Overview of Financial Markets and Instruments

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Outline

Bibliography

J. Hull.
Options, Futures, and other Derivatives, 6-th edition.
Prentice Hall, 2005.

J. C. Cox and M. Rubinstein.
Options Markets.
Prentice Hall, 1985.

Z. Bodie, A. Kane and A. J. Marcus.
Investments, 6-th edition.
McGraw-Hill/Irwin, 2004.

Financial Markets and Instruments

- Financial instruments (assets, securities): claim to future cash-flows.
- **Financial** assets vs. **real** assets.
- Financial securities are **traded** in Financial markets.
- Financial market: ‘place’ where **supply** and **demand** of financial assets meet.
- Role of financial markets:
  - ▷ provide investment opportunity for agents with surplus (buy securities);
  - ▷ provide financing opportunity to agents with need of capital (issue securities).

  Hence financial markets permit to redistribute wealth and consumption over time.

Financial Markets Players

- Agents differ according to their needs/preferences/behaviour.
- Some agents invest; some issue securities; some act as intermediaries (not exclusive roles).
- HOUSEHOLDS.
- FIRMS.
- BANKS; other FINANCIAL institutions.
- MUTUAL funds.
- PENSION funds; INSURANCE companies.
- SOVEREIGN entities.
- ...

Financial Markets

- Securities can be traded on:
  - Primary market, i.e. when first issued, or
  - after issuance, i.e. in the Secondary market.
- We distinguish between
- Organized markets (exchange):
  - Only specialized agents (members) can operate;
  - traded securities must satisfy given conditions;
  - trades are standardized;
  - demand and supply do not directly meet.
- **O**ver **T**he **C**ounter (OTC) markets:
  - Less restrictive rules for agents and traded assets;
  - counterparties directly meet;
  - trades are not standardized.
- Primary securities: Stocks and fixed income (bonds).

Stocks

- Common Stock or equities: represent **ownership** of a corporation
- Shareholders are entitled to
  - **voting rights** in shareholders'meetings: take part in corporate governance;
  - receiving part of firm profits as **dividends**.
- Features of common stock:
  - **residual claim** to firm assets;
  - **limited liability**: loss is limited to original investment.
- A publicly held corporation stock is traded in a market (otherwise: privately held corporation).
  A stock traded on organized exchange is listed,
  e.g. NYSE, AMEX, LSE, ...
- Preferred stock: entitled to receive fixed dividends/no voting rights.

Fixed Income Securities

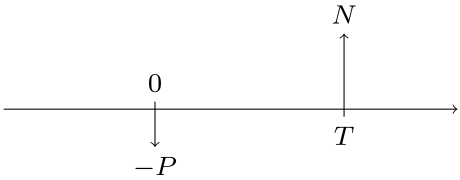
- Financial assets generating cash-flows computable according with **prespecified rules**.
- Loan is split in many obligations (bonds):
  one borrower, many lenders (bondholders).
- Bonds can then be **transferred** in financial markets.
- Bondholders receive periodical payments of interest (coupon) and capital at **maturity**.
- Credit Risk: failure of payments (interest or capital) or change in credit quality.
- Distinguish between
  - ▷ money market securities: short term, low risk;
  - ▷ bonds: longer maturities, higher risk.

. . . Fixed Income Securities

- Many bonds provide fixed (**known**) cash-flows; e.g. zero-coupon bonds and coupon bonds.
- Some bonds pays coupons and/or nominal linked to economic variables:
  - interest rates (floater);
  - market index;
  - currencies;
  - inflation;
  - stocks.
- Other bonds contains options: callable and convertible bonds.
- Some bonds are traded on organized exchanges; most are OTC.

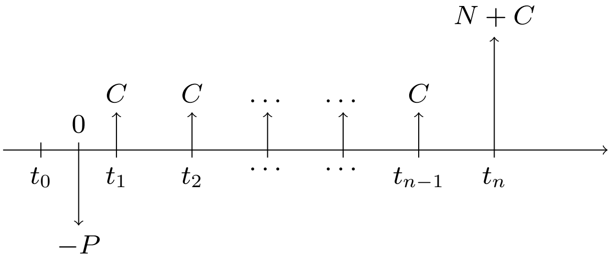
Zero-Coupon Bonds (ZCB)

- Simplest type of financial security.
- Bonds that pays no coupon; ‘sells at discount’
  $\rightsquigarrow$ **pure discount bond**.
- Investor pays price P at time 0; receives nominal (face, par) value N at maturity T.
  Think of P/N as riskless **discount factor**.
- Clearly, $P < N$; $N - P$ is the discount.
- Typically maturity is 1 year or less.



Coupon Bonds

- Also known as bullet bonds/coupon bearing bonds.
- Payments occur at times $0 < t_1 < t_2 < \ldots < t_n = T$;
  T is the maturity;
  typically $t_i - t_{i-1} = \Delta$ (e.g. $\Delta = 1/2, 1/4, \ldots$).
- Investor pays price P at time 0.
- Receive coupons C at $t_1, \ldots, t_n$ and nominal (face, par) value N at t_n.



Coupon Bonds

- If $P <, =, > N$, bond sells below, at, above par.
- discount $= N - P > 0$; premium $= P - N > 0$.
- Coupon measured as percentage of nominal: $c = C/N$ coupon rate, $c' = c\Delta$ nominal rate.
- Bond is frequently quoted as with its clean price (flat price) Q, related to the price actually paid P (dirty or full price) through

$$P = Q + A,$$

where A is the accrued interest:

$$A = \frac{-t_0}{t_1 - t_0} C$$

t_0: issue or last coupon date.

