Embedding is equivalent to a condition

$$f(X^A) = 0. \tag{12.30}$$

In the example of S^2, we had $f(X^A) = X^2 + Y^2 + Z^2 - 1$, $X^A = (X, Y, Z)$. Note that f is not unique. In any chart on M, one can use the embedding condition to solve the $d+1$ embedding coordinates in terms of d x^μ:

$$X^A = F^A(x), \tag{12.31}$$

for some functions F^A. Conversely, given the set of F^A one can derive the embedding function. One solutions is

$$f(X^A) = X^1 - F^1(x(\{X^2, X^3, \cdots, X^{d+1}\})) \tag{12.32}$$

where we used the fact that in a non-degenerate situation we can use d of $d+1$ equations (12.31) to solve for d x^μ in terms of $\{X^2, \cdots, X^{d+1}\}$. Unless it causes ambiguity, I will denote the functions $F^A(x)$ as $X^A(x)$ in the future.

The tangent spaces of M at different points are just d dimensional flat sections of the embedding space (they are called **Codimension-1** hypersurfaces). Therefore, any vector v in the tangent space of M is naturally associated to a vector V in the embedding space. In coordinates:

$$V^A = \frac{\partial X^A(x)}{\partial x^\mu} v^\mu. \tag{12.33}$$

On the other hand, any covector in the embedding space naturally gives rise to a covector on M:

$$\omega_\mu = \frac{\partial X^A}{\partial x^\mu} \Omega_A. \tag{12.34}$$

Similarly, all covariant structure of the embedding space is induced on M.

Induced Metric. When thinking of manifolds as embedded in a higher dimensional Euclidean space, like the apple in $3d$ space, its hardly possible to separate the notion of a distance from manifolds. The little ants inherit a metric from the Euclidean metric of the embedding space, since all tangent spaces are planes in $\mathbb{R}^3$. This metric is called the induced metric. More generally, if the embedding space has metric G, then the induced metric is defined by

$$g(v, w) = G(V, W), \tag{12.35}$$

where V, W are the same as v, w viewed as vectors in the embedding space. In any coordinate system x^μ the components of the induced metric can be easily related to the components of the embedding space metric in coordinates X^A in the following way. One considers a generic infinitesimal displacement dx^μ on the manifold. This results in a displacement dX^A as a function of $\{dx^\mu\}$. By definition of the induced metric

$$g_{\mu\nu} dx^\mu dx^\nu = G_{AB} dX^A(x) dX^B(x). \tag{12.36}$$

Of course, this is nothing but a special example of the above general statement about the mapping of cotangent spaces.

6. In f_{z+} chart of S^2 the embedding coordinates are related to x^1, x^2 as

$$X = x^1, \qquad Y = x^2, \qquad Z = \sqrt{1 - (x^1)^2 - (x^2)^2}, \tag{12.37}$$

hence for an infinitesimal displacement on S^2

$$dZ = -\frac{x^1 dx^1 + x^2 dx^2}{\sqrt{1 - (x^1)^2 - (x^2)^2}} \tag{12.38}$$

and

$$ds^2_{S^2} = \frac{1}{1 - (x^1)^2 - (x^2)^2}[(1 - (x^2)^2)(dx^1)^2 + (1 - (x^1)^2)(dx^2)^2 + 2x^1 x^2 dx^1 dx^2]. \tag{12.39}$$

13 Curvature

Reading: *Wald 3.1, 3.2, 3.3*

In the last lecture, we introduced manifolds and metric. Equipped with the metric, we study the local structure of manifolds.

Parallel Transport. Imagine we want to compare two vectors belonging, respectively, to the tangent spaces of two different points: $v \in V_p$, $w \in V_q$. Since $V_p \neq V_q$, such a comparison requires associating an image $\tilde{v} \in V_q$ to v. For instance, we could fix a chart and take

$$\tilde{v} = v^\mu \frac{\partial}{\partial x^\mu}\bigg|_q . \tag{13.1}$$

This is not very satisfactory, since depending on the choice of the chart we get different answers for $\tilde{v}$. We would like to find a more intrinsic way of doing so: The little ants on the apple can follow the following procedure. First, draw a curve that connects p and q. For each curve, one can find an image of v in V_q by following a rule that determines how to transport vectors infinitesimally along the curve. Whatever the rule is, it should be linear in v, so that it fixes the failure of $v^\mu(x)$ to be a vector at $x + dx$, and linear in the displacement dx. The most general rule for such an infinitesimal transformation in a fixed chart is therefore

