1 Schwarzschild Metric

Reading: *LL §100*

It is generally hard to find analytic solutions of a nonlinear system of equations, like those of Einstein

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.1}$$

This is a hard problem even in vacuum, where $T_{\mu\nu} = 0$. Symmetries usually simplify the task. In particular, assuming spherical symmetry allows us to find a simple and yet very important analytic solution. Spherical symmetry means that we can write the metric as

$$ds^2 = -h(t,r)dt^2 + 2k(t,r)dtdr + l(t,r)dr^2 + r^2 d\Omega^2, \tag{1.2}$$

where $d\Omega^2$ is the metric of a unit round sphere

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{1.3}$$

Here we have chosen one of the coordinates (r) to describe the area of the symmetric spheres ($4\pi r^2$). This is must be familiar from the spherical coordinates in $3d$ Euclidean geometry where

$$ds^2 = dr^2 + r^2 d\Omega^2. \tag{1.4}$$

However, it is a special feature of $\mathbf{R}^3$ that r also measures the distance to the origin. For instance, the unit 3-sphere metric can be written as

$$ds^2 = \frac{dr^2}{1-r^2} + r^2 d\Omega^2. \tag{1.5}$$

Here $4\pi r^2$ determines the area of 2-spheres at coordinate r, but it does not measure the distance between them and the $r = 0$ point. One can think of the coordinate system (1.5) as picking an arbitrary point (say the North pole) on the 3-sphere and foliating the manifold with 2-spheres at constant distance from the North pole. These are a higher dimensional analogs of the latitudes on the surface of the Earth; so I'll call them such.

1. Show that the above metric is equivalent to

$$ds^2 = d\chi^2 + \sin^2\chi d\Omega^2, \tag{1.6}$$

   and find the distance between a point at r and the North pole ($r = 0$).

There is a coordinate singularity at $r = 1$ in (1.5). This is the analog of the equator. The 3-sphere is not singular there, it is just the area of the latitudes that is not a good coordinate for the entire manifold because it ranges over the same values on the northern and southern hemispheres. Similarly, the coordinates in (1.2) might not cover the entire manifold.

We can set $g_{tr} = 0$ in (1.2) by the coordinate transformation

$$t \to f(t,r) \qquad \text{with} \qquad \partial_r f(t,r) = \frac{k(t,r)}{h(t,r)}. \tag{1.7}$$

The g_{tt} and g_{tr} components will change under this transformation, but they remain functions of t and r. It is convenient to parametrize the new metric as

$$ds^2 = -e^{\nu(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + r^2 d\Omega^2. \tag{1.8}$$

2. Find the transformation that takes the metric

$$ds^2 = -dt^2 + 2tdtdr + dr^2 \tag{1.9}$$

to one with $g_{tr} = 0$, and $g_{tt} = e^{2r}$. What is the new g_{rr}?

It is relatively easy to compute the Riemann tensor and hence $R_{\mu\nu}$, R and $G_{\mu\nu}$ for the metric (1.8). For instance

$$G_{tr} = \frac{\dot{\lambda}}{r}, \tag{1.10}$$

$$G_{tt} = e^{\nu-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{e^{\nu}}{r^2}, \tag{1.11}$$

$$G_{rr} = \left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{e^{\lambda}}{r^2}, \tag{1.12}$$

where dot denotes d/dt and prime d/dr. In vacuum $G_{\mu\nu} = 0$. Therefore, from (1.10) we learn that λ is a function of r only. From the combination

$$0 = e^{\lambda-\nu}G_{tt} + G_{rr} = \frac{\nu' + \lambda'}{r} \tag{1.13}$$

we learn that

$$\nu(t,r) = -\lambda(r) + \tilde{\nu}(t). \tag{1.14}$$

But we can set $\tilde{\nu}(t) = 0$ by a coordinate redefinition of $t \to \tilde{t}$:

$$\tilde{t} = \int^t e^{\tilde{\nu}(s)/2} ds. \tag{1.15}$$

Note that this does not involve r and hence does not change g_{tr} or g_{rr}. I will drop the tilde and call the new coordinate t, in terms of which $\nu = -\lambda$. Finally, we can integrate (1.11) to find

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} + r^2 d\Omega^2, \tag{1.16}$$

where r_g (the gravitational radius) is an integration constant.

