Overview of Financial Markets and Instruments

Outline

1. Financial Markets and Primary Securities
2. Term Structure of Interest Rates
3. Forward and Futures Contracts
4. Swaps
5. Options
Bibliography

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Financial Markets and Instruments

- **Financial instruments** (assets, securities): claim to future cash-flows.
- **Financial** assets vs. **real** assets.
- Financial securities are **traded** in Financial markets.
- Financial market: ‘place’ where supply and demand of financial assets meet.
- Role of financial markets:
  - provide **investment** opportunity for agents with surplus (buy securities);
  - provide **financing** opportunity to agents with need of capital (issue securities).

Hence financial markets permit to redistribute wealth and consumption over time.
Financial Markets Players

- Agents differ according to their needs/preferences/behaviour.
- Some agents invest; some issue securities; some act as intermediaries (not exclusive roles).
- HOUSEHOLDS.
- FIRMS.
- BANKS; other FINANCIAL institutions.
- MUTUAL funds.
- PENSION funds; INSURANCE companies.
- SOVEREIGN entities.
- ...
Financial Markets

- Securities can be traded on:
  - **Primary** market, i.e. when first issued, or
  - after issuance, i.e. in the **Secondary** market.

- We distinguish between

- **Organized** markets (exchange):
  - Only specialized agents (members) can operate;
  - traded securities must satisfy given conditions;
  - trades are standardized;
  - demand and supply do not directly meet.

- **Over The Counter** (OTC) markets:
  - Less restrictive rules for agents and traded assets;
  - counterparties directly meet;
  - trades are not standardized.

- **Primary securities**: Stocks and fixed income (bonds).
Overview of Financial Markets and Instruments
Financial Markets and Primary Securities

Stocks

- **Common Stock** or **equities**: represent **ownership** of a corporation.
- **Shareholders** are entitled to:
  - Voting rights in shareholders’ meetings: take part in corporate governance;
  - Receiving part of firm profits as **dividends**.
- Features of common stock:
  - Residual claim to firm assets;
  - Limited liability: loss is limited to original investment.
- A **publicly held** corporation stock is traded in a market (otherwise: **privately held** corporation).
  A stock traded on organized exchange is **listed**, e.g. NYSE, AMEX, LSE, …
- **Preferred stock**: entitled to receive fixed dividends/no voting rights.
Fixed Income Securities

- Financial assets generating cash-flows computable according with **prespecified rules**.
- Loan is split in many obligations (**bonds**): one borrower, many lenders (**bondholders**).
- Bonds can then be **transferred** in financial markets.
- Bondholders receive periodical payments of interest (**coupon**) and **capital** at **maturity**.
- **Credit Risk**: failure of payments (interest or capital) or change in credit quality.
- Distinguish between
  - **money market** securities: short term, low risk;
  - **bonds**: longer maturities, higher risk.
...Fixed Income Securities

- Many bonds provide fixed *(known)* cash-flows; e.g. zero-coupon bonds and coupon bonds.
- Some bonds pays coupons and/or nominal linked to economic variables:
  - interest rates (floater);
  - market index;
  - currencies;
  - inflation;
  - stocks.
- Other bonds contains options: callable and convertible bonds.
- Some bonds are traded on organized exchanges; most are OTC.
Zero-Coupon Bonds (ZCB)

- Simplest type of financial security.
- Bonds that pays no coupon; ‘sells at discount’
  \[ \rightarrow \text{pure discount bond.} \]
- Investor pays price \( P \) at time 0; receives nominal (face, par) value \( N \) at maturity \( T \).
  Think of \( P/N \) as riskless discount factor.
- Clearly, \( P < N \); \( N - P \) is the discount.
- Typically maturity is 1 year or less.
Coupon Bonds

- Also known as bullet bonds/coupon bearing bonds.
- Payments occur at times $0 < t_1 < t_2 < \ldots < t_n = T$; $T$ is the maturity; typically $t_i - t_{i-1} = \Delta$ (e.g. $\Delta = 1/2, 1/4, \ldots$).
- Investor pays price $P$ at time 0.
- Receive coupons $C$ at $t_1, \ldots, t_n$ and nominal (face, par) value $N$ at $t_n$.

\begin{align*}
0 & \quad \downarrow \quad t_0 \\
C & \quad \uparrow \\
| & \quad t_1 \\
C & \quad \uparrow \\
| & \quad t_2 \\
\ldots & \quad \uparrow \\
\ldots & \quad \uparrow \\
C & \quad \uparrow \\
| & \quad t_{n-1} \\
N + C & \quad \uparrow \\
0 & \quad \downarrow \\
- P & \quad \downarrow \\
| & \quad t_0 \\
\ldots & \quad \uparrow \\
\ldots & \quad \uparrow \\
C & \quad \uparrow \\
| & \quad t_n
\end{align*}
Coupon Bonds

- If $P <,\leq, > N$, bond sells below, at, above par.
- discount $= N - P > 0$; premium $= P - N > 0$.
- Coupon measured as percentage of nominal:
  $c = C/N$ coupon rate, $c' = c\Delta$ nominal rate.
- Bond is frequently quoted as with its clean price (flat price) $Q$, related to the price actually paid $P$ (dirty or full price) through

$$P = Q + A,$$

where $A$ is the accrued interest:

$$A = \frac{-t_0}{t_1 - t_0} C$$

$t_0$: issue or last coupon date.
Market Indices

- Pure numbers reflecting market behaviour.
- Convey information for investor on market trend.
- **Benchmark** for mutual funds/derivatives/...
- Usually computed as weighted arithmetic average of market prices.
- Example: S&P500 (Standard & Poor’s 500); average of 500 large US corporate common stock prices, weighted by number of shares:

\[ \text{S&P500}_t = \frac{I_t}{I_{t0}}, \]

where \( I_t = \sum_{i=1}^{500} n^i_t S^i_t \) and \( n^i_t, S^i_t \) are number of shares outstanding and market price at \( t \) of \( i \)-th stock.
Derivatives

- **DERIVATIVE CONTRACTS**: as opposed to primary securities. (combined together sometimes)

- Financial instruments whose cash-flows depend on the value of one or more **underlying** economic variables.

