## FRW Cosmology

Reading: *Wald 5*

Friedmann-Robertson-Walker cosmology is a model inspired by the large scale homogeneity and isotropy of our universe. It is a spacetime in which there is a set of preferred observers (called *comoving* observers) who all go through the same experience, and every spacetime point is passed by the worldline of one such observer. This means that after synchronizing their clocks at some initial time, there are preferred, maximally symmetric time-slices on which their clocks all show the same time $t$. Up to an overall rescaling, there are three possible choices for the metric of these $3d$ spatial manifolds, depending on whether their scalar Ricci $R^{(3)}$ is zero, positive, or negative. Maximal symmetry means that there is no preferred point or direction, so $R^{(3)}$ fully specifies the curvature. The full metric can be written as

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{1}$$

where $a(t)$ is called the scale factor, and by an appropriate redefinition of $a(t)$ and $r$, $k$ can be taken to be 0 if the spatial slices are "flat" ($3d$ Euclidean), 1 if they are "closed" (spherical/positively curved), or $-1$ if they are "open" (hyperbolic/negatively curved).

By definition, comoving observers are fixed spatial coordinates (which are called comoving coordinates) and they are moving along geodesics, with 4-velocity $u^\mu = (1,0,0,0)$. Consider two nearby observers at comoving distance $\chi$. The physical distance between them is

$$x_{\rm ph} = a\chi, \tag{2}$$

and hence

$$\dot{x}_{\rm ph} = \frac{\dot{a}}{a} x_{\rm ph}. \tag{3}$$

This relation between velocity and distance is called the "Hubble law". The Hubble parameter is thus defined by

$$H \equiv \frac{\dot{a}}{a}. \tag{4}$$

The homogeneity and isotropy of the FRW model restricts the stress-energy tensor to be a function only of $t$ and diagonal

$$T^0_0 = -\rho(t), \qquad T^i_j = p(t)\ \delta^i_j. \tag{5}$$

This is how the stress-tensor of a perfect fluid with energy density $\rho$ and pressure $p$ looks like, though the matter content of the universe might not be an actual fluid.

The ultimate goal in cosmology is to figure out the history of the universe. In the FRW model this corresponds to finding $a(t)$, $\rho(t)$ and $p(t)$ given the matter content and the current state of the universe. One equation that governs this evolution is the energy-momentum conservation

$\nabla_\nu T^\nu_\mu = 0$. The only nontrivial component in this case is $\mu = 0$ component, which gives

$$\dot\rho = -3H(\rho + p). \tag{6}$$

Another equation follows from the $tt$ component of the Einstein equation, and is called the **Friedmann equation**

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{7}$$

The other components are either trivial or derivable from (6) and (7). Finally, we need a relation between $p$ and $\rho$ to solve the system. This is called the **equation of state**, and depends on the microscopic details of the theory. In general the relation can be quite complicated, but it is useful and often a sufficiently accurate approximation to consider a mixture of a few components each with a linear equation of state

$$\rho_i = w_i p_i. \tag{8}$$

If these components are decoupled, then their stress-tensors are separately conserved and (6) holds for each of them separately. Under this assumption, we can integrate $\dot\rho_i = -3H(1 + w_i)\rho_i$ to get

$$\rho_i \propto a^{-3(1+w_i)}. \tag{9}$$

Let's consider some important examples.

- A thermal gas of relativistic particles has $p = \rho/3$. It is commonly denoted as **radiation**

$$\rho_r \propto a^{-4}. \tag{10}$$

- Non-raltivistic matter has negligible pressure, $w = 0$. In cosmology, **matter** usually means non-relativistic matter

$$\rho_m \propto a^{-3}. \tag{11}$$

- Cosmological constant if interpreted as a part of stress-energy tensor corresponds to $T_{\mu\nu} = -\Lambda g_{\mu\nu}$. Hence, on FRW metric it has $\rho_\Lambda = -p_\Lambda = \Lambda$:

$$\rho_\Lambda = \text{const}. \tag{12}$$

Our universe contains all of these three components. It is seen that in an expanding universe radiation dominates at earlier times, while cosmological constant at late times.

There are theoretical reasons to believe not all equations of state can have a physical microscopic description. These lead to various energy conditions that are imposed on components of $T_{\mu\nu}$. **Null Energy Condition** is perhaps the weakest condition, and it is expected to be satisfied classically.

It says that $k^\mu k^\nu T_{\mu\nu} \geq 0$ for all null vectors $k^\mu$. In FRW cosmology, this implies

$$\rho + p \geq 0 \Rightarrow w \geq -1. \tag{13}$$

Cosmological constant saturates the bound, and hence nothing can dominate a $\Lambda$-dominated cosmology in the future.

One common way to specify the energy content of the universe is in terms of the fractional contribution to the Friedmann equation at time $t_0$ (which is usually taken to be the present):

$$\Omega_i \equiv \frac{8\pi G \rho_i(t_0)}{3H_0^2}. \tag{14}$$

Also defining

$$\Omega_K \equiv -\frac{k}{a_0^2 H_0^2}, \tag{15}$$

we have

$$\sum_i \Omega_i + \Omega_K = 1. \tag{16}$$

Hence, it is enough to know $\Omega_i$ and $H_0$ to determine $k$, and also $a_0$ if $k \neq 0$.

1. Find the age of a matter dominated universe $\Omega_m = 1$, with Hubble $H_0$.
   **Solution:** The Friedmann equation is in this case

   $$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{a_0}{a}\right)^3. \tag{17}$$

   Since $k = 0$, we can rescale the comoving coordinates to set $a_0 = 1$. Integrating the above equation we'll find

   $$a(t) = \left(\frac{3}{2} H_0 t\right)^{2/3}, \tag{18}$$

   where we chose the integration constant such that $a(0) = 0$. The Hubble rate is

   $$H = \frac{2}{3t}, \tag{19}$$

   and therefore the age of the universe $t_0 = 2/3H_0$. At $t = 0$ the Hubble rate diverges and there is a curvature singularity. This is called the **Big Bang** singularity.

**Conformal time.** We have seen that transforming the metric into a form that is conformal to Minkowski (in all directions or at least in two directions) is useful in understanding the causal structure of the spacetime. In such coordinates, null rays travel at 45°. This is even more relevant in cosmology because we observe the universe through such null signals, i.e. photons, and recently also gravitons. Hence we introduce the conformal time as

$$\tau(t) = \int^{t} \frac{dt'}{a(t')}, \tag{20}$$

in terms of which the metric looks like

$$ds^2 = a^2(\tau)(-d\tau^2 + \partial\chi^2 + r^2(\chi)d\Omega^2) \tag{21}$$

where $r(\chi) = \sin\chi, \chi, \sinh\chi$ for $k = 1, 0, -1$, respectively. Picking our conformal coordinate as the origin, $\chi = 0$, our past lightcone becomes simply $\tau + \chi = \tau_0$, the present conformal time. As an example, in matter-dominated universe

$$\tau = 3\left(\frac{2}{3H_0}\right)^{2/3} t^{1/3}, \tag{22}$$

where I chose the integration constant such that the big bang occurs at $\tau = 0$.

**To be continued ...**