GR HW 2

Due Nov 28 at 11am, in class

Goals: Symmetries and conserved quatities, Stress-energy tensor, Forces mediated by particles, Redshift

1. For a point-like particle derive conserved quantities corresponding by Noether theorem to invariance under Lorentz transformations $X^\mu \to X^\mu + \omega^\mu_\nu X^\nu$. Do it following the procedure we outlined in class, namely by considering Lorentz transformations with time dependent parameters $\omega^\mu_\nu \to \omega^\mu_\nu(\sigma)$.

2. In addition to the physical time translation invariance $X^0 \to X^0 + \epsilon$ the action for a point-like particle is obviously invariant also under the worldline time translation (a special case of worldline reparametrization),
$$X^\mu(\sigma) \to X^\mu(\sigma + \epsilon) \ .$$
Normally, by Noether theorem one expects to obtain the conserved "worldline energy", corresponding to this symmetry. Calculate this energy.

3. Derive the equations of motion following from the action
$$S_{\text{Polyakov}} = \frac{1}{2}\int d\sigma e^{-1}\left(\dot{X}^\mu \dot{X}_\mu - e^2 m^2\right), \qquad \dot{X}^\mu = \frac{dX^\mu}{d\sigma},$$
where e (which is called the **worldline metric**) is an additional dynamical variable. Are these equations equivalent to the equation of motion for a point-particle? Note that in this action (unlike the Nambu-Goto action), one can set $m = 0$. Does it reproduce what you expect for a massless particle? What is the transformation of e under reparametrizations of σ?

4. The stress-energy tensor of a relativistic **perfect fluid** is given by
$$T^\mu_\nu = (\rho + p)u^\mu u_\nu + p\delta^\mu_\nu \tag{1}$$
where ρ, p and u^μ are respectively fluid density, pressure and 4-velocity. u^μ is a unit time-like vector $u_\mu u^\mu = -1$. For a non-relativistic fluid $\rho = mn + \rho_K$ where m is the mass of fluid particles and n is their number density and ρ_K is the density of kinetic plus potential energy. In this limit $p \sim \rho_K \sim mnv^2 \ll mn$ where v is the fluid 3-velocity. The equation of state $p(n)$ fixes the pressure in terms of number density. Take the non-relativistic limit of fluid stress-energy conservation $\partial_\mu T^\mu_\nu = 0$ to find the time-evolution of ρ and $\boldsymbol{v}$.

5. Find the number of degrees of freedom (polarizations) of dilaton (massless scalar), photon (massless spin 1), and graviton (massless spin 2) in $d = 5$ spacetime dimensions. (Are there photons or gravitons in $d = 2$ or $d = 3$?)

6. Twin brothers are separated at birth: one stays at sea level the other lives in mountains at altitude 2km. Which one will outlive his brother (will see him die) and by how much?

7. What's the temperature difference (in thermodynamic equilibrium at room temperature) between the floor and the ceiling (3m high room, due to gravity)?

Bonus problems:

8. Calculate the differential scattering cross-section in attractive $1/r$ potential with strength C. Use Born approximation.

9. Calculate the differential scattering cross-section for two non-relativistic particles of mass m, M ($m \ll M$), due to the tree-level exchange of a scalar field φ_c coupled universally to the trace of the stress-energy tensor, i.e. $S_{\text{int}} = \sqrt{4\pi G} \int d^4x \varphi_c T^\mu_\mu$. Is there a choice of C in the previous problem that gives the same answer as what you find here in the rest frame of the mass-M particle?