Market Indices

- Pure numbers reflecting market behaviour.
- Convey information for investor on market trend.
- Benchmark for mutual funds/derivatives/...
- Usually computed as weighted arithmetic average of market prices.
- Example: S&P500 (Standard & Poor's 500); average of 500 large US corporate common stock prices, weighted by number of shares:

$$\text{S\&P500}_t = \frac{I_t}{I_{t_0}},$$

where $I_t = \sum_{i=1}^{500} n_t^i S_t^i$ and n_t^i, S_t^i are number of shares outstanding and market price at t of i-th stock.

Derivatives

- DERIVATIVE CONTRACTS: as opposed to primary securities. (combined together sometimes)
- Financial instruments whose cash-flows depend on the value of one or more underlying economic variables.
- Underlying:
  - stock
  - fixed income
  - interest rate
  - market index
  - commodity
  - currency
  - derivative
  - credit risk
  - ...

Perfect Markets

- Very convenient for analysis to assume that markets are perfect:
- AGENTS:
  - **rational**, nonsatiated;
  - **price-takers**;
  - share same information;
  - default riskless (no credit risk).
- MARKETS: frictionless
  - continuously open;
  - securities are **infinitely divisible**;
  - **short selling** is allowed;
  - no **taxation**;
  - no **transaction costs**;
- Enough if these hypotheses hold for some (large) investor.

. . . Perfect Markets

- STRATEGIES: there are no arbitrage opportunities (**NA**).
  Arbitrage opportunity (or free lunch): strategy involving available securities providing
  - nonnegative cash-flows at every time and state of nature;
  - a positive cash-flow at some time, with positive probability.

  Hence it is a riskless strategy that may result in a profit.
- NA is the key property:
  - necessary for equilibrium;
  - implies the Law of one price (LOP): two strategies providing the **same cash-flows** have the **same value**; if not, sell the more expensive, buy the cheaper (arbitrage).

Long/Short Positions

- When faced with a security, an investor could take a long position or a short position.
- LONG POSITION: buying a security
  - profit if price rises;
  - **unlimited** potential gain, **limited** liability.
- SHORT POSITION: short selling, i.e. selling a security that is not owned.
  - Security is borrowed form a third party's portfolio;
  - short seller gains if price falls;
  - short seller must provide any cash-flow paid by security;
  - **limited** potential gain; **unlimited** potential loss;
  - requires margin as collateral.

Outline

Time Value of Money

- Lending/borrowing provides interest/has a cost.
- €C today (**principal**) is the same as €$C\,b(t)$ at t;
  $b(t)$: **accumulation factor**.
  €M at t are the same as €$M\,d(t)$ at 0 (today);
  $d(t)$: **discount factor**.
  $b = 1/d$; $b \uparrow$, $d \downarrow$; $b(0) = d(0) = 1$.
- $C\,b(t)$ is the accumulated value of C;
  $M\,d(t)$ is the present (discounted) value of M.
- $b(t) - 1$ is interest per unit of principal (interest rate).
- Different rules for computing interest: $b(t) =?$, $d(t) =?$.
- ▷ Simple Interest;
  ▷ Discretely Compound Interest;
  ▷ Continuously compound Interest.

Simple Interest

- Interest is **proportional** to time:

$$\boxed{b(t) = 1 + L\,t; \;\; d(t) = (1 + L\,t)^{-1}.}$$

- L: 1 year interest rate.
- Example: LIBOR (**L**ondon **I**nter**B**ank **O**ffered **R**ate) rates:
  - ▷ Rates at which large UK banks lend/borrow deposits between them (EURIBOR in the € area);
  - ▷ standard reference rate for many other contracts/derivatives;
  - ▷ maturities: 1 day to 1 year.
- Example: 6 months LIBOR: $L_{0,1/2} = 4\%$. Borrow now 1000000£, pay $1000000(1 + 0.04 \cdot 1/2) = 1020000$£ in 6 months.

Discretely Compound Interest

- Fix $k > 0$. After each $\frac{1}{k}$-th (e.g. $k = 1, 2, 4, 12, \ldots$) of year interest is compounded i.e. added to principal.
- After n periods, accumulated value is
$$\boxed{b(n) = \left(1 + \frac{R^k}{k}\right)^n;}$$
interpolate linearly between points to get $b(t)$, all $t \geq 0$.
- Equivalent 1 year interest rate is R given by
$1 + R = \left(1 + \frac{R^k}{k}\right)^k;$
$\rightsquigarrow R^k = k[(1+R)^{1/k} - 1].$
- Example: €1000 invested for 6 months at 7% compounded monthly gives
$1000\,(1 + \frac{1}{12}0.07)^6 = 1035.51$€. Equivalent 1-year rate is 7.23%.

Continuous Compounding

- Consider discrete compounding when $k \to \infty$: interest is compounded continuously.
- For fixed 1 year interest rate R, $R^k \to Y \doteq \log(1+R)$ as $k \to \infty$.
- $(1 + \frac{R^k}{k})^{kn} \to e^{Yn}$ as $k \to \infty$;
- $\rightsquigarrow$ $\boxed{b(t) = e^{Yt} = (1+R)^t; d(t) = e^{-Yt} = (1+R)^{-t}}$
- Y: force of interest;
  $b'(t)/b(t) = Y$, i.e. $b(t + \Delta t) \approx b(t)(1 + Y\Delta t)$.
- Note that $b(t+s) = b(t)b(s)$.
- Example: 500\$ invested at 2% force of interest for 1 and 1/2 years gives $500\, e^{0.02 \cdot 1.5} = 515.23\$$.
  Corresponding interest rate is 2.02%.