$$\tilde{v}^\mu = v^\mu - \Gamma^\mu_{\alpha\beta} dx^\alpha v^\beta \tag{13.2}$$

for some matrix $\Gamma^\mu_{\alpha\beta}$. I don't call it a tensor because v^μ is not a vector at $x^\mu_p + dx^\mu$, but $\tilde{v}$ is, so the second term must compensate. $\Gamma^\mu_{\alpha\beta}$ is called an **Affine Connection**. Note that it is not unique. If there is a $\Gamma^\mu_{\alpha\beta}$ such that the above sum is a vector, then it is so for any

$$\Gamma'^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + C^\mu_{\alpha\beta} \tag{13.3}$$

where $C^\mu_{\alpha\beta}$ is a tensor. Conversely, the difference of any two affine connection $\Gamma^\mu_{\alpha\beta}$ and $\Gamma'^\mu_{\alpha\beta}$ is a tensor. This is because $\tilde{v} - \tilde{v}'$ is a vector and so is

$$(\Gamma^\mu_{\alpha\beta} - \Gamma'^\mu_{\alpha\beta}) dx^\alpha v^\beta . \tag{13.4}$$

Since there is already a factor of dx, the difference between transformation laws of vectors at x and $x + dx$ gives an $\mathcal{O}(dx^2)$ contribution, which must be discarded.

We can also transport 1-forms (covectors) or higher rank tensors belonging to the point p. Recall that a 1-form ω is a map from vectors to real numbers. Thus we can impose a further requirement on the rules of transportation, namely

$$\tilde{\omega}(\tilde{v}) = \omega(v). \tag{13.5}$$

This fixes

$$\tilde{\omega}_\mu = \omega_\mu + \Gamma^\beta_{\alpha\mu} dx^\alpha \omega_\beta, \tag{13.6}$$

with the obvious generalization to higher rank tensors.

Therefore, given a transportation rule, we can start from an arbitrary vector (tensor) at a given point p and an arbitrary curve with tangent $T^\mu(t) = dx^\mu/dt$ that passes through this point and associate a vector (tensor) at all points along the curve. This is called parallel transport. Suppose $C(t=0) = p$ then the parallel transport $v(t)$ of a vector $v \in V_p$ is the solution to

$$\frac{d}{dt} v^\nu(t) = -\Gamma^\mu_{\alpha\beta} T^\alpha(t) v^\beta(t), \qquad v(0) = v. \tag{13.7}$$

Covariant Derivative. Once we find a rule for infinitesimal transportation of vectors it is possible to define a covariant derivative of vector fields:

$$\begin{aligned} dx^\nu \nabla_\nu v^\mu(x) =& v^\mu(x+dx) - \tilde{v}^\mu(x+dx) = v^\mu(x+dx) - v^\mu + \Gamma^\mu_{\alpha\beta} dx^\alpha v^\beta \\ =& dx^\nu [\partial_\nu v^\mu + \Gamma^\mu_{\nu\beta} v^\beta]. \end{aligned} \tag{13.8}$$

Unlike the ordinary derivative the covariant derivative of a vector is rank $(1,1)$ tensor. Covariant derivative of other tensors is defined in a similar fashion. The compatibility of parallel transport with the action of vectors, covectors, and tensors (equation (13.6)) ensures that covariant derivative satisfies the Leibniz rule.

In terms of the covariant derivative the parallel transportation of a vector v along a curve (i.e. $v(t)$ with $v(0) = v$) with tangent $T^\mu = dx^\mu/dt$ can be formulated as the condition

$$T^\mu \nabla_\mu v^\nu = 0. \tag{13.9}$$

Torsion. The antisymmetric part of affine connection is called torsion:

$$T^\mu_{\alpha\beta} \equiv \Gamma^\mu_{[\alpha\beta]} = \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\beta\alpha}. \tag{13.10}$$

Torsion is a tensor field. This follows from the fact that for any two vector fields

$$v^\mu \nabla_\mu w^\nu - w^\mu \nabla_\mu v^\nu = [v,w]^\nu + T^\nu_{\alpha\beta} v^\alpha w^\beta. \tag{13.11}$$