- **Underlying**:
  - stock
  - fixed income
  - interest rate
  - market index
  - commodity
  - currency
  - derivative
  - credit risk
  - ...
Perfect Markets

- Very convenient for analysis to assume that markets are perfect:
- **Agents:**
  - rational, nonsatiated;
  - price-takers;
  - share same information;
  - default riskless (no credit risk).
- **Markets:** frictionless
  - continuously open;
  - securities are infinitely divisible;
  - short selling is allowed;
  - no taxation;
  - no transaction costs;
- Enough if these hypotheses hold for some (large) investor.
...Perfect Markets

- **STRATEGIES:** there are no arbitrage opportunities (NA).

  Arbitrage opportunity (or free lunch): strategy involving available securities providing
  - nonnegative cash-flows at every time and state of nature;
  - a positive cash-flow at some time, with positive probability.

  Hence it is a riskless strategy that may result in a profit.

- **NA is the key property:**
  - necessary for equilibrium;
  - implies the **Law of one price (LOP):** two strategies providing the same cash-flows have the same value; if not, sell the more expensive, buy the cheaper (arbitrage).
Long/Short Positions

- When faced with a security, an investor could take a **long** position or a **short** position.
- **LONG POSITION**: buying a security
  - ▶ profit if price rises;
  - ▶ *unlimited* potential gain, *limited* liability.
- **SHORT POSITION**: **short selling**, i.e. selling a security that is not owned.
  - ▶ Security is borrowed from a third party’s portfolio;
  - ▶ short seller gains if price falls;
  - ▶ short seller must provide any cash-flow paid by security;
  - ▶ *limited* potential gain; *unlimited* potential loss;
  - ▶ requires margin as collateral.
Overview of Financial Markets and Instruments

Term Structure of Interest Rates

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Term Structure of Interest Rates

Time Value of Money

- Lending/borrowing provides interest/has a cost.
- $C$ today (principal) is the same as $C \cdot b(t)$ at $t$;
  $b(t)$: accumulation factor.
- $M$ at $t$ are the same as $M \cdot d(t)$ at $0$ (today);
  $d(t)$: discount factor.
- $b = 1/d$; $b \uparrow$, $d \downarrow$; $b(0) = d(0) = 1$.
- $C \cdot b(t)$ is the accumulated value of $C$;
  $M \cdot d(t)$ is the present (discounted) value of $M$.
- $b(t) - 1$ is interest per unit of principal (interest rate).
- Different rules for computing interest: $b(t) = ?, d(t) = ?$.
  
  ▶ Simple Interest;
  ▶ Discretely Compound Interest;
  ▶ Continuously Compound Interest.
Simple Interest

- Interest is **proportional** to time:
  \[ b(t) = 1 + L t; \quad d(t) = (1 + L t)^{-1}. \]
- \( L \): 1 year interest rate.
- **Example**: LIBOR (**London InterBank Offered Rate**) rates:
  - Rates at which large UK banks lend/borrow deposits between them (**EURIBOR** in the € area);
  - standard reference rate for many other contracts/derivatives;
  - maturities: 1 day to 1 year.
- **Example**: 6 months LIBOR: \( L_{0,1/2} = 4\% \). Borrow now \( 1000000£ \), pay \( 1000000(1 + 0.04 \cdot 1/2) = 1020000£ \) in 6 months.
Discretely Compound Interest

- Fix $k > 0$. After each $\frac{1}{k}$-th (e.g. $k = 1, 2, 4, 12, \ldots$) of year interest is **compounded** i.e. added to principal.
- After $n$ periods, accumulated value is
  \[ b(n) = \left(1 + \frac{R}{k}\right)^n; \]
  interpolate linearly between points to get $b(t)$, all $t \geq 0$.
- Equivalent 1 year interest rate is $R$ given by
  \[ 1 + R = \left(1 + \frac{R}{k}\right)^k; \]
  \[ \Rightarrow R^k = k[(1 + R)^{1/k} - 1]. \]
- Example: €1000 invested for 6 months at 7% compounded monthly gives
  \[ 1000 \left(1 + \frac{1}{12}0.07\right)^6 = 1035.51€. \] 
  Equivalent 1-year rate is 7.23%.
Continuous Compounding

- Consider discrete compounding when \( k \to \infty \): interest is compounded continuously.
- For fixed 1 year interest rate \( R \), \( R^k \to Y = \log(1 + R) \) as \( k \to \infty \).
- \((1 + \frac{R^k}{k})kn \to e^{Yn} \) as \( k \to \infty \);
- \( b(t) = e^{Yt} = (1 + R)^t; d(t) = e^{-Yt} = (1 + R)^{-t} \)
- \( Y \): force of interest;
  \( b'(t)/b(t) = Y \), i.e. \( b(t + \Delta t) \approx b(t)(1 + Y \Delta t) \).
- Note that \( b(t + s) = b(t)b(s) \).
- Example: 500$ invested at 2% force of interest for 1 and 1/2 years gives \( 500 e^{0.02 \cdot 1.5} = 515.23 \)$. Corresponding interest rate is 2.02\%. 
Term Structure of Interest Rates