The fact that spherical symmetry uniquely fixes the vacuum solution up to one parameter r_g (also known as the Birkhoff's theorem) is reminiscent of electromagnetism, where the role of r_g is played by the electric charge Q. This is the consequence of the fact that there is no spherically symmetric electromagnetic or gravitational waves, which in turn results from the nonzero helicity of photons and gravitons.

Newtonian limit. The Schwarzschild metric describes the spacetime outside spherically symmetric matter distribution. As a simple model for the matter distribution consider a non-relativistic, static star. This means that there is a choice of coordinates where $T_{i0} = 0$ and

$$T_{00} = \rho(r), \qquad \rho(r) \underset{r\to\infty}{=} 0. \tag{1.17}$$

In these coordinates we continue to have $g_{tr} = 0$ and $\dot{\nu} = \dot{\lambda} = 0$ inside the star. In the non-relativistic limit, the spacetime is close to Minkowski, so we can linearize in ν and λ, moreover $T_{ij} \approx 0$. We can now use

$$8\pi G\rho = G_{tt} = \frac{1}{r^2}(\lambda + r\lambda') + \mathcal{O}(\nu^2, \lambda^2, \nu\lambda), \tag{1.18}$$

and the trace of the Einstein equation

$$R = -8\pi G T^{\mu}_{\mu} \approx 8\pi G e^{-\nu}\rho, \tag{1.19}$$

where using $\dot{\nu} = \dot{\lambda} = 0$, and to linear order in ν, λ

$$R^{(1)} = \frac{2\lambda}{r^2} - \frac{1}{r}(r\nu'' + 2\nu' - 2\lambda'). \tag{1.20}$$

Eliminating λ gives

$$\nu'' + 2\frac{\nu'}{r} \approx 8\pi G\rho \tag{1.21}$$

which is the Poisson equation for the Newtonian potential $\phi = \frac{\nu}{2}$. Its solutions is

$$\phi = -\frac{GM}{r}, \qquad M \equiv \int_0^\infty 4\pi r^2 \rho(r) dr. \tag{1.22}$$

3. What is the necessary condition on the radius of the star for the non-relativistic approximation to be valid?

Orbits. Free test particles move on geodesics, described by

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}, \tag{1.23}$$

where τ is the proper time for massive particles and the affine parameter for the massless ones.

4. Derive the geodesic equation by varying the point particle action with respect to x^μ

$$S_{\rm pp} = -m \int d\sigma \sqrt{-g_{\mu\nu}(x)\ \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}. \tag{1.24}$$

At large distances, the orbits of the Schwarzschild metric are well approximated by Keplerian orbits, however when $r \sim r_g$ there are significant differences. For instance, there is an innermost circular orbit as we will see next.

Without loss of generality, we can choose the circular geodesic to be in the $\theta = \frac{\pi}{2}$ plane. Then

$$u^\mu = (\dot{t}, 0, 0, \dot{\varphi}), \qquad \dot{t} \equiv \frac{dt}{d\tau}, \quad \dot{\varphi} \equiv \frac{d\varphi}{d\tau}. \tag{1.25}$$

Note that we changed the definition of dot from d/dt to $d/d\tau$.

Since the metric is independent of t and φ, we have two conservation laws:

$$\frac{d}{d\tau}u_t = 0 \Rightarrow \dot{t} = \frac{e}{1 - \frac{r_g}{r}}, \tag{1.26}$$

$$\frac{d}{d\tau}u_\varphi = 0 \Rightarrow \dot{\varphi} = \frac{\ell}{r^2}, \tag{1.27}$$

where the constants e and ℓ are, respectively, the energy and angular momentum per unit mass. The normalization of the 4-velocity $u^\mu u_\mu = -1$ implies

$$e^2 = \left(1 - \frac{r_g}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right). \tag{1.28}$$

Finally, the r component of (1.23) implies

$$-\frac{r_g}{r^2}\dot{t}^2 + 2r\dot{\varphi}^2 = 0. \tag{1.29}$$

Substituting the above expressions for $\dot{t}, \dot{\varphi}$, and e gives

$$\ell^2 = \frac{r_g r}{2\left(1 - \frac{3r_g}{2r}\right)}. \tag{1.30}$$

The minimum value for r for this to have a solutions is $r_{\rm min} = \frac{3}{2}r_g$.