The l.h.s. as well as the first term on the r.h.s. (Lie derivative defined in the previous lecture) are vectors, so the last term should also be a vector. Since v, w are arbitrary $T^\nu_{\alpha\beta}$ must be a tensor. It is natural (though not necessary) to set this tensor to zero. This is equivalent to requiring the covariant derivatives commute on scalar fields:

$$\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f. \tag{13.12}$$

Metric Compatibility. As mentioned above there is no unique choice for the affine connection. We can always add a $(1,2)$ tensor field to it. However, there is a natural choice when there is a metric. One asks for the inner product of the vectors to be preserved under parallel transport:

$$T^\mu(t)\nabla_\mu(g_{\alpha\beta}(x(t))v^\alpha(t)w^\beta(t)) = 0. \tag{13.13}$$

The Leibniz rule and the definition of parallel transport (13.9) imply that this equation is satisfied for all T, v, w if and only if

$$\nabla_\mu g_{\alpha\beta} = 0. \tag{13.14}$$

Assuming zero torsion this can be solved for the connection coefficients:

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\sigma}}{2}(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}). \tag{13.15}$$

Geodesics. The generalization of the notion of a straight line to curved manifolds is a curve whose tangent is parallel transported along itself and it is called an affinely parametrized geodesic. This means that

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{d^2x^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{13.16}$$

This is the familiar equation of motion for a free particle we encountered before, parametrized in terms of the proper time. Note however that in Lorentzian manifolds there are null geodesics which satisfy the same equation but the affine paramter τ no longer measures the invariant distance (which is zero). For a different parametrization of the curve $\sigma(\tau)$, we get the condition that

$$T^\mu \nabla_\mu T^\nu = cT^\nu, \qquad c = -\left(\frac{d\tau}{d\sigma}\right)^2 \frac{d^2\sigma}{d\tau^2}. \tag{13.17}$$

This is the equation that we obtained by extremizing the worldline action S_{pp}, with an arbitrary parametrization σ. Therefore, geodesics are curves with (locally) extreme length between two points on the manifold. Null, spacelike, or timelike, it is always possible to choose affine parametrization for geodesics.

1. * Show that the geodesic equation in the affine parametrization can alternatively be obtained from the action

   $$S = \int d\tau g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{13.18}$$

   The Lagrangian is L_{pp}^2. Why is it possible to replace $L \to f(L)$ (where f is a smooth function)?

2. Find geodesics on a $2d$ sphere.

3. * Let us consider a two-dimensional surface M_2 embedded in a three dimensional Euclidean space. Let us define a parallel transport on this surface according to the following rule. If a and b are two points of M_2 and v is a tangent vector at a, then one first considers a parallel transport of v from a to b as defined in the three dimensional space, and then projects on the tangent plane at b. What are the connection coefficients associated with this parallel transport?

   Come up with a different geometric rule for the parallel transport in this setup, such that the torsion is nonzero. Confirm this by direct calculation of the Christofell symbols. Is induced metric covariantly constant with this new rule? If not, can you modify the rule to make induced metric covariantly constant?

Riemann Curvature Tensor. Unlike ordinary derivatives, covariant derivatives of vectors and tensors do not necessarily commute, though they do commute on scalar fields if torsion is zero. This non-commutation is an unambiguous measure of the curvature of space. Let us apply two derivatives on a 1-form field at some point p and in two different orders:

$$[\nabla_\mu, \nabla_\nu]\omega_\alpha \equiv \nabla_\mu \nabla_\nu \omega_\alpha - \nabla_\nu \nabla_\mu \omega_\alpha. \tag{13.19}$$

One can show that this only depends on the value of ω at p. To see this multiply ω by a smooth function f. All derivatives of f cancel in the commutator and we get $f[\nabla_\mu, \nabla_\nu]\omega_\alpha$. So the commutator is a map from a 1-form to a $(0, 3)$ tensor (since there are three lower indices and the object is clearly a tensor). This is a rank $(1, 3)$ tensor called Riemann tensor

$$[\nabla_\mu, \nabla_\nu]\omega_\alpha = R_{\mu\nu\alpha}{}^\beta \omega_\beta. \tag{13.20}$$

There is a direct connection between the Riemann tensor and the path dependence of parallel transport. Recall the original question we started with: *How can one map vectors in V_p to those in V_q, where p, q are finitely separated points?* We introduced parallel transport to allow little ants move vectors from p to q. However, the result can definitely depend on the path. Equivalently, parallel transportation along a closed loop might not bring the vector back to itself.