- Fix time 0: today.
- Suppose a discount function \( d(\cdot) \equiv d_0(\cdot) \) (price of hypothetical ZCB for any maturity) is given.
- Assume continuous compounding; interest rate prevailing for borrowing/lending up to time \( t \) is \( R_{0,t} \) (\( Y_{0,t} \) corresponding force of interest);
  \[ d(t) = (1 + R_{0,t})^{-t} = e^{-Y_{0,t}t}. \]
- The function \( t \to R_{0,t} \) is the **Term Structure of Interest Rates** (at 0); it can take several shapes: flat, **normal** (increasing), **inverted** (decreasing), humped, spoon-shaped.
- Knowing \( d(\cdot) \) you get \( R_{0,\cdot} \), and viceversa.
Forward Rates

- $1 + R_{0,s}$ weighted geometric average of $1 + R_{0,t}, 1 + R_{0,t,s}$;
- $Y_{0,s}$ weighted arithmetic average of $Y_{0,t}, Y_{0,t,s}$.

Let $s \downarrow t$ in $Y_{0,t,s}$. Get instantaneous forward rates:

$$r_{0,t} = \lim_{s \downarrow t} Y_{0,t,s} = -\frac{d}{dt} \log d(t) = -\frac{d'(t)}{d(t)}.$$  

$t \rightarrow r_{0,t}$: term structure of instantaneous forward rates

- Also: $d(t) = e^{-\int_0^t r_{0,u} du}$, $Y_{0,t,s} = \frac{1}{s-t} \int_t^s r_{0,u} du$,
  $Y_{0,t} = \frac{1}{t} \int_0^t r_{0,u} du$  
  $\leadsto$ knowing $r_{0,\cdot}$, recover $R_{0,\cdot}, Y_{0,\cdot}$ and $d_0(\cdot)$.

- Prove $r_{0,t} = Y_{0,t} + t \frac{d}{dt} Y_{0,t}$.
Term Structure of Simple Rates

- We could work with **simple** instead of **compounded** rates.

- Spot rate $L_{0,t}$:
  
  $$d_0(t) = (1 + L_{0,t}t)^{-1}$$

  $t \to L_{0,t}$: ‘term structure of LIBOR rates’.

- $0 \leq t < s$; $L_{0,t,s}$: **forward rates** in 0 for $[t, s]$ defined by
  
  $$1 + L_{0,s}s = (1 + L_{0,t}t)(1 + L_{0,t,s}(s - t)).$$

- When $s \downarrow t$ one gets
  
  $$l_{0,t} = \lim_{s \downarrow t} L_{0,t,s} = r_{0,t},$$

  i.e. no difference between simple and compounded instantaneous rates.
Term Structure of Interest Rates

- As time goes on, the term structure moves (and change shape).
- Given $d_t(s) \ (s \geq t)$, discount function at $t$, i.e. price at $t$ of a riskless pure discount bond with unit face value $\Rightarrow$ derive with obvious definitions $R_{t,s}, Y_{t,s}, R_{t,s,u}, Y_{t,s,u}$ and $r_{t,s}$.
- Many stochastic approaches to the term structure models the short (i.e. spot, instantaneous) rate $r_t = r_{t,t}$ (one factor models; e.g. Vasicek, Cox-Ingersoll-Ross ...); others model the instantaneous forward rates $r_{t,s}$ (e.g. Heath-Jarrow-Morton).
Intuition suggests that forward rates (determined by ‘short’ and ‘long’ rates) convey information about expected future spot rates.

(Pure) Expectations Theory: forward rates are unbiased expectations of future spot rates, i.e.
\[ R_{t,s,u} = E_t[R_{s,u}] \]

Liquidity Preference Theory: forward rates are biased (upward) expectations; difference is premium for liquidity, i.e. preference for shorter investments.

Market Segmentation Theory: bond markets are segmented; agents with different horizons invest in different segments; short and long rates are not directly related.
Overview of Financial Markets and Instruments

Term Structure of Interest Rates

Pricing of Cash-flows

- Given today the term structure \( d_0(\cdot) \equiv d(\cdot) \).
- Consider a security producing cash-flows \( C_i \) (>\, < 0) at time \( t_i \) \((i = 1, \ldots, n)\),
with initial value \( V_0 \).

\[
V_0 = \sum_{i=1}^{n} C_i d(t_i)
\]

**Proof.** The portfolio consisting of \(|C_i|\) ZCB with maturity \( t_i \) (long if \( C_i > 0 \), short if \( C_i < 0 \)), produces the same cash-flow as security. Apply LOP.

- Example: coupon bearing bond paying \( C \) at \( t_1, \ldots, t_n \) and \( N \) in \( t_n \). Price \( P \) given by

\[
P = C \sum_{i=1}^{n} d(t_i) + N d(t_n).
\]
Pricing of a Floater

- A (plain-vanilla) **floater** pays coupons linked to LIBOR: $C_i = N \Delta L_{t_{i-1},t_i}$ at $t_i = i \Delta$ ($i = 1, \ldots, n$) and $N$ at $t_n$.
- Coupons are **predetermined**: $C_i$ is known at $t_{i-1}$.
- Denote $FL_t$ price in $t$ of floater.
  We have $FL_{t_i} = N$ ($i = 0, \ldots, n - 1$) i.e. floater trades at par at reset dates.

