Cosmological Perturbations

Reference: *Weinberg Cosmology 6.2, 6.3, 6.4*

Our universe is not exactly homogenous and isotropic. The late universe is extremely inhomogenous at short distances, but as we go to cosmological scales, or to earlier times the homogeneities becomes smaller and can be described as perturbtaions on FRW. Observation of these perturbations at various scales can teach us about (1) the initial state of our universe, and since we have to evolve them to the observer, also about (2) the cosmological parameters of the FRW background, and (3) microscopic physics.

Fortunately, the initial perturbations turn out to be extremely small and hence linear perturbation theory gives a very accurate description at large scales and early times. So we start with a toy model of a massless scalar field on flat FRW

$$S = \frac{1}{2}\int d\tau d^3\boldsymbol{x}\ a^2[\phi'^2 - |\nabla\phi|^2], \tag{1}$$

where τ is the conformal time, a the scale factor and prime denotes $d/d\tau$. Since the equation of motion is linear and $\boldsymbol{x}$ independent, it diagonalizes in the momentum basis as

$$\phi''_{\boldsymbol{k}} + 2\mathcal{H}\phi'_{\boldsymbol{k}} + k^2\phi_{\boldsymbol{k}} = 0, \tag{2}$$

where $\mathcal{H} \equiv a'/a = \dot{a} = aH$ is the comoving Hubble parameter, $k^2 = \sum_i k_i^2$ and

$$\phi(\tau, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\phi_{\boldsymbol{k}}(\tau)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{3}$$

$\boldsymbol{k}$ is called the comoving momentum. Note that it is a constant vector. At large k (geometric optics limit), it can be thought of as the covariant components of the ϕ particle 4-momentum, which is conserved.

Equation (2) has to extreme regimes:

1. **Superhorizon** $k \ll \mathcal{H}$: the solution is a superposition of a "growing mode" and a "decaying mode"
$$\phi_{\boldsymbol{k}}(\tau) = c_g + c_d\int^\tau \frac{d\tau'}{a^2}. \tag{4}$$
2. **Subhorizon** $k \gg \mathcal{H}$: the solution is wavelike with an adiabatically changing amplitude,
$$\phi_{\boldsymbol{k}}(\tau) = \frac{1}{a}\left(c_+e^{ik\tau} + c_-e^{-ik\tau}\right). \tag{5}$$

Consider an FRW cosmology with a single energy component with $p = w\rho$. The Friedmann equation in this case gives $H^2 = H_0^2 a^{-3(1+w)}$, from which we learn

$$\frac{1}{\mathcal{H}} \propto a^{\frac{1}{2}(1+3w)}. \tag{6}$$

$\mathcal{H}^{-1}$ is called the **comoving horizon**. It grows if $w > -1/3$. Of course this conclusion also holds if a cosmology contains multiple components, but it is dominated by one or a subset of the components with $w_i > -1/3$. This is the case in our universe during its hot phase, and for most of the subsequent evolution until $z \sim 0.2$, where Λ starts dominating. Hence, the perturbations of our hypothetical field ϕ start in the superhorizon regime at early enough time, and (unless k is extremely small) they will enter the subhorizon regime. In order for the perturbations to remain finite as $\tau \to 0$, we must have $c_d = 0$. This will fix the ratio c_+/c_- in the subhorizon regime.

A useful way to think about the superhorizon regime is the **separate universe** picture. In this regime the gradients are much smaller than the curvature length. Therefore, one can think of points separated by $r \sim 1/k$ as different FRW cosmologies that evolve independently with different homogeneous initial conditions for ϕ and ϕ'. Any initial ϕ' quickly redshifts, and one is left with a time-independent configuration. When $k > \mathcal{H}$, the gradients become relevant and $\phi_{\boldsymbol{k}}$ starts oscillating.

Adiabatic Perturbations

What is the initial condition for perturbations in our universe? Let's consider the simplest option. Imagine at some early time all constituents of the universe are in thermal equilibrium and there is no conserved charge. Then T fully determines the state of an FRW cosmology. Moreover, T is just a time variable. Different values of T correspond to different times in the evolution of the same cosmology. If instead of having $T =$ constant on an early time slice, we let it fluctuate at superhorizon scales, then, as long as these temperature fluctuations are superhorizon, *different patches of the universe go through the same history but with a relative time-shift.* This is called adiabatic initial condition. We can generalize it to include conserved charges, or other energy contents, by ensuring the basic property of having identical (but shifted with respect to one another) histories at superhorizon scales.

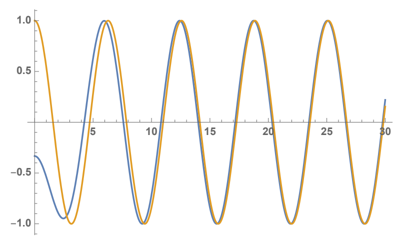
During radiation domination, it is easy to solve for the evolution of adiabatic fluctuations. At subhorizon scales these are just the sound waves satisfying

$$\delta_{\boldsymbol{k}}'' + c_s^2 k^2 \delta_{\boldsymbol{k}} \approx 0, \tag{7}$$

where $\delta_{\boldsymbol{k}} = \delta\rho_{\boldsymbol{k}}/\bar{\rho}$, and $c_s^2 = 1/3$. To match with the superhorizon initial condition, one has to include metric fluctuations and linearize Einstein equations and $\nabla_\mu T^\mu_\nu = 0$ to find

$$\delta_{\boldsymbol{k}} = A_{\boldsymbol{k}} \frac{\omega\tau(\omega^2\tau^2 - 2)\cos(\omega\tau) - 2(\omega^2\tau^2)\sin(\omega\tau)}{\omega^3\tau^3}, \tag{8}$$

where $\omega = c_s k$ and $A_{\boldsymbol{k}}$ is the unknown amplitude. After the modes cross the *sound horizon*, $\omega\tau > 1$, the solution approaches $A_{\boldsymbol{k}} \cos(\omega\tau)$ which is just the solution to (7). The solution (8) and its asymptotic limit both at $A_{\boldsymbol{k}} = 1$ are plotted respectively in blue and orange