4. Take a north-pointing vector at $\theta = \pi/2$, $\phi = 0$ on a sphere. Parallel transport it to the north pole along the longitude $\phi = 0$. Transport it back to the equator along $\phi = \pi/2$ longitude, and finally bring it back to the original point along the equator. Now it points Eastward, a rotation by $\pi/2$.

The Riemann tensor gives a local measure of the path dependence of parallel transport. To see this consider two infinitesimal displacement dx_1, dx_2, and apply our parallel transport rule in two different orders: first dx_1 followed by dx_2 and then dx_2 followed by dx_1. The difference between the final vectors is

$$\delta v^\mu = [\partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\mu_{\beta\lambda}\Gamma^\lambda_{\nu\alpha}] dx_1^\alpha dx_2^\beta v^\nu. \tag{13.21}$$

The tensor in square brackets is the same as $-R_{\alpha\beta\nu}{}^\mu$ defined above. Alternatively, we could use an arbitrary 1-form field ω and ask how $\omega(v)$ changes. This is

$$dx_1^\alpha dx_2^\beta [\nabla_\beta, \nabla_\alpha]\omega(v) = -dx_1^\alpha dx_2^\beta v^\nu \omega_\mu R_{\alpha\beta\nu}{}^\mu, \tag{13.22}$$

where we used the rule of parallel transport $dx^\mu \nabla_\mu v = 0$.

Symmetries of Riemann. $R_{\mu\nu\alpha}{}^\beta$ is by definition antisymmetric in $\mu \leftrightarrow \nu$. If the derivative operator ∇_μ is metric compatible, then parallel transport preserves the norm. Hence, the last two components of Riemann must be the generator of a rotation (or a Lorentz transformation). Therefore, lowering the contravariant index by metric, we get

$$R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}. \tag{13.23}$$

Another property of the Riemann follows from the fact that exact forms are closed $d^2\omega = 0$. First note that

$$(d\omega)_{\mu\nu} = \nabla_{[\mu}\omega_{\nu]}, \tag{13.24}$$

where square brackets on indices means antisymmetrization, e.g. $A_{[\mu\nu]} = A_{\mu\nu} - A_{\nu\mu}$ for any matrix $A_{\mu\nu}$. Taking another derivative and antisymmetrizing, we get

$$0 = (d^2\omega)_{\mu\nu\rho} = \nabla_{[\mu}\nabla_\nu\omega_{\rho]} = R_{[\mu\nu\rho]}{}^\sigma \omega_\sigma, \qquad \forall\omega. \tag{13.25}$$

It follows that

$$R_{[\mu\nu\rho]}{}^\sigma = 0. \tag{13.26}$$

This combined with the other two symmetries imply that

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}. \tag{13.27}$$

5. * Prove that in two dimensions $R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}$. The Ricci tensor and Ricci scalar are defined as

$$R = g^{\mu\nu} R_{\mu\nu}, \qquad R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho}. \tag{13.28}$$

6. Calculate the Riemann tensor on a sphere in terms of the metric. Hint: use symmetries.

Theorem: *The knowledge of Riemann tensor and its derivatives at any given point on a smooth manifold are sufficient to uniquely reconstruct the metric in a finite vicinity of that point, up to* *freedom in the choice of coordinates.*

The theorem can be proven by giving an operational way of constructing the metric, for instance by using **Gaussian Normal Coordinates**.

7. * a) Prove that in two dimensions

$$\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} R \tag{13.29}$$

   is a total derivative, and so does not depend on the metric.

   b) Find χ for a sphere and a torus.

8. * Recalling the symmetries of Riemann tensor (antisymmetric in first and second pair of indices, symmetric for exchange of the pairs, cyclic sum of any three indices is zero), find the number of independent components of Riemann in d dimensions.

