Relativistic magnetotransport in graphene, at quantum criticality and in black holes

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in collaboration with

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Taiwan String Workshop - Taipei 21 Dec, 2009

The challenge of strong coupling in condensed matter theory

- Electrons have strong bare interactions (Coulomb)

- But: non-interacting quasiparticle picture (Landau-Fermi liquid) works very well for most metals
  Reason: RG irrelevance of interactions,
  ↔ screening and dressing of quasiparticles

- Opposite extreme: Interactions much stronger than the Fermi energy → Mott insulators with localized e's

- Biggest challenge: strong coupling physics close to quantum phase transitions.
  Maximal competition between wave and particle character (e.g.: high Tc superconductors, heavy fermions, cold atoms, graphene)

The challenge of strong coupling in condensed matter theory

Idea and Philosophy:

Study [certain] strongly coupled CFTs (= QFT's for quantum critical systems) by the AdS-CFT correspondence

➔ Learn about physical properties of strongly coupled theories.

Extract the general/universal physics form the particular examples to make the lessons useful for condensed matter theory.

Outline

- Relativistic physics in graphene, quantum critical systems and conformal field theories
- Strong coupling features in collision-dominated transport – as inspired by AdS-CFT results
- Comparison with strongly coupled fluids (via AdS-CFT)
- Graphene: an almost perfect quantum liquid

Quantum critical systems in condensed matter

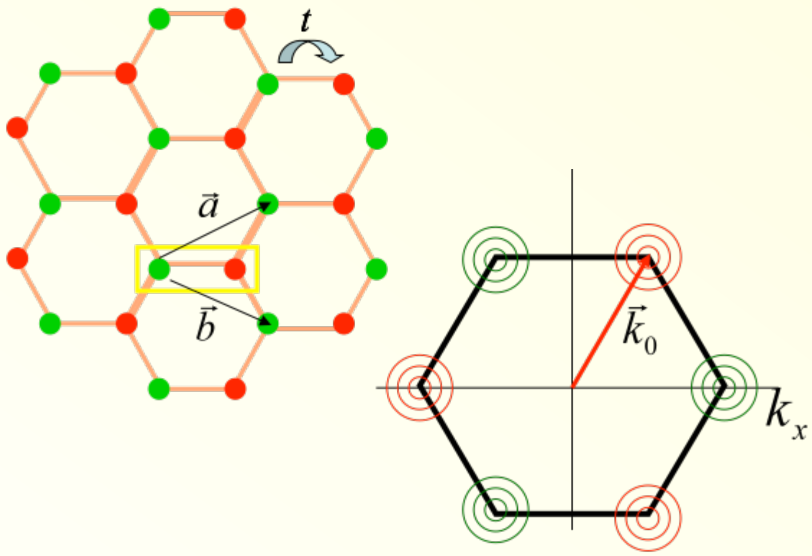
A few examples

- Graphene
- High Tc
- Superconductor-to-insulator transition (interaction driven)

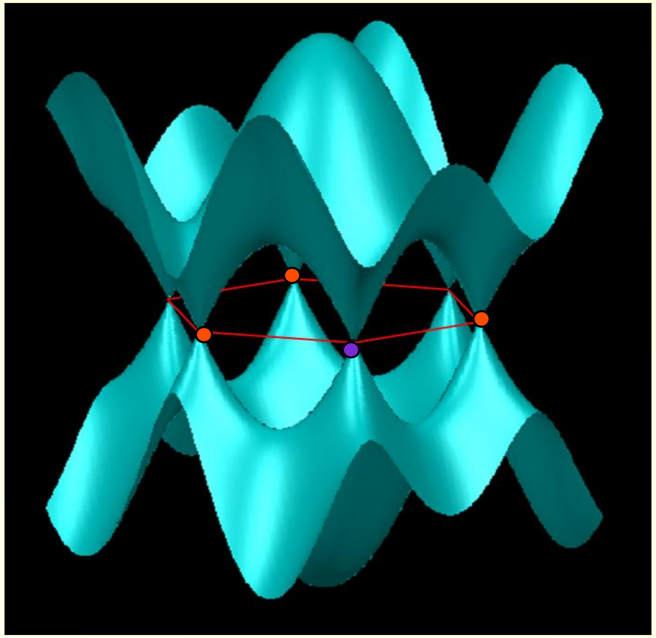
Dirac fermions in graphene

(Semenoff '84, Haldane '88)