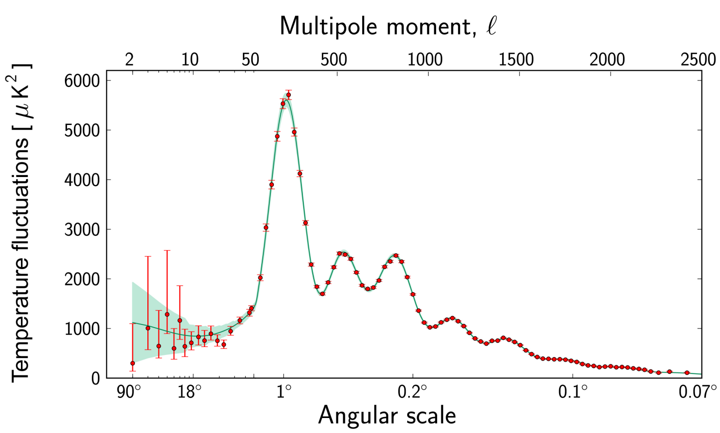


We have good evidence that the perturbations in our universe are predominantly adiabatic, and their amplitude $A_{\boldsymbol{k}}$ is an approximately scale-invariant Gaussian random variable,

$$\langle A_{\boldsymbol{k}} A_{\boldsymbol{k}'} \rangle = \frac{\Delta}{k^{4-n_s}} (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k}'), \tag{9}$$

with $1 - n_s \ll 1$, $\Delta \sim 10^{-9}$.

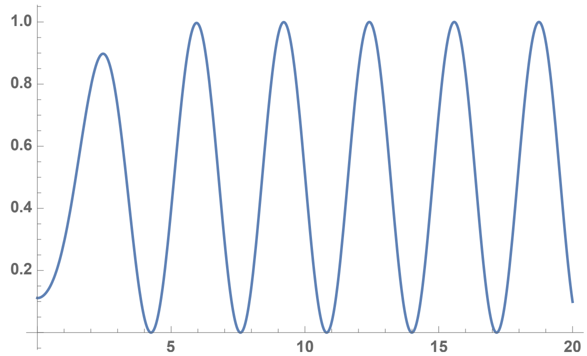
One important manifestation of these perturbations is in the CMB temperature anisotropy, i.e the variation of the CMB temperature as we look in different directions in the sky. This arises from fluctuations in different locations on the last scatter surface and along the trajectory of photons from there to us along our past lightcone. Their statistical average as a function of angular scale looks like



(10)

The solid line is the prediction of the ΛCDM cosmological model, with an adiabatic initial condition.

Let's now imagine taking an snapshot at time τ_* of all k modes in (8), square them, correlate them and multiply by k^3 to cancel the $1/k^3$ in (9). Since the result only depends on $k\tau$ (up to $1-n_s$ corrections), the k dependence of the result would be the same as the τ dependence of $(\delta_{\boldsymbol{k}}(\tau)/A_{\boldsymbol{k}})^2$ at fixed $\boldsymbol{k}$:



(11)

We see that it bears some resemblance with (10). Of course (11) is too simplistic because (i) we only looked at perturbations at a fixed time rather than different angles along our past lightcone, (ii) we neglected $\Omega_m \neq \Omega_b \neq 0$, (iii) $\delta T/T$ is not exactly $\delta\rho/\rho$, (iv) there is a nonzero damping of the sound waves, ...

It is the sensitivity of the map (10) to all these details and the smallness of the error-bars that make CMB such a great probe of the cosmological model. For instance, keeping Ω_m fixed but changing Ω_b would change the relative height of the peaks, making CMB one of the best sources of evidence for the existence of dark matter.

Finally, two examples of less minimal initial conditions are (i) fluctuations in the chemical potentials, and (ii) fluctuations of a light scalar field, both at fixed T.

Structure Formation

Most large scale structures (galaxies, clusters of galaxies, etc) are formed during the matter domination. They are the result of **Jeans instability**, i.e. the dominance of gravitational attraction over the pressure for a large enough body of mass. Since we are dealing with non-relativistic matter the basic idea can be understood in Newtonian gravity. Consider a ball of gas with radius R, density ρ, and $\frac{\partial p}{\partial \rho} = c_s^2 \ll 1$. The Euler equation for the gas velocity reads

$$\dot{\boldsymbol{v}} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p - \rho\nabla\Phi \tag{12}$$

where Φ is the Newtonian potential. We can estimate

$$|\nabla p| \sim \frac{c_s^2\rho}{R}, \qquad |\rho\nabla\Phi| \sim \rho\frac{GM}{R^2} \sim G\rho^2 R. \tag{13}$$

The maximum size at fixed ρ, where it is possible to have a pressure supported distribution can then be estimated to be

$$R_J = \frac{c_s}{\sqrt{G\rho}}. \tag{14}$$

For $R \gg R_J$ the matter distribution will collapse.

A flat matter dominated FRW never collapses. The Hubble expansion gives just enough kinetic energy to the elements to escape the gravitational well. However, in the presence of perturbations over-dense regions will collapse. Using $G\rho \sim H^2$, we find

$$R_J \sim \frac{c_s}{H} \ll \frac{1}{H}, \tag{15}$$

which implies that there is a large range of comoving scales where perturbations are subhorizon but "super-Jeans". They grow in time and form the structures like the ones we observe.