Einstein-Hilbert Action. The common approach to GR is to start from the Equivalence Principle, which asserts that gravity is equivalent to geometry. Once we decide that the metric is the dynamical variable, there are very few possibilities for writing an action in terms of geometric invariants that characterize local properties of manifolds. We know that the action must be the volume integral with measure $d^4x\sqrt{-g}$ of some scalar quantity. But according to the above theorem apart from the trivial case of a constant Λ the only scalar quantities made just out of the metric field have to expressible in terms of products of derivatives of Riemann. Riemann tensor is itself second order in derivatives. Therefore, at zeroth order in derivatives there is only the cosmological constant:

$$S_{\text{cc}} = \int d^4x \sqrt{-g} \Lambda. \tag{13.30}$$

At second order in derivatives, there has to be only a single Riemann tensor with no derivatives. Due to the symmetries of Riemann tensor there is a unique scalar that can be obtained by taking traces of a single Riemann, namely the Ricci scalar:

$$R \equiv R^{\mu}_{\mu}, \qquad R_{\mu\nu} \equiv R_{\mu\beta\nu}{}^{\beta}. \tag{13.31}$$

Hence, the Einstein-Hilbert action is the unique answer

$$S_{\text{EH}} = \kappa \int d^4x \sqrt{-g} R. \tag{13.32}$$

There is no nontrivial invariant that is made only of first derivatives of metric, because it would vanish in the local inertial frame and make it zero. R is linear in second derivative of $g_{\mu\nu}$, but one can decompose it into a total derivative term and a term that only contains first derivatives (see

LL §93 for the derivation). This **First Order** form is useful to set up the canonical formalism. Higher order terms in derivatives are often irrelevant phenomenologically.

9. Derive the cosmological constant term by varying $S_{\rm cc}$.

10. Derive the Einstein tensor by varying $S_{\rm EH}$ with respect to $g_{\mu\nu}$.

I should again emphasize that spin-2 gravity is not the unique answer to this problem. One could follow the same approach but assume that the determinant of the metric is the only dynamical variable by writing

$$g_{\mu\nu} = e^{2\varphi}\eta_{\mu\nu}. \tag{13.33}$$

For an appropriate choice of κ this would give a nonlinear completion of spin-0 gravity.

11. * Calculate the Ricci scalar for the metric (13.33). What is κ in scalar gravity?

12. * Express variation with respect to φ, i.e. $\delta S/\delta\varphi$ in terms of variation with respect to metric $\delta S/\delta g^{\mu\nu}$.

14 Two-Dimensional De Sitter and Anti-De-Sitter

These are some problems to be discussed in class.

1. De Sitter$_2$ is the $1+1$-dimensional spacetime of constant positive curvature. The metric may be written
$$ds^2 = L^2(-dt^2 + \cosh^2 t \; d\phi^2), \tag{14.1}$$
where ϕ has period 2π. In general dS$_d$ can be embedded as a hyperboloid in $d+1$ Minkowski.

   [Commentary: the $t > 0$ patch of de Sitter$_4$ is a good model for the exponential expansion of the universe during cosmic inflation (with L of perhaps 10^{-30} or 10^{-20} meters), and also for the new inflationary phase we are believed to be entering now due to Dark Energy (with L of about 10 billion light-years).]

   (a) Two nearby observers follow comoving geodesics $\phi = 0$ and $\phi = \epsilon$. What is the rate of change of their distance s at $t = 0$? What is $d^2s/d\tau^2$ to leading order in ϵ? [Commentary: positive curvature makes timelike geodesics diverge.]

   (b) A timelike geodesic passes $\phi = 0$ at $t = 0$ with initial velocity $\frac{d\phi}{d\tau}(0) = \beta$. Calculate $\frac{d\phi}{d\tau}(t)$. [Commentary: expanding universes have "friction".]

   (c) A null geodesic passes $\phi = 0$ at $t = 0$. How many times does it circumnavigate the circle by $t = \infty$? [Commentary: in a rapidly expanding universe, observers become causally disconnected so that it eventually becomes impossible to send a signal from one to the other.]

2. Anti-de Sitter$_2$ is the $1+1$-dimensional spacetime of constant negative curvature. The metric may be written
$$ds^2 = L^2\left(-(1+\chi^2)dt^2 + \frac{d\chi^2}{1+\chi^2}\right). \tag{14.2}$$
Negative curvature makes spacelike geodesics diverge and timelike geodesics converge. A timelike geodesic leaving $\chi = 0$ will eventually return to $\chi = 0$; calculate how long this takes in

   (a) proper time τ along the geodesic

   (b) coordinate time t

   as a function of the geodesic's initial rapidity $\frac{d\chi}{d\tau}(0) = \beta$.

   (c) What is the maximum value of χ reached as a function of β? [Commentary: anti-de Sitter is a perfect lens.]

3. What is the proper acceleration required to stay at fixed χ in AdS$_2$? In what direction do you need to accelerate? In what direction is the "gravitational field" pointing?