Honeycomb lattice of C atoms



Tight binding dispersion



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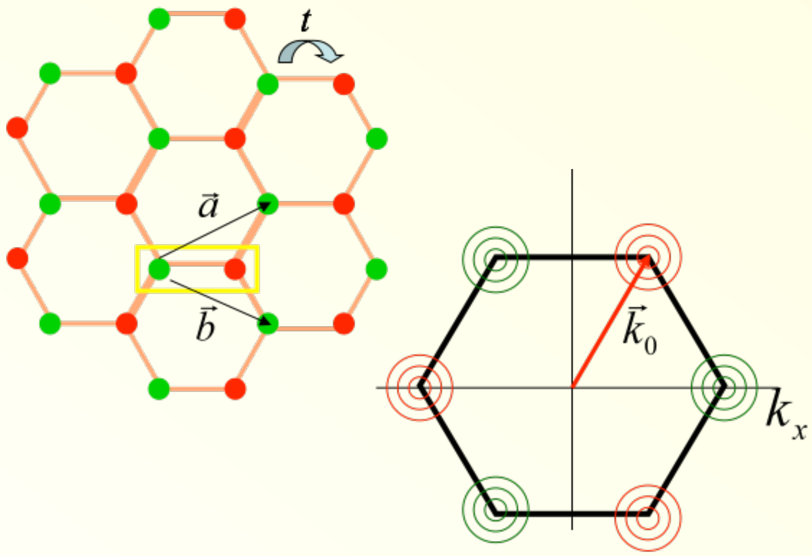

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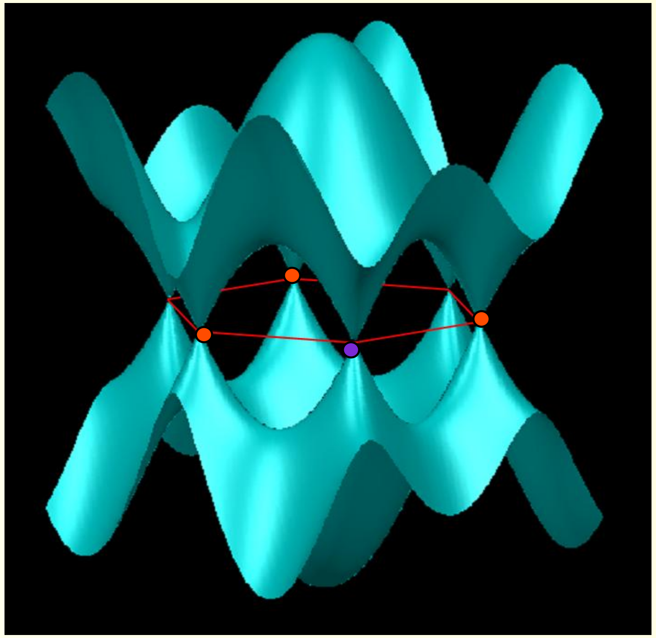

2 massless Dirac cones in the Brillouin zone: (Sublattice degree of freedom ↔ pseudospin)

Close to the two Fermi points **K**, **K'**:

$$H \approx v_F \left(\vec{\mathbf{p}} - \vec{\mathbf{K}}\right) \cdot \vec{\sigma}_{\text{sublattice}}$$

$$\rightarrow \quad E_{\mathbf{p}} = v_F \left|\vec{\mathbf{p}} - \mathbf{K}\right|$$

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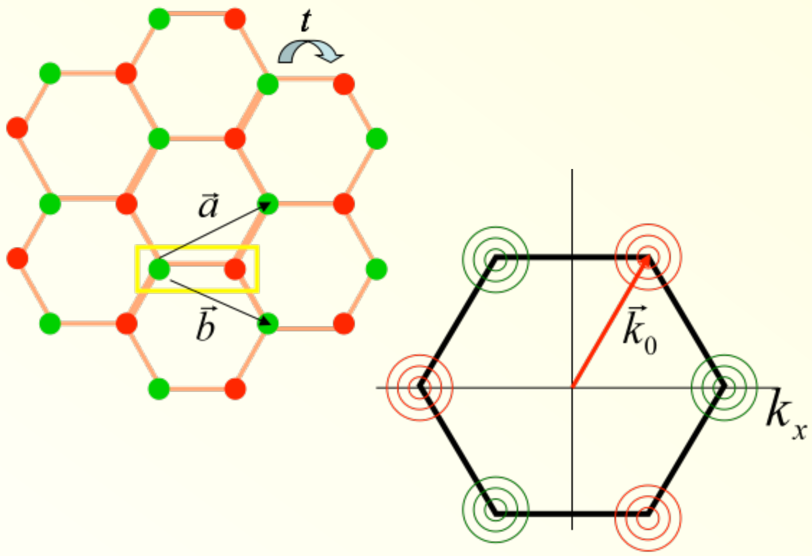