**Proof** Consider the **dynamic** strategy (roll-over) at $t_i$:
  start with $N$; invest $N$ until $t_{i+1}$, get $N + C_{i+1}$; reinvest $N$ until $t_{i+2}$, ... at $t_n$ get $N + C_n$. This strategy produces same cash-flow as floater. Initial value is $N$.
  Apply LOP.

- If $t_{i-1} < t < t_i$, then $FL_t = (N + C_i) \frac{d_t(t_i)}{d_t(t)}$, i.e. next coupon (already known) plus value after paying coupon (par), discounted to $t$. 

Bond Return Measure: IRR

- **Internal Rate of Return** (IRR) of a bond is a popular measure of its return.

- If a bond pays coupons $C_i$ at time $t_i$ ($i = 1, \ldots, n$), and price is $P$, IRR (with continuous compounding) is $R^*$

$$P = \sum_{i=1}^{n} C_i (1 + R^*)^{-t_i}$$

or the corresponding force of interest $Y^* = \log(1 + R^*)$.

- IRR: if term structure is flat at $R^*$, price is present value of coupons

$\leadsto R^*$ is average of $R_{0,t_i}$ ($i = 1, \ldots, n$).

- For a ZCB maturity $T$, $R^* = R_{0,T}$ and $Y^* = Y_{0,T}$

$\leadsto Y_{0,T}$ is also known as **yield to maturity**.

- For a coupon bond $R^*$ has to be found numerically.
Bootstrapping the Term Structure

- Most bonds pay coupons; ZCB available only for short maturities.
  How to extract $d_0(\cdot) \equiv d(\cdot)$?
- Example: suppose we have 4 bonds (1 ZCB, 3 coupon bearing, all with face value 100):
  - ZCB maturity 6 months, price 98;
  - bond with semiannual coupons, nominal rate 4%, maturity 1 year, price 99.88;
  - bond with semiannual coupons, nominal rate 6%, maturity 18 months, price 103.155;
  - bond paying annual coupons, coupon rate 4.5%, maturity 2 and 1/2 years, next coupon date in 6 months, price 105.325.

- Translate these cash-flows as relationships between discount factors.
... Bootstrapping the Term Structure

- Maturities involved are $\frac{1}{2}, 1, \frac{3}{2}, \frac{5}{2}$.
  Let $d(1/2) = x$, $d(1) = y$, $d(3/2) = z$, $d(5/2) = w$.
- We must have

$$
100x = 98 \\
2x + 102y = 99.88 \\
3x + 3y + 103z = 103.155 \\
4.5x + 4.5z + 104.5w = 105.325
$$

- Solving the system one gets $x = 0.98$, $y = 0.96$, $z = 0.945$, $w = 0.925$.
- Extend to all maturities through interpolation (e.g. use interpolating splines).
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Forward and Futures Contracts

- Agreement between two parties to exchange a real or financial asset (underlying) at a future date (delivery date) and at a given price (delivery price).
  - Long party buys the asset;
  - Short party sells the asset.
- Both parties have an obligation.
- Long/short party gains if price of underlying rises/falls.
- No cash-flow at inception.
- Settlement: cash or physical.
- Motivation:
  - HEDGING: investors exposed to the risk of movements of the underlying can mitigate this risk by taking a position in forward/futures;
  - SPECULATION: investors having an expectation on underlying price movements can exploit with a position in futures.
Forward vs. Futures

- Forward are OTC instruments/Futures trade on organized markets (CBOT, ...). Futures are standardized/Forward are not.
- Futures are marked-to-market: every gain/loss is settled at the end of each trading day, through the margin system.
  - The value of a Futures contract is always 0. In a Forward contract gains and losses are realized at delivery date.
  - Futures contract are much less affected by credit risk than Forward.
- Every investor’s counterparty is the Clearing House.
- Forward usually reach maturity (physical settlement)/Futures are usually closed before maturity (cash settlement), by taking an opposite position.
Forward Contracts: payoff

- Today: 0; delivery date: \( T \).
- \((S_t)\): spot price process of underlying; \( K \): delivery price.
- PAYOFF at delivery is

\[
S_T - K \quad \text{long position}; \quad K - S_T \quad \text{short position}
\]
Forward Contract Analysis

- **Long Forward contract value**: \((V_t)\);
  - Short Forward contract value: \((-V_t)\);
- **Forward price at** \(t\) for **\(T\)**: \((F^T_t) =\) delivery price at \(t\) for \(T\).

\[
\begin{align*}
V_0 &= 0, \quad V_T = S_T - K; \\
F^T_0 &= K, \quad F^T_T = S_T \text{ (convergence)}.
\end{align*}
\]

- **\(V_t =?\), \(F^T_t =?\)** for \(0 \leq t \leq T\).
- Assume that **owing the underlying asset generates no income/outcome** during the life of the contract (commodities requires storage costs/stocks pays dividends/bonds pays coupons).
Forward Contract Analysis

\[ V_t = S_t - K d_t(T), \]

\[ \sim \quad F_t^T = S_t b_t(T) \text{ and in particular} \]

\[ K = S_0 b_0(T). \]

- Forward price = accumulated value of spot price.

**Proof.** At time \( t < T \), long position in the underlying, lend \( K d_t(T) \) until \( T \). Strategy produces same cash-flows as forward. Apply LOP.