Term Structure of Interest Rates

- Fix time 0: today.
- Suppose a discount function $d(\cdot) \equiv d_0(\cdot)$ (price of hypothetical ZCB for any maturity) is given.
- Assume continuous compounding; interest rate prevailing for borrowing/lending up to time t is $R_{0,t}$ ($Y_{0,t}$ corresponding force of interest); $d(t) = (1 + R_{0,t})^{-t} = \mathrm{e}^{-Y_{0,t}t}$.
- The function $t \to R_{0,t}$ is the Term Structure of Interest Rates (at 0); it can take several shapes: flat, **normal** (increasing), **inverted** (decreasing), humped, spoon-shaped.
- Knowing $d(\cdot)$ you get $R_{0,\cdot}$, and viceversa.

Forward Rates

- $R_{0,t}$ is a spot rate, i.e. a rate prevailing now for $[0, t]$.
- Fix $0 \leq t < s$; $R_{0,t,s}$: forward rate contracted now, for borrowing/lending over $[t, s]$ ($Y_{0,t,s}$: corresponding force of interest).
- Buy 1 ZCB maturity s, sell $d(s)/d(t)$ ZCB maturity t; no cash-flow in 0; cash-flow in t: $-d(s)/d(t)$; cash-flow in s: 1. Hence

$$\frac{d(s)}{d(t)}\left(1 + R_{0,t,s}\right)^{s-t} = 1.$$

$$\rightsquigarrow \boxed{(1 + R_{0,t})^t (1 + R_{0,t,s})^{s-t} = (1 + R_{0,s})^s.}$$

- Forward rates are 'implied' by the term structure now; $R_{0,t,t} = R_{0,t}$, $Y_{0,t,t} = Y_{0,t}$.

Forward Rates

- ▷ $1 + R_{0,s}$ weighted geometric average of $1 + R_{0,t}$, $1 + R_{0,t,s}$;
  ▷ $Y_{0,s}$ weighted arithmetic average of $Y_{0,t}$, $Y_{0,t,s}$.
- Let $s \downarrow t$ in $Y_{0,t,s}$. Get instantaneous forward rates:

$$r_{0,t} \doteq \lim_{s \downarrow t} Y_{0,t,s} = -\frac{\mathrm{d}}{\mathrm{d}t} \log d(t) = -\frac{d'(t)}{d(t)}.$$

  $t \longrightarrow r_{0,t}$: term structure of instantaneous forward rates
- Also: $d(t) = \mathrm{e}^{-\int_0^t r_{0,u} \mathrm{d}u}$, $Y_{0,t,s} = \frac{1}{s-t} \int_t^s r_{0,u} \mathrm{d}u$, $Y_{0,t} = \frac{1}{t} \int_0^t r_{0,u} \mathrm{d}u$
  $\rightsquigarrow$ knowing $r_{0,\cdot}$, recover $R_{0,\cdot}$, $Y_{0,\cdot}$ and $d_0(\cdot)$.
- Prove $r_{0,t} = Y_{0,t} + t \frac{\mathrm{d}}{\mathrm{d}t} Y_{0,t}$.

Term Structure of Simple Rates

- We could work with **simple** instead of **compounded** rates.
- Spot rate $L_{0,t}$:
  $\boxed{d_0(t) = (1 + L_{0,t}t)^{-1}}$
  $t \to L_{0,t}$: 'term structure of LIBOR rates'.
- $0 \leq t < s$; $L_{0,t,s}$: **forward rates** in 0 for $[t, s]$ defined by $1 + L_{0,s}s = (1 + L_{0,t}t)(1 + L_{0,t,s}(s - t))$.
- When $s \downarrow t$ one gets

$$l_{0,t} \doteq \lim_{s \downarrow t} L_{0,t,s} = r_{0,t},$$

i.e. no difference between simple and compounded instantaneous rates.

Term Structure of Interest Rates

- As time goes on, the term structure moves (and change shape).
- Given $d_t(s)$ $(s \geq t)$, discount function at t, i.e. price at t of a riskless pure discount bond with unit face value $\rightsquigarrow$ derive with obvious definitions $R_{t,s}$, $Y_{t,s}$, $R_{t,s,u}$, $Y_{t,s,u}$ and $r_{t,s}$.
- Many stochastic approaches to the term structure models the short (i.e. spot, instantaneous) rate $r_t = r_{t,t}$ (one factor models; e.g. Vasicek, Cox-Ingersoll-Ross …); others model the instantaneous forward rates $r_{t,s}$ (e.g. Heath-Jarrow-Morton).

Theories of the Term Structure

- Intuition suggest that forward rates (determined by ‘short’ and ‘long’ rates) convey information about expected future spot rates.
- (Pure) Expectations Theory: forward rates are unbiased expectations of future spot rates, i.e. $R_{t,s,u} = E_t[R_{s,u}]$
- Liquidity Preference Theory: forward rates are biased (upward) expectations; difference is premium for liquidity, i.e. preference for shorter investments.
- Market Segmentation Theory: bond markets are segmented; agents with different horizons invest in different segments; short and long rates are not directly related.

Pricing of Cash-flows

- Given today the term structure $d_0(\cdot) \equiv d(\cdot)$.
- Consider a security producing cash-flows C_i $(>, < 0)$ at time t_i $(i = 1, \ldots, n)$,
  with initial value V_0.
- $$\boxed{V_0 = \sum_{i=1}^{n} C_i d(t_i)}$$
  Proof. The portfolio consisting of $|C_i|$ ZCB with maturity t_i (long if $C_i > 0$, short if $C_i < 0$), produces the same cash-flow as security. Apply LOP.
- Example: coupon bearing bond paying C at $t_1, \ldots, t_n$ and N in t_n. Price P given by

$$P = C \sum_{i=1}^{n} d(t_i) + N d(t_n).$$

Pricing of a Floater

- A (plain-vanilla) **floater** pays coupons linked to LIBOR: $C_i = N\,\Delta\,L_{t_{i-1},t_i}$ at $t_i \doteq i\Delta$ $(i = 1, \dots, n)$ and N at t_n.
- Coupons are predetermined: C_i is known at t_{i-1}.
- Denote FL_t price in t of floater.
  We have $FL_{t_i} = N$ $(i = 0, \dots, n-1)$ i.e. floater trades at par at reset dates.
  Proof Consider the **dynamic** strategy (roll-over) at t_i: start with N; invest N until t_{i+1}, get $N + C_{i+1}$; reinvest N until t_{i+2}, ... at t_n get $N + C_n$. This strategy produces same cash-flow as floater. Initial value is N. Apply LOP.
- If $t_{i-1} < t < t_i$, then $FL_t = (N + C_i)\,d_t(t_i)$, i.e. next coupon (already known) plus value after paying coupon (par), discounted to t.