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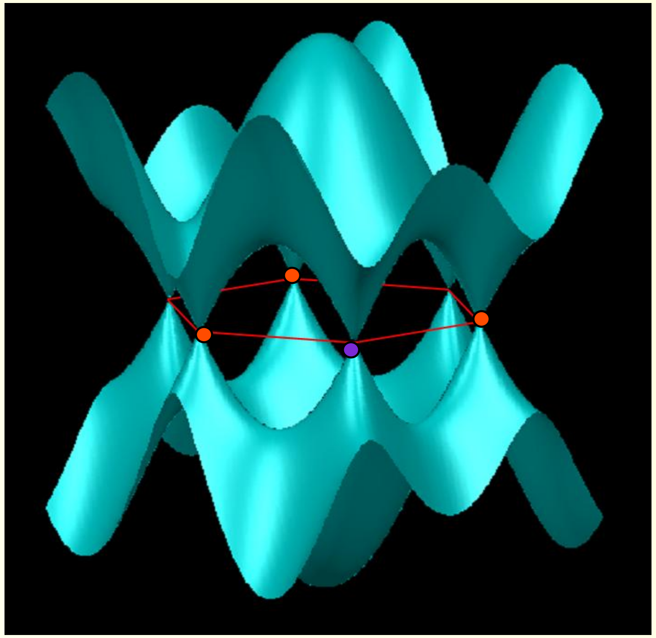

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Fermi velocity (speed of light")

$$v_F \approx 1.1 \cdot 10^6 \text{ m/s} \approx \frac{c}{300}$$

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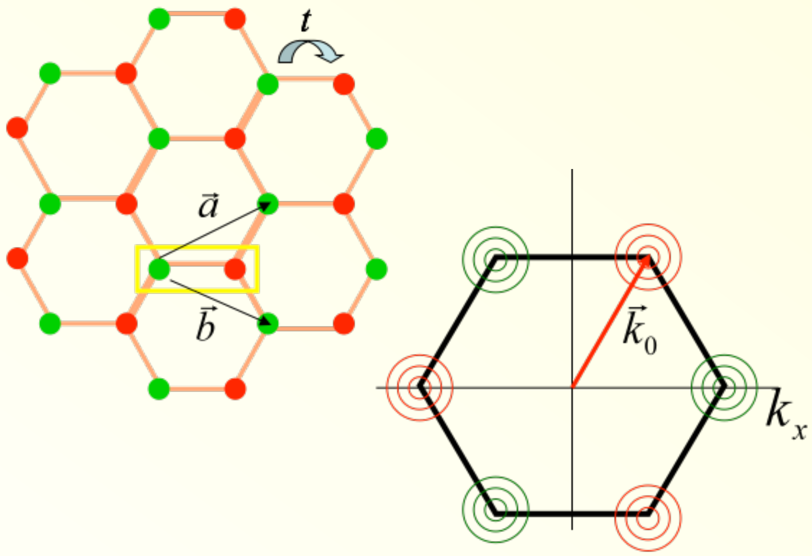

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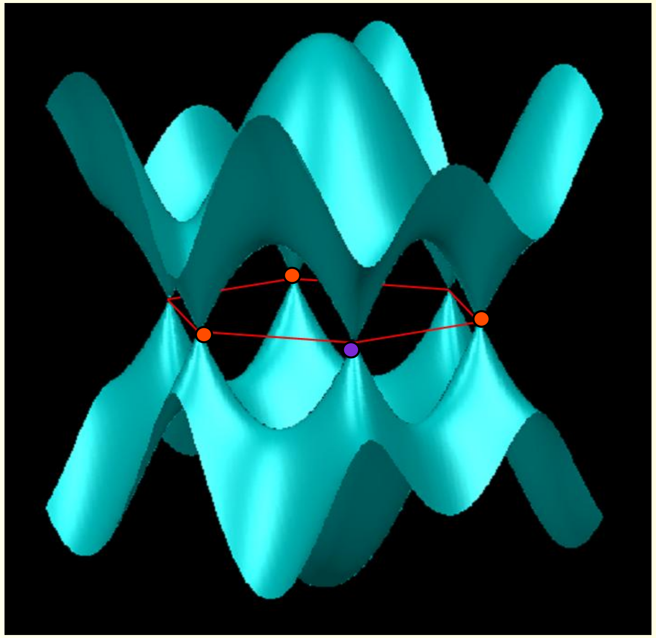

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