\( F_t^T \) defined by \( K = F_t^T \) and \( V_t = 0 \).
Consider a stock worth now $S_0 = 100€$, $T = \frac{1}{4}$, force of interest is $Y_{0,\frac{1}{4}} = 3\%$. The stock pays no dividend. Delivery price is then $K = 100.753$. Enter 1000 such contracts for buying stock at €100.753 in 3 months.

After 1 month ($t = \frac{1}{12}$), stock price is $S_{\frac{1}{12}} = 100.5$ and force of interest is $Y_{\frac{1}{12},\frac{1}{4}} = 2.5\%$. Value of one forward contract for long position is $V_{\frac{1}{12}} = 0.166$. Market value is then 166€.

At delivery date, stock price is $S_{\frac{1}{4}} = 101$. Payoff for one contract is $S_{\frac{1}{4}} - K = 0.247$, so payoff is 247.18€.
Forward Contract Analysis: extension

- Suppose that *owing the security generates a known income/outcome* over $[0, T]$.

- Let $Q_{t,T}$ be present value at $t$ of income/outcome over $[t, T]$.

  Then $V_t = S_t - Q_{t,T} - K d_t(T)$
  $\Rightarrow F_t^T = (S_t - Q_{t,T}) b_t(T)$
  $\Rightarrow K = (S_0 - Q_{0,T}) b_0(T)$.

- **Proof.** Same as before, but add: ‘borrow $Q_{t,T}$ for relevant maturities, so as to reproduce cash-flow generated by holding the underlying’.

- Examples: forward contracts on stocks paying known dividends or on coupon bearing bonds.
Forward Contracts on Currencies

- Agreement on exchange rate to be applied at future date, on a given nominal.
- \((S_t) \in/\$\) (spot) exchange rate, i.e. price of \(\in\) in \$, \(\in\) is \(S_t\); 1$ is \(\in/S_t\).
- Parties agree on \(\in/\$\) exchange rate \(K\) to be applied at \(T\) (i.e. to exchange \(\in\) for \(K\)); payoff in $ is \(S_T - K\).
- \(V_t = S_t d_t^\$ (T) - K d_t^\€ (T)\) in $, where \(d_t^c (T) = e^{-Y_{t,T}^c (T-t)}\) for \(c=\$, \€;\)

**proof.** Borrow \(K\) in $ up to \(T\); lend \(d_t^\€ (T)\) up to \(T\), convert in $ at prevailing rate.

\(\leadsto \quad K = S_0 e^{-Y_{0,T}^\€ - Y_{0,T}^\$}\) interest rates parity relation.
Forward Rate Agreements (FRA)

- Agreement between two parties to lend/borrow a nominal $N$ at a given simple interest rate $L_{FRA}$ over $[T, U]$ ($0 < T < U$; $T$ settlement date, $U$ maturity).
- Long party borrow/short party lend.
- Convention is to settle at $T$ the difference between $L_{FRA}$ and prevailing LIBOR rate $L_{T,U}$; payoff in $T$ is

$$N \left( U - T \right) \left( L_{T,U} - L_{FRA} \right) d_T(U).$$

- Could be seen as forward contract on ZCB.
- Initial value is zero $\implies L_{FRA} = L_{0,T,U}$. 
Forward Rate Agreements (FRA)

- $V_t^{\text{FRA}}$ value of the FRA ($0 \leq t \leq T$).

A FRA can be valued as if forward rates are realized:

$$V_t^{\text{FRA}} = N (U - T) (L_{t,T,U} - L_{\text{FRA}}) d_t(U).$$

**Proof.** Build the strategy at $t$:
- borrow $N (1 + L_{\text{FRA}}(U - T)) d_t(U)$ until $U$;
- invest $d_t(T)$ until $T$, then reinvest proceedings until $U$ at prevailing rate $L_{T,U}$.

- $V_t^{\text{FRA}} \iff 0$ according to $L_{t,T,U} \iff 0$.

<table>
<thead>
<tr>
<th>$T \times U$</th>
<th>EURIBOR FRA rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 $\times$ 6</td>
<td>3.78</td>
</tr>
<tr>
<td>6 $\times$ 9</td>
<td>3.84</td>
</tr>
<tr>
<td>9 $\times$ 12</td>
<td>3.84</td>
</tr>
<tr>
<td>6 $\times$ 12</td>
<td>3.86</td>
</tr>
<tr>
<td>12 $\times$ 18</td>
<td>3.77</td>
</tr>
</tbody>
</table>
Consider the $9 \times 12$ FRA equal to 3.84%, and a nominal amount of 1000000€.

Suppose in 6 months EURIBOR term structure (simple rates) is flat at 3.5%. Then forward rates equal spot rates. Value of the FRA is negative:

$$V_{\frac{1}{2}}^{\text{FRA}} = 1000000 \frac{\frac{1}{4}(0.035 - 0.0384)}{1 + \frac{1}{2}0.035} = -835€.$$

If at settlement (in 9 months) the prevailing 3 months EURIBOR has risen to $L_{\frac{3}{4},1} = 4.32\%$, then long party receives

$$1000000 \frac{\frac{1}{4}(0.0432 - 0.0384)}{(1 + 0.25 \cdot 0.0432)} = 1187.18€.$$
Futures: Marking-to-Market

- Delivery price: Futures price.
- An investor trading in futures must make an initial deposit in a margin account with its broker.
- At the end of each trading day, investor gains/losses (difference between closing and initial futures prices) increases/decreases the margin account; → contract value is 0 at the end of trading day; actually, any amount above initial margin can be withdrawn by the investor.
- If margin account falls below a maintenance margin → margin call: investor must deposit a variation margin and restore the initial margin.
- Broker has to maintain a similar account with clearinghouse.
Futures: Marking-to-Market

- Example: futures on gold (adapted from [Hull, 2006]).
- 1 futures contract on gold is for delivery of 100 ounces; futures price quoted (in $) is for 1 ounce; initial margin is 2000$ per contract; maintenance margin is 1500$ per contract.
- Long position in 10 futures contracts \( \rightarrow \) initial/maintenance margin is 20000$/15000$.