Bond Return Measure: IRR

- Internal Rate of Return (IRR) of a bond is a popular measure of its return.
- If a bond pays coupons C_i at time t_i ($i = 1, \dots, n$), and price is P, IRR (with continuos compounding) is R^* solution of
$$\boxed{P = \sum_{i=1}^{n} C_i (1 + R^*)^{-t_i}}$$
or the corresponding force of interest $Y^* = \log(1 + R^*)$.
- IRR: if term structure is flat at R^*, price is present value of coupons
$\rightsquigarrow$ R^* is average of R_{0,t_i} ($i = 1, \dots, n$).
- For a ZCB maturity T, $R^* = R_{0,T}$ and $Y^* = Y_{0,T}$
$\rightsquigarrow$ $Y_{0,T}$ is also known as yield to maturity.
- For a coupon bond R^* has to be found numerically.

Bootstrapping the Term Structure

- Most bonds pays coupons; ZCB available only for short maturities.
  How to extract $d_0(\cdot) \equiv d(\cdot)$?
- Example: suppose we have 4 bonds (1 ZCB, 3 coupon bearing, all with face value 100):
  - ZCB maturity 6 months, price 98;
  - bond with semiannual coupons, nominal rate 4%, maturity 1 year, price 99.88;
  - bond with semiannual coupons, nominal rate 6%, maturity 18 months, price 103.155;
  - bond paying annual coupons, coupon rate 4.5%, maturity 2 and 1/2 years, next coupon date in 6 months, price 105.325.
- Translate these cash-flows as relationships between discount factors.

. . . Bootstrapping the Term Structure

- Maturities involved are $\frac{1}{2}$, 1, $\frac{3}{2}$, $\frac{5}{2}$.
  Let $d(1/2) = x$, $d(1) = y$, $d(3/2) = z$, $d(5/2) = w$.
- We must have

$$\begin{aligned}
100x & &= 98\\
2x + 102y & &= 99.88\\
3x + 3y \quad + 103z & &= 103.155\\
4.5x \quad + 4.5z + 104.5w & &= 105.325
\end{aligned}$$

- Solving the system one gets $x = 0.98$, $y = 0.96$, $z = 0.945$ e $w = 0925$.
- Extend to all maturities through interpolation (e.g. use interpolating splines).

Outline

Forward and Futures Contracts

- Agreement between two parties to exchange a real or financial asset (underlying) at a future date (delivery date) and at a given price (delivery price).
- ▷ Long party buys the asset;
  ▷ Short party sells the asset.
- Both parties have an **obligation**.
- Long/short party gains if price of underlying rises/falls.
- No cash-flow at inception.
- Settlement: cash or physical.
- Motivation:
  - ▷ HEDGING: investors exposed to the risk of movements of the underlying can mitigate this risk by taking a position in forward/futures;
  - ▷ SPECULATION: investors having an expectation on underlying price movements can exploit with a position in futures.

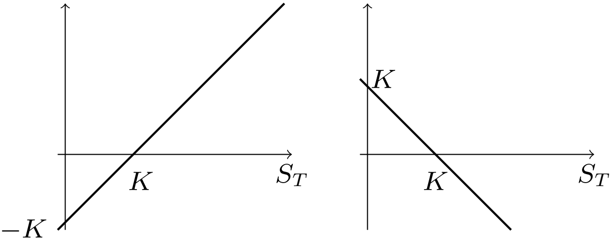
Forward vs. Futures

- Forward are **OTC** instruments/Futures trade on **organized** markets (CBOT, . . .). ⇝ Futures are **standardized**/Forward are not.
- Futures are marked-to-market: every gain/loss is settled at the end of each trading day, through the margin system.
  ⇝ The value of a Futures contract is always 0. In a Forward contract gains and losses are realized at delivery date.
  ⇝ Futures contract are much less affected by credit risk than Forward.
- Every investor's counterparty is the Clearing House.
- Forward usually reach maturity (physical settlement)/Futures are usually closed before maturity (cash settlement), by taking an opposite position.

Forward Contracts: payoff

- Today: 0; delivery date: T.
- (S_t): spot price process of underlying; K: delivery price.
- PAYOFF at delivery is

$$S_T - K \;\text{ long position; }\; K - S_T \;\text{ short position}$$



Forward Contract Analysis

- Long Forward contract value: (V_t);
  Short Forward contract value: $(-V_t)$;
  Forward price at t for T: (F_t^T) = delivery price at t for T.
- $V_0 = 0$, $V_T = S_T - K$;
  $F_0^T = K$, $F_T^T = S_T$ (**convergence**).
- $V_t =?$, $F_t^T =?$ for $0 \leq t \leq T$.
- Assume that owing the underlying asset generates no income/outcome during the life of the contract (commodities requires storage costs/stocks pays dividends/bonds pays coupons).

