

# Problem Sheet 6

## I. THE FERMI GAS

### A. Sommerfeld expansion

The evaluation of thermodynamic properties of Fermi systems involves the calculation of integrals of the type

$$I = \int_0^{\infty} \frac{g(\epsilon)}{e^{(\epsilon-\mu)/T} + 1} d\epsilon, \quad (1.1)$$

where  $g(\epsilon)$  is a product of the density of states  $\nu(\epsilon)$  and some energy dependent observable, e.g.,  $g(\epsilon) = \nu(\epsilon)$  for the density  $I = N/V = n$ ; or  $g(\epsilon) = \nu(\epsilon)\epsilon$  for the energy density  $I = E/V$ .

Derive the systematic expansion of such integrals in powers of  $T/\mu \ll 1$ ! Recipe: Note that at  $T = 0$

$$f_{FD}(\epsilon, T) = \frac{1}{e^{(\epsilon-\mu)/T} + 1} \xrightarrow{T \rightarrow 0} \Theta(\epsilon - \mu), \quad (1.2)$$

where  $\Theta(x) = 1$  if  $x > 0$  and 0 otherwise. Substitute  $f_{FD}(\epsilon, T) = \Theta(\epsilon - \mu) + f_{FD}(\epsilon, T) - \Theta(\epsilon - \mu)$ . Write the  $T$ -dependent part as a single integral over the distance from the chemical potential, divided by  $T$ ,  $z = |\epsilon - \mu|/T$  and neglect that the bottom of energy is finite (under which circumstances is that allowed?). Expand the integrand in  $T$  and obtain the first two terms! (Look up the integrals, or remember the Riemann  $\zeta$  function!)

Obtain the final result:

$$I = \int_0^{\mu} g(\epsilon) d\epsilon + \frac{\pi^2}{6} T^2 g'(\mu) + \frac{7\pi^4}{360} T^4 g'''(\mu) + \dots, \quad (1.3)$$

### B. Application: specific heat

Consider the Fermi sea at low  $T$ : Estimate roughly the fraction of the electrons which are "thermally active". By thermally active I mean electrons in single particle states which are only partly occupied, say with  $0.1 < n_{\epsilon} < 0.9$ . It is always these 'active' electrons that contribute to thermodynamic response! What behavior (power law of  $T$ ) do you thus expect for the specific heat at low  $T \rightarrow 0$ ?

Calculate the specific heat with the help of Sommerfeld's expansion! (Calculate  $E(T)$  and derive it with respect to  $T$ !).

### C. Pauli paramagnetism

In a magnetic field the electrons of a metal have energies  $\epsilon_{k,s=\pm 1} = (\hbar k)^2/2m + s\mu_B H$ , where  $\mu_B = \frac{e\hbar}{2mc}$  is the Bohr magneton. The magnetic field thus favors spins with orientation opposite to  $H$ .

The spin up and spin down electrons separately form a Fermi sea up to wavevectors  $k_F^\pm$  such that  $\mu = (\hbar k_F^\pm)^2/2m \pm \mu_B H$ . The magnetization is equal  $\mu_B$  times the excess of spin up over spin down electrons,

$$M = \mu_B(N_\uparrow - N_\downarrow), \quad (1.4)$$

with  $N = N_\uparrow + N_\downarrow$ . Calculate the Pauli susceptibility  $\chi_{\text{para}} = \partial M/\partial H$  at  $T = 0$ !

### D. Graphene: Density of states

Graphene is a newly discovered two-dimensional semimetal (a monolayer of graphite) which causes a lot of excitement these days, among others because of its electronic properties and its special band structure: the tight-binding dispersion  $\epsilon(\mathbf{k} = (k_x, k_y))$  is such that the energy surface in momentum space forms two cones, centered at two inequivalent points  $K, K'$  in the Brillouin zone. The energy is counted from the energy level of these points, that is we choose a reference point for energy such that  $\epsilon_{K,K'} = 0$ . Around these points, the dispersion can be approximated as

$$\epsilon(\mathbf{k} = \mathbf{K} + \mathbf{p}) = \hbar v_F |\mathbf{p}| \quad (1.5)$$

with the "Fermi velocity" or group velocity  $v_F \equiv |\partial\epsilon/\partial\hbar\mathbf{k}|$  which has the value  $v_F = 10^6 \text{m/s} = c/300$ . Note that electrons go really fast in metals! A lot of the excitement about graphene is due to the fact that this dispersion is the same as for ultrarelativistic particles (or photons) whose mass is negligible as compared to energy,  $mc^2 \ll E$ .

Neutral graphene has the special property that all the electronic states are filled up to the tip of the cones. The Fermi surface consists in just one point there! However, one can introduce more electrons (or less electrons) with an external gate voltage, controlling their density  $n$ . One thus fills more ( $n > 0$ ) or less levels ( $n < 0$ ) in the cones in the Brillouin zone. This shifts the chemical potential  $\mu$  away from the cone tip and creates a finite Fermi surface, like in a metal. Show that when the temperature is bigger than  $\mu$ , there are both "electron" ( $\epsilon > 0$ ), and "hole" states ( $\epsilon < 0$ ) which are only partly occupied, say with  $0.1 < n_\epsilon < 0.9$ . In this regime graphene behaves like an ultrarelativistic fluid of particles and antiparticles!

i) Calculate the density of states of this material! Why do you not find  $\nu(\epsilon) \sim \epsilon^{d/2-1} = \text{const.}$ , even though you are in two dimensions?

ii) Find the chemical potential  $\mu$  (i.e., the Fermi energy) at  $T = 0$  as a function of the gate-controlled extra density of electrons  $n$ ! Before calculating: draw a picture of the DOS and the Fermi function and predict whether you will obtain an increase or a decrease of  $\mu$ !

iii) Calculate the change in chemical potential at finite  $T$ , to lowest order in  $T \ll \mu$ . (Hint: Use the Sommerfeld expansion to calculate  $n(\mu, T)$  and determine  $\mu(T)$  imposing a fixed density  $n$ !)

You will find a result which is similar to a Fermi sea in 3d with parabolic dispersion  $\epsilon(k) = (\hbar k)^2/2m$ . How will the chemical potential behave in 1d and 2d systems with parabolic dispersion?