<table>
<thead>
<tr>
<th>day</th>
<th>Futures price</th>
<th>Daily gain/loss</th>
<th>Margin account</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>400</td>
<td>—</td>
<td>20000</td>
</tr>
<tr>
<td>2</td>
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<td>+100</td>
<td>21000</td>
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<tr>
<td>3</td>
<td>399</td>
<td>−200</td>
<td>19000</td>
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<tr>
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<td>394</td>
<td>−350</td>
<td>14000</td>
</tr>
<tr>
<td>6</td>
<td>393.5</td>
<td>−50</td>
<td>19500</td>
</tr>
</tbody>
</table>
Forward and Futures Prices

- Assume interest rates are flat and nonrandom \((Y_{t,T} = Y, d_0(\cdot) = d_t(\cdot))\). Then futures prices equal forward prices.

- \(0 < t_1 < t_2 < \ldots < t_n = T; 0: \) today, \(T: \) maturity; futures contract is marked to market at \(t_i, i = 1, \ldots, n\) (e.g. \(t_i - t_{i-1} = \Delta_i = 1/365\)).

- Let \(f_t \equiv f_{t}^{T}\) futures price in \(t; F_{0}^{T}\) forward price, and \(S_t\) spot price; by convergence, \(f_T = S_T\).

- **First strategy:** start at \(t_0 = 0,\) long \(b(t_1)\) futures; at \(t_i,\) cash-flow \(b(t_i) (f_{t_i} - f_{t_{i-1}})\), increase long position to \(b(t_{i+1})\) contracts \((i = 1, \ldots, n - 1)\).
Forward and Futures Prices

- At any date $t_i$, invest cash-flow $b(t_i)(f_{t_i} - f_{t_{i-1}})$ up to $t_n = T$ at risk-free rate; get $b(T)(f_{t_i} - f_{t_{i-1}})$.
- Globally, get $\sum_{i=1}^{n} b(T)(f_{t_i} - f_{t_{i-1}}) = b(T)(f_{t_n} - f_{t_0}) = b(T)(S_T - f_0)$.
- At 0, lend $f_0$ up to $T$, get $b(T)f_0$; entire strategy gives $b(T)S_T$ at $T$, and initial value $f_0$.
- **second strategy**: long $b(T)$ forward contracts at 0, lend $F^T_0$ at risk free rate up to $T$; payoff in $T$ $b(T)(S_T - F^T_0) + b(T)F^T_0 = b(T)S_T$; initial value $F^T_0$.
- LOP $\Rightarrow f_0 = F^T_0$. 
Swaps

- Among most popular OTC derivatives.
- Agreement between two parties to exchange regular cash-flows based on some economic variable.
- Typically, one party pays fixed cash-flows, the other variable cash-flows.
- Can be seen as combinations (portfolios) of forward contracts.
- Cash-flows computed based on *notional* amount. Only net cash-flows are actually exchanged.
- Focus on standard type of swaps: interest rate swaps (IRS).
- Swaps are used for
  - hedging;
  - asset/liability *transformation*;
  - exploiting *comparative advantages*; see [Hull, 2006].
Interest Rate Swaps (IRS)

- **Plain Vanilla** fixed-for-floating swap: agreement to exchange payments based on a fixed rate (fixed branch), known as swap rate \( L_{SWAP} \), against payments based on a variable rate (floating branch), e.g. LIBOR, applied to same notional amount.
- Long party: fixed rate payer/short part: fixed rate receiver.
- Payments at \( t_i = i\Delta \) (e.g. \( \Delta = 1/2 \)), \( i = 1, \ldots, n \). Net cash-flow at \( t_i \) (long position) is \( N \Delta (L_{t_{i-1},t_i} - L_{SWAP}) \) known at reset date \( t_{i-1} \).
- Maturity ranging from 2 to 30 years.
- \( L_{SWAP} \) is fixed so that initial contract value is 0: no cash-flow at \( t_0 = 0 \).
Overview of Financial Markets and Instruments

Swaps

Fixed/floating branches

Floating branch

\[ N \Delta L_{t_0,t_1}, N \Delta L_{t_1,t_2}, \ldots, N \Delta L_{t_{n-1},t_n} \]

Fixed branch:

\[ -N \Delta L_{SWAP_{t_0}}, -N \Delta L_{SWAP_{t_1}}, \ldots, -N \Delta L_{SWAP_{t_n}} \]
Overview of Financial Markets and Instruments

Swaps

**IRS as portfolio of FRAs**

- Cash-flow at \( t_i \) is that of FRA with settlement \( t_{i-1} \) and maturity \( t_i \), FRA rate = SWAP rate.
- \( V_t^{SWAP} \) value of swap at \( t \) for long position:

\[
V_t^{SWAP} = \sum_{i=1}^{n} V_t^{FRA_i}
\]

Using results for FRAs, get

\[
L_{swap} = \frac{\sum_{i=1}^{n} L_{0,t_{i-1},t_i} d_0(t_i)}{\sum_{i=1}^{n} d_0(t_i)}.
\]

- Swap rate is weighted average of forward rates. Single FRAs would not be worth 0 at beginning but balance on average.
IRS as fixed coupon/floater exchange

- Add notional amount $N$ at $t_n$ for both fixed and floating branch. Net cash-flows remain unchanged.
- ▶ fixed branch $\equiv$ fixed coupon (coupon rate $= L_{SWAP}$) bearing bond;
  ▶ floating branch $\equiv$ floater.