...Forward Contract Analysis

- $$\boxed{V_t = S_t - K\,d_t(T),}$$

  $\rightsquigarrow$ $F_t^T = S_t\,b_t(T)$ and in particular

  $$\boxed{K = S_0\,b_0(T).}$$

- Forward price = accumulated value of spot price.
- **Proof**. At time $t < T$, long position in the underlying, lend $K\,d_t(T)$ until T. Strategy produces same cash-flows as forward. Apply LOP.
  F_t^T defined by $K = F_t^T$ and $V_t = 0$.

... Forward Contract: example

- Consider a stock worth now $S_0 = 100$€, $T = \frac{1}{4}$, force of interest is $Y_{0,\frac{1}{4}} = 3\%$. The stock pays no dividend. Delivery price is then $K = 100.753$. Enter 1000 such contracts for buying stock at €100.753 in 3 months.
- After 1 month ($t = \frac{1}{12}$), stock price is $S_{\frac{1}{12}} = 100.5$ and force of interest is $Y_{\frac{1}{12},\frac{1}{4}} = 2.5\%$. Value of one forward contract for long position is $V_{\frac{1}{12}} = 0.166$. Market value is then 166€.
- At delivery date, stock price is $S_{\frac{1}{4}} = 101$. Payoff for one contract is $S_{\frac{1}{4}} - K = 0.247$, so payoff is 247.18€.

Forward Contract Analysis: extension

- Suppose that owing the security generates a known income/outcome over $[0, T]$.
- Let $Q_{t,T}$ be present value at t of income/outcome over $[t, T]$.
  Then $V_t = S_t - Q_{t,T} - K\, d_t(T)$
  $\rightsquigarrow F_t^T = (S_t - Q_{t,T})\, b_t(T)$
  $\rightsquigarrow K = (S_0 - Q_{0,T})\, b_0(T)$.
- **Proof.** Same as before, but add: 'borrow $Q_{t,T}$ for relevant maturities, so as to reproduce cash-flow generated by holding the underlying'.
- Examples: forward contracts on stocks paying known dividens or on coupon bearing bonds.

Forward Contracts on Currencies

- Agreement on exchange rate to be applied at future date, on a given nominal.
- (S_t) €/\$ (spot) exchange rate, i.e. price of €1 in \$; €1 is S_t\$; 1\$ is €$1/S_t$.
- Parties agree on €/\$ exchange rate K to be applied at T (i.e. to exchange €1 for K\$); payoff in \$ is $S_T - K$.
- $V_t = S_t\, d_t^{\$}(T) - K\, d_t^{€}(T)$ in \$, where $d_t^{c}(T) = e^{-Y_{t,T}^{c}(T-t)}$ for c=\$, €;
  proof. Borrow K in \$ up to T; lend $d_t^{€}(T)$ up to T, convert in \$ at prevailing rate.
- $\rightsquigarrow$ $\boxed{K = S_0 e^{-(Y_{0,T}^{€} - Y_{0,T}^{\$})}}$ interest rates parity relation.

Forward Rate Agreements (FRA)

- Agreement between two parties to lend/borrow a nominal N at a given simple interest rate L_{FRA} over $[T, U]$ ($0 < T < U$; T settlement date, U maturity).
- Long party borrow/short party lend.
- Convention is to settle at T the difference between L_{FRA} and prevailing LIBOR rate $L_{T,U}$; payoff in T is

$$N\,(U-T)\,(L_{T,U}-L_{\mathrm{FRA}})\,d_T(U).$$

- Could be seen as forward contract on ZCB.
- Initial value is zero $\rightsquigarrow$ $L_{\mathrm{FRA}} = L_{0,T,U}$.

Forward Rate Agreements (FRA)

- V_t^{FRA} value of the FRA ($0 \leq t \leq T$).
  A FRA can be valued as if forward rates are realized:

$$V_t^{\text{FRA}} = N\,(U-T)\,(L_{t,T,U} - L_{\text{FRA}})d_t(U).$$

  Proof. Build the strategy at t:
  borrow $N\,(1 + L_{\text{FRA}}(U-T))\,d_t(U)$ until U;
  invest $d_t(T)$ until T, then reinvest proceedings until U at prevailing rate $L_{T,U}$.

- V_t^{FRA} <=> 0 according to $L_{t,T,U}$ <=> 0.

$T \times U$	EURIBOR FRA rate
3×6	3.78
6×9	3.84
9×12	3.84
6×12	3.86
12×18	3.77

. . . FRA

- Consider the 9×12 FRA equal to 3.84%, and a nominal amount of 1000000€.
- Suppose in 6 months EURIBOR term structure (simple rates) is flat at 3.5%. Then forward rates equal spot rates. Value of the FRA is negative:

$$V_{\frac{1}{2}}^{\mathrm{FRA}} = 1000000 \, \frac{\frac{1}{4}(0.035 - 0.0384)}{1 + \frac{1}{2}0.035} = -835€.$$

- If at settlement (in 9 months) the prevailing 3 months EURIBOR has risen to $L_{\frac{3}{4},1} = 4.32\%$, then long party receives

$$1000000 \frac{\frac{1}{4}(0.0432 - 0.0384)}{(1 + 0.25 \cdot 0.0432)} = 1187.18€.$$

Futures: Marking-to-Market

- Delivery price: Futures price.
- An investor trading in futures must make an initial deposit in a margin account with its broker.
- At the end of each trading day, investor gains/losses (difference between closing and initial futures prices) increases/decreases the margin account;
  $\rightsquigarrow$ contract value is 0 at the end of trading day; actually, any amount above initial margin can be withdrawn by the investor.
- If margin account falls below a maintenance margin $\rightsquigarrow$ margin call: investor must deposit a variation margin and restore the initial margin.
- Broker has to maintain a similar account with clearinghouse.

Futures: Marking-to-Market

- Example: futures on gold (adapted from [Hull, 2006]).
- 1 futures contract on gold is for delivery of 100 ounces; futures price quoted (in $) is for 1 ounce; initial margin is 2000$ per contract; maintenance margin is 1500$ per contract.
- Long position in 10 futures contracts $\rightsquigarrow$ initial/maintenance margin is 20000$/15000$.

day	Futures price	Daily gain/loss	Margin account
1	400	—	20000
2	401	+100	21000
3	399	−200	19000
4	397.5	−150	17500
5	394	−350	14000
6	393.5	−50	19500

Forward and Futures Prices

- Assume interest rates are flat and nonrandom ($Y_{t,T} = Y$, $d_0(\cdot) = d_t(\cdot)$). Then futures prices equal forward prices.
- $0 < t_1 < t_2 < \ldots < t_n = T$; 0: today, T: maturity; futures contract is marked to market at t_i, $i = 1, \ldots, n$ (e.g. $t_i - t_{i-1} = \Delta_i = 1/365$).
- Let $f_t \equiv f_t^T$ futures price in t; F_0^T forward price, and S_t spot price; by convergence, $f_T = S_T$.
- **First strategy**: start at $t_0 = 0$, long $b(t_1)$ futures; at t_i, cash-flow $b(t_i)\,(f_{t_i} - f_{t_{i-1}})$, increase long position to $b(t_{i+1})$ contracts ($i = 1, \ldots, n-1$).

Forward and Futures Prices

- At any date t_i, invest cash-flow $b(t_i)(f_{t_i} - f_{t_{i-1}})$ up to $t_n = T$ at risk-free rate; get $b(T)\,(f_{t_i} - f_{t_{i-1}})$.
- Globally, get $\sum_{i=1}^{n} b(T)\,(f_{t_i} - f_{t_{i-1}}) = b(T)\,(f_{t_n} - f_{t_0}) = b(T)\,(S_T - f_0)$.
- At 0, lend f_0 up to T, get $b(T)\,f_0$;
  entire strategy gives $b(T)\,S_T$ at T, and initial value f_0.
- **second strategy**: long $b(T)$ forward contracts at 0,
  lend F_0^T at risk free rate up to T;
  payoff in T $b(T)\,(S_T - F_0^T) + b(T)\,F_0^T = b(T)\,S_T$;
  initial value F_0^T.
- LOP $\rightsquigarrow$ $f_0 = F_0^T$.

Outline

Swaps

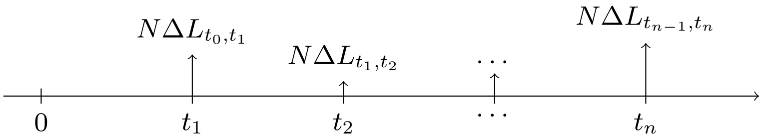
- Among most popular OTC derivatives.
- Agreement between two parties to exchange regular cash-flows based on some economic variable.
- Typically, one party pays fixed cash-flows, the other variable cash-flows.
- Can be seen as combinations (portfolios) of forward contracts.
- Cash-flows computed based on notional amount. Only net cash-flows are actually exchanged.
- Focus on standard type of swaps: interest rate swaps (IRS).
- Swaps are used for
  - hedging;
  - asset/liability **transformation**;
  - exploiting **comparative advantages**; see [Hull, 2006].

Interest Rate Swaps (IRS)

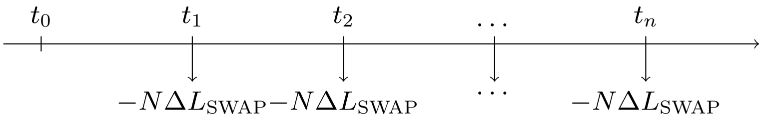
- Plain Vanilla fixed-for-floating swap: agreement to exchange payments based on a fixed rate (fixed branch), known as swap rate L_{SWAP}, against payments based on a variable rate (floating branch), e.g. LIBOR), applied to same notional amount.
- Long party: fixed rate payer/short part: fixed rate receiver.
- Payments at $t_i = i\Delta$ (e.g. $\Delta = 1/2$), $i = 1, \dots, n$. Net cash-flow at t_i (long position) is $N\Delta(L_{t_{i-1},t_i} - L_{\mathrm{SWAP}})$ known at reset date t_{i-1}.
- Maturity ranging from 2 to 30 years.
- L_{SWAP} is fixed so that initial contract value is 0: no cash-flow at $t_0 = 0$.

Fixed/floating branches

Floating branch



Fixed branch:



IRS as portfolio of FRAs

- Cash-flow at t_i is that of FRA with settlement t_{i-1} and maturity t_i, FRA rate = SWAP rate.
- V_t^{SWAP} value of swap at t for long position:

  $$\boxed{V_t^{\text{SWAP}} = \sum_{i=1}^{n} V_t^{\text{FRA}_i}}$$

  Using results for FRAs, get

  $$L_{\text{swap}} = \frac{\sum_{i=1}^{n} L_{0,t_{i-1},t_i} d_0(t_i)}{\sum_{i=1}^{n} d_0(t_i)}.$$

- Swap rate is weighted average of forward rates. Single FRAs would not be worth 0 at beginning but balance on average.

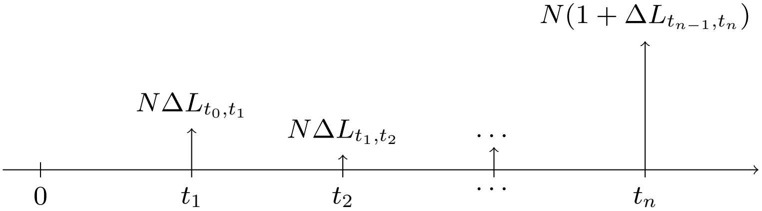
IRS as fixed coupon/floater exchange

- Add notional amount N at t_n for both fixed and floating branch.
  Net cash-flows remians unchanged.
- ▷ fixed branch $\equiv$ fixed coupon (coupon rate $= L_{\text{SWAP}}$) bearing bond;
  ▷ floating branch $\equiv$ floater.
- CB_t, FL_t prices of coupon bond and floater at t;
  $\boxed{V_t^{\text{SWAP}} = FL_t - CB_t.}$
- Recall floater trades at par at reset dates: $\rightsquigarrow$
  $V_{t_i}^{\text{SWAP}} = N - CB_{t_i}$.
- L_{SWAP} is such that $CB_0 = FL_0 = N$; i.e. coupon rate such that coupon bond trades at par: par rate:

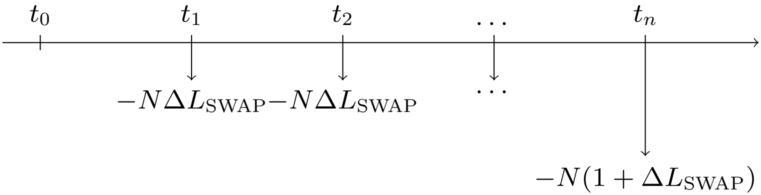
$$L_{\text{SWAP}} = \frac{1 - d_0(t_n)}{\Delta \sum_{i=1}^{n} d_0(t_i)}.$$

IRS as fixed coupon/floater exchange

Modified loating branch ≡ floater:



Modified fixed branch ≡ fixed coupon bond:



Outline

Options

- Agreement between two parties: one party (long position, or option holder) has the right to buy/sell the underlying at a given price (strike price or strike price), from/to the other party (short position, option writer).
- A call option gives the holder the right to buy, a put option gives the holder the right to sell.
- Unlike forward (or futures, or swaps), options gives rights, not obligations. The writer must stand the holder decision.
- Deciding whether or not to buy/sell is known as exercising the option.
- An option is European/American if exercise can take place at maturity/at any time before maturity.
- Since an option confer a right, the holder has to pay a price (option premium) at inception.

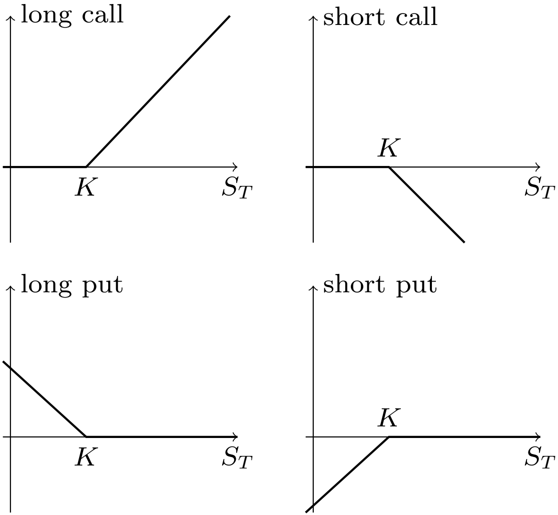
Option Markets

- Options can be used for hedging and speculation; unlike forward, allow to make profits without incurring any loss.
- Options are traded both on exchanges (CBOE, ...) (with a margin system like for futures) and OTC.
- Underlying can be: stock/indices/currencies/commodities/futures/Swaps (Swaptions)...
- Most options are American; European are simpler to analyze.
- Typically, several strikes and maturities are quoted at any trading date.
- Standard options are called plain-vanilla; many other types of options are exotics.

Options payoff

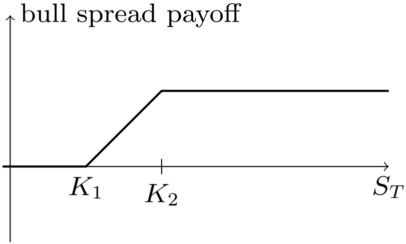
- 0: today; T: maturity;
- (S_t): spot price of underlying; K strike price.
  C_t, P_t: prices of American call/put,
  c_t, p_t: prices of European call/put,
- At any time $0 < t < T$, holder can (i) sell the option (ii) if American, exercise the option (early exercise) (iii) do nothing;
  at maturity T, holder can (j) exercise the option (jj) do nothing.
- Since holder is rational, at maturity T exercise call/put iff $S_T > K / S_T < K$.
  Call payoff (Call value): $\boxed{C_T = c_T = \max\{S_T - K, 0\}}$
  Put payoff (Put value): $\boxed{P_T = p_T = \max\{K - S_T, 0\}}$
- Call/put Writer payoff: $\min\{0, K - S_T\}$ and $\min\{0, S_T - K\}$.

Options payoff



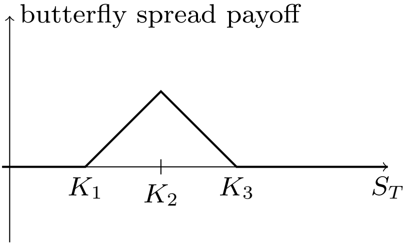
Option Strategies

- One can build many strategies using options.
- EXAMPLE I. Long put+long stock: $\max\{K - S_T, 0\} + S_T = \max\{S_T, K\}$ (minimum guarantee).
- EXAMPLE II. Long call strike K_1, short call strike $K_2 (> K_1)$ (bull spread):



. . . Option Strategies

- EXAMPLE III. Long call strike K_1 and K_3, short 2 call strike K_2 ($K_1 < K_2 < K_3$): (butterfly spread):



Option Moneyness

- At any time $0 \leq t \leq T$ the option is in/at/out of the money if immediate exercise (only hypothetical for European options) generates a positive/null/negative cash-flow.
- Hence a call option is in/at/out of the money according to $S_t >, =, < K$. A put option is in/at/out of the money according to $S_t <, =, > K$.
- A necessary condition for exercise is that the option be in the money (also sufficient at maturity).
- The intrinsic value of an option at $0 \leq t \leq T$ is $\max\{S_t - K, 0\}$ for a call, $\max\{K - S_t, 0\}$ for a put.
- $C_t \geq \max\{S_t - K, 0\}$, $P_t \geq \max\{K - S_t, 0\}$ (otherwise, buy option and exercise). The difference, if positive, is the time value of the option.

