# GR HW 2

Due Nov 21 at 11am

*Goals: Symmetries and conserved quatities, Stress-energy tensor, Point-particle action, Time-reversal*

1. For a point-like particle derive conserved quantities corresponding by Noether theorem to invariance under Lorentz transformations $X^\mu \to X^\mu + \omega^\mu_\nu X^\nu$. Do it following the procedure we outlined in class, namely by considering Lorentz transformations with time dependent parameters $\omega^\mu_\nu \to \omega^\mu_\nu(\sigma)$.

2. In addition to the physical time translation invariance $X^0 \to X^0 + \epsilon$ the action for a point-like particle is obviously invariant also under the worldline time translation (a special case of worldline reparametrization),
$$X^\mu(\sigma) \to X^\mu(\sigma + \epsilon) \ .$$
Normally, by Noether theorem one expects to obtain the conserved "worldline energy", corresponding to this symmetry. Calculate this energy.

3. Derive the equations of motion following from the action
$$S_{\text{Polyakov}} = \frac{1}{2}\int d\sigma e^{-1}\left(\dot{X}^\mu \dot{X}_\mu - e^2 m^2\right), \qquad \dot{X}^\mu = \frac{dX^\mu}{d\sigma},$$
where $e$ (which is called the **worldline metric**) is an additional dynamical variable. Are these equations equivalent to the equation of motion for a point-particle? Note that in this action (unlike the particle Nambu-Goto action), one can set $m = 0$. Does it reproduce what you expect for a massless particle? What is the transformation of $e$ under reparametrizations of $\sigma$?

4. For a system of particles, time-reversal symmetry (**T**) means that if all velocities are reversed in the final state, they trace back their trajectories to the initial positions. Equivalently, the action is invariant under $x^i(t) \to x^i(-t)$.

   (a) Is the motion of a charged particle in a time-independent, external electric field **T**-invariant?

   (b) What about the motion in a time-independent, external magnetic field?

   (c) In reality, electromagnetic fields are sourced by charged particles. Applying **T** to these sources, find the transformation of $\mathbf{E}, \mathbf{B}$ and $A_\mu$ under **T**.

   (d) Does your answer to (a),(b) change if electromagnetic field is transformed as in (c)?

5. The stress-energy tensor of a relativistic **perfect fluid** is given by
$$T^\mu_\nu = (\rho + p)u^\mu u_\nu + p\delta^\mu_\nu \tag{1}$$
where $\rho, p$ and $u^\mu$ are respectively fluid density, pressure and 4-velocity. $u^\mu$ is a unit time-like vector $u_\mu u^\mu = -1$. For a non-relativistic fluid $\rho = mn + \rho_K$ where $m$ is the mass of fluid particles and $n$ is their number density and $\rho_K$ is the density of kinetic plus potential energy. In this limit $mnv^2 < p \sim \rho_K \ll mn$ where $v$ is the fluid 3-velocity. The equation of state $p(n)$ fixes the pressure in terms of number density. Take the non-relativistic limit of fluid stress-energy conservation $\partial_\mu T^\mu_\nu = 0$ to find the time-evolution of $\rho$ and $\boldsymbol{v}$.