- $CB_t$, $FL_t$ prices of coupon bond and floater at $t$;
  \[ V_t^{SWAP} = FL_t - CB_t. \]

- Recall floater trades at par at reset dates: $\rightsquigarrow$
  \[ V_{t_i}^{SWAP} = N - CB_{t_i}. \]

- $L_{SWAP}$ is such that $CB_0 = FL_0 = N$; i.e. coupon rate such that coupon bond trades at par: **par rate**:
  \[ L_{SWAP} = \frac{1 - d_0(t_n)}{\Delta \sum_{i=1}^{n} d_0(t_i)}. \]
IRS as fixed coupon/floater exchange

Modified floating branch $\equiv$ floater:

$$N \Delta L_{t_0,t_1} \quad N \Delta L_{t_1,t_2} \quad \ldots \quad N(1 + \Delta L_{t_{n-1}, t_n})$$

Modified fixed branch $\equiv$ fixed coupon bond:

$$-N \Delta L_{SWAP} \quad -N \Delta L_{SWAP} \quad \ldots \quad -N(1 + \Delta L_{SWAP})$$
Overview of Financial Markets and Instruments

Options

Outline

1. Financial Markets and Primary Securities
2. Term Structure of Interest Rates
3. Forward and Futures Contracts
4. Swaps
5. Options
Options

- Agreement between two parties: one party (long position, or option holder) has the right to buy/sell the underlying at a given price (strike price or strike price), from/to the other party (short position, option writer).
- A call option gives the holder the right to buy, a put option gives the holder the right to sell.
- Unlike forward (or futures, or swaps), options gives rights, not obligations. The writer must stand the holder decision.
- Deciding whether or not to buy/sell is known as exercising the option.
- An option is European/American if exercise can take place at maturity/at any time before maturity.
- Since an option confer a right, the holder has to pay a price (option premium) at inception.
Option Markets

- Options can be used for hedging and speculation; unlike forward, allow to make profits without incurring any loss.
- Options are traded both on exchanges (CBOE, ...) (with a margin system like for futures) and OTC.
- Underlying can be: stock/indices/currencies/commodities/futures/Swaps (Swaptions)...
- Most options are American; European are simpler to analyze.
- Typically, several strikes and maturities are quoted at any trading date.
- Standard options are called plain-vanilla; many other types of options are exotics.
Options payoff

- 0: today; \( T \): maturity;
- \((S_t)\): spot price of underlying; \( K \) strike price.
- \( C_t, P_t \): prices of American call/put,
- \( c_t, p_t \): prices of European call/put,
- At any time \( 0 < t < T \), holder can (i) sell the option (ii) if American, exercise the option (early exercise) (iii) do nothing;
  at maturity \( T \), holder can (j) exercise the option (jj) do nothing.
- Since holder is rational, at maturity \( T \) exercise call/put iff \( S_T > K / S_T < K \).

Call payoff (Call value):
\[
C_T = c_T = \max\{S_T - K, 0\}
\]

Put payoff (Put value):
\[
P_T = p_T = \max\{K - S_T, 0\}
\]

Call/put Writer payoff: \( \min\{0, K - S_T\} \) and \( \min\{0, S_T - K\} \).
Options payoff

- **Long Call**: Gain increases with the increase in the stock price $S_T$ above the strike price $K$.
- **Short Call**: Gain is limited to the premium received, and it decreases as $S_T$ increases above $K$.
- **Long Put**: Gain increases with the decrease in the stock price $S_T$ below the strike price $K$.
- **Short Put**: Gain is limited to the premium received, and it increases as $S_T$ decreases below $K$.
Option Strategies

- One can build many strategies using options.
- **EXAMPLE I.** Long put + long stock:
  \[\max\{K - S_T, 0\} + S_T = \max\{S_T, K\}\] (minimum guarantee).

- **EXAMPLE II.** Long call strike \(K_1\), short call strike \(K_2(> K_1)\) (bull spread):

![Bull spread payoff diagram]
Overview of Financial Markets and Instruments
Options

... Option Strategies

- **Example III.** Long call strike $K_1$ and $K_3$, short 2 call strike $K_2$ ($K_1 < K_2 < K_3$): (butterfly spread):

  ![butterfly spread payoff diagram](image)

  - butterfly spread payoff

  $S_T$
Option Moneyness

- At any time $0 \leq t \leq T$ the option is in/at/out of the money if immediate exercise (only hypothetical for European options) generates a positive/null/negative cash-flow.

- Hence a call option is in/at/out of the money according to $S_t >, =, < K$. A put option is in/at/out of the money according to $S_t <, =, > K$.

- A necessary condition for exercise is that the option be in the money (also sufficient at maturity).

- The intrinsic value of an option at $0 \leq t \leq T$ is $\max\{S_t - K, 0\}$ for a call, $\max\{K - S_t, 0\}$ for a put.