Option Bounds

- If an American option has time value, **exercise is not convenient** (better wait or sell it!), even if option is deep in the money.
- Pure no-arbitrage reasonings lead only to bounds for option prices.
- Clearly, $C_t \geq c_t$ and $P_t \geq p_t$, for $0 \leq t \leq T$.
- Bounds for options on non-dividend paying stock: for $0 \leq t \leq T$,

  BOUNDS FOR CALL OPTIONS:
  $\max\{S_t - K\, d_t(T), 0\} \leq c_t \leq C_t \leq S_t$;
  BOUNDS FOR EUROPEAN PUT OPTIONS:
  $\max\{K\, d_t(T) - S_t, 0\} \leq p_t \leq K\, d_t(T)$;
  BOUNDS FOR AMERICAN PUT OPTIONS:
  $\max\{K - S_t, 0\} \leq P_t \leq K$;
  Proof. If an inequality does not hold, build an arbitrage.

. . . Option Bounds

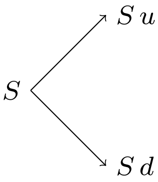
- Note that $C_t \geq \max\{S_t - K\, d_t(T), 0\} > \max\{S_t - K, 0\}$ (provided $Y_{t,T} > 0$ and $S_t > K$), so that it is **never convenient to early exercise an American call option on a non dividend paying** stock $\rightsquigarrow C_t = c_t$; instead, early exercise of an American put may be convenient.
- European Put-Call Parity: for all $0 \leq t \leq T$
  $\boxed{p_t + S_t = c_t + K\, d_t(T)}$
  Proof. Long position on put and stock = long position on call+lending $K\, d_t(T)$.
  American Put-Call relation: for all $0 \leq t \leq T$
  $C_t + K\, d_t(T) \leq P_t + S_t \leq C_t + K.$

... Option Bounds

- Previous inequalities extend to stocks paying known dividends.
- If $Q(t,T)$ denotes present value in t of dividends paid in $[t,T]$, put-call European parity becomes:
  $p_t + S_t - Q(t,T) = c_t + K\, d_t(T)$.
- Early exercise of American call may be convenient only immediately before dividend paying dates.
- Early exercise of American put may be convenient only immediately after dividend paying dates.

Binomial Model

- In order to value options, we have to set up a model.
- Simplest type is binomial: uncertainty can evolve with 2 possible scenarios.
- Consider a market with two assets, stock and a bond.
- Only 2 dates (1 period), 0 and 1. Price today is $S_0 = S$, price in 1 is S_1 with either $S_1(u) = S\,u$ or $S_1(d) = S\,d$ with $u > d(> 0)$. Interest rate for $[0, 1]$ is R.



. . . Binomial Model

- A strategy is $(\phi, \eta) \in \mathbb{R}^2$; the value at 0 is $V_0^{\phi,\eta} = \phi S_0 + \eta$; value at 1 is $V_1^{\phi,\eta} = \phi S_1 + \eta(1+R)$.
- An arbitrage opportunity is a strategy (ϕ, η) such that $V_0^{\phi,\eta} = 0$ and $V_1^{\phi,\eta}(u) \geq 0$, $V_1^{\phi,\eta}(d) \geq 0$ (with one of the two inequalities strict).
- No arbitrage iff

$$0 < q \equiv \frac{(1+R) - d}{u - d} < 1$$

iff $d < 1 + R < u$.

- Think of q as probability of 'u' movement, $1 - q$ of 'd' movement. Call Q this probability.

. . . Binomial Model

- We have $S_0 = E^Q\left[\frac{S_1}{1+R}\right]$.
  $\rightsquigarrow$ Q is Risk Neutral probability.
- $C = (C(u), C(d)) \in \mathbb{R}^2$ contingent claim (e.g. $C = \max\{S_1 - K, 0\}$); there exists a strategy (ϕ, η) such that $V_1^{\phi,\eta} = C$; indeed the **hedge ratios** are

$$\phi = \frac{C(u) - C(d)}{S_0\,(u-d)}, \quad \eta = \frac{C(u)\,d - C(d)\,u}{(1+R)(u-d)}$$

We say that markets are complete (any contingent claim is replicated). Moreover,

$$V_0^{\phi,\eta} = E^Q\left[\frac{C}{1+R}\right]$$

$\rightsquigarrow$ $V_0^{\phi,\eta}$ is the initial price one should pay for C.

Exotic Options

- Barrier Options: option is activated/cancelled if a barrier is reached/not reached during option's life.
- Lookback Options: underlying or strike is the minimum or maximum value of stock during stock's life.
- Asian Options: underlying or strike is the average value of stock during stock's life.
- Compound Options: option on an option.
- Binary Options: receive a fixed amount in case of exercise.
- Basket Options: options on maximum or minimum of several assets.
- Exchange Options: options to exchange an asset for another asset.

Embedded Options

- Many securities contain embedded options:
- Callable bonds: issuer may retire the bond (has a call option).
- Convertible bonds: bondholder can convert bond into issuer company's stock.
- Minimum guarantees: insurance contracts are often equity-linked with minimum guarantee.
- Surrender option: policyholder may surrender an insurance contract.
- Prepayment options: mortgagors have the right to prepay mortgage.
- Executive stock options: options to incentivate corporate managers.
- ...