- $C_t \geq \max\{S_t - K, 0\}$, $P_t \geq \max\{K - S_t, 0\}$ (otherwise, buy option and exercise). The difference, if positive, is the time value of the option.
Option Bounds

- If an American option has time value, exercise is not convenient (better wait or sell it!), even if option is deep in the money.
- Pure no-arbitrage reasonings lead only to bounds for option prices.
- Clearly, $C_t \geq c_t$ and $P_t \geq p_t$, for $0 \leq t \leq T$.
- Bounds for options on non-dividend paying stock: for $0 \leq t \leq T$,
  - **Bounds for Call Options:**
    $$\max\{S_t - K d_t(T), 0\} \leq c_t \leq C_t \leq S_t;$$
  - **Bounds for European Put Options:**
    $$\max\{K d_t(T) - S_t, 0\} \leq p_t \leq K d_t(T);$$
  - **Bounds for American Put Options:**
    $$\max\{K - S_t, 0\} \leq P_t \leq K;$$
- **Proof.** If an inequality does not hold, build an arbitrage.


**Options**

... Option Bounds

- Note that \( C_t \geq \max\{S_t - K d_t(T), 0\} > \max\{S_t - K, 0\} \)
  (provided \( Y_{t,T} > 0 \) and \( S_t > K \)), so that it is **never convenient to early exercise an American call option on a non dividend paying stock** \( \rightsquigarrow C_t = c_t \); instead, early exercise of an American put may be convenient.

- **European Put-Call Parity**: for all \( 0 \leq t \leq T \)
  \[
  p_t + S_t = c_t + K d_t(T)
  \]

  **Proof.** Long position on put and stock = long position on call+lending \( K d_t(T) \).

- **American Put-Call relation**: for all \( 0 \leq t \leq T \)
  \[
  C_t + K d_t(T) \leq P_t + S_t \leq C_t + K.
  \]
Previous inequalities extend to stocks paying known dividends.

If $Q(t, T)$ denotes present value in $t$ of dividends paid in $[t, T]$, put-call European parity becomes:

$$pt + St - Q(t, T) = ct + K dt(T).$$

Early exercise of American call may be convenient only immediately before dividend paying dates.

Early exercise of American put may be convenient only immediately after dividend paying dates.
In order to value options, we have to set up a model.

Simplest type is binomial: uncertainty can evolve with 2 possible scenarios.

Consider a market with two assets, stock and a bond.

Only 2 dates (1 period), 0 and 1. Price today is $S_0 = S$, price in 1 is $S_1$ with either $S_1(u) = Su$ or $S_1(d) = Sd$ with $u > d(> 0)$. Interest rate for $[0, 1]$ is $R$. 

\[ S 
\quad \rightarrow \quad Su 
\quad \rightarrow \quad Sd \]
A strategy is \((\phi, \eta) \in \mathbb{R}^2\); the value at 0 is 
\[ V_{0}^{\phi,\eta} = \phi S_0 + \eta; \] value at 1 is 
\[ V_{1}^{\phi,\eta} = \phi S_1 + \eta(1 + R). \]

An arbitrage opportunity is a strategy \((\phi, \eta)\) such that 
\[ V_{0}^{\phi,\eta} = 0 \] and 
\[ V_{1}^{\phi,\eta}(u) \geq 0, \ V_{1}^{\phi,\eta}(d) \geq 0 \] (with one of the two inequalities strict).

No arbitrage iff 
\[
0 < q \equiv \frac{(1 + R) - d}{u - d} < 1
\]
iff \(d < 1 + R < u\).

Think of \(q\) as probability of ‘\(u\)’ movement, \(1 - q\) of ‘\(d\)’ movement. Call \(Q\) this probability.
...Binomial Model

- We have \( S_0 = E^Q \left[ \frac{S_1}{1+R} \right] \).
  \( \Rightarrow \) \( Q \) is Risk Neutral probability.

- \( C = (C(u), C(d)) \in \mathbb{R}^2 \) contingent claim (e.g. \( C = \max\{S_1 - K, 0\} \)); there exists a strategy \((\phi, \eta)\) such that \( V_{1}^{\phi,\eta} = C \); indeed the hedge ratios are

\[
\phi = \frac{C(u) - C(d)}{S_0 (u - d)}, \quad \eta = \frac{C(u) d - C(d) u}{(1 + R)(u - d)}
\]

We say that markets are complete (any contingent claim is replicated). Moreover,

\[
V_{0}^{\phi,\eta} = E^Q \left[ \frac{C}{1+R} \right]
\]

\( \Rightarrow \) \( V_{0}^{\phi,\eta} \) is the initial price one should pay for \( C \).
Exotic Options

- **Barrier Options**: option is activated/cancelled if a barrier is reached/not reached during option’s life.
- **Lookback Options**: underlying or strike is the minimum or maximum value of stock during stock’s life.
- **Asian Options**: underlying or strike is the average value of stock during stock’s life.
- **Compound Options**: option on an option.
- **Binary Options**: receive a fixed amount in case of exercise.
- **Basket Options**: options on maximum or minimum of several assets.
- **Exchange Options**: options to exchange an asset for another asset.
Many securities contain embedded options:

- **Callable bonds**: issuer may retire the bond (has a call option).
- **Convertible bonds**: bondholder can convert bond into issuer company’s stock.
- **Minimum guarantees**: insurance contracts are often equity-linked with minimum guarantee.
- **Surrender option**: policyholder may surrender an insurance contract.
- **Prepayment options**: mortgagors have the right to prepay mortgage.
- **Executive stock options**: options to incentivize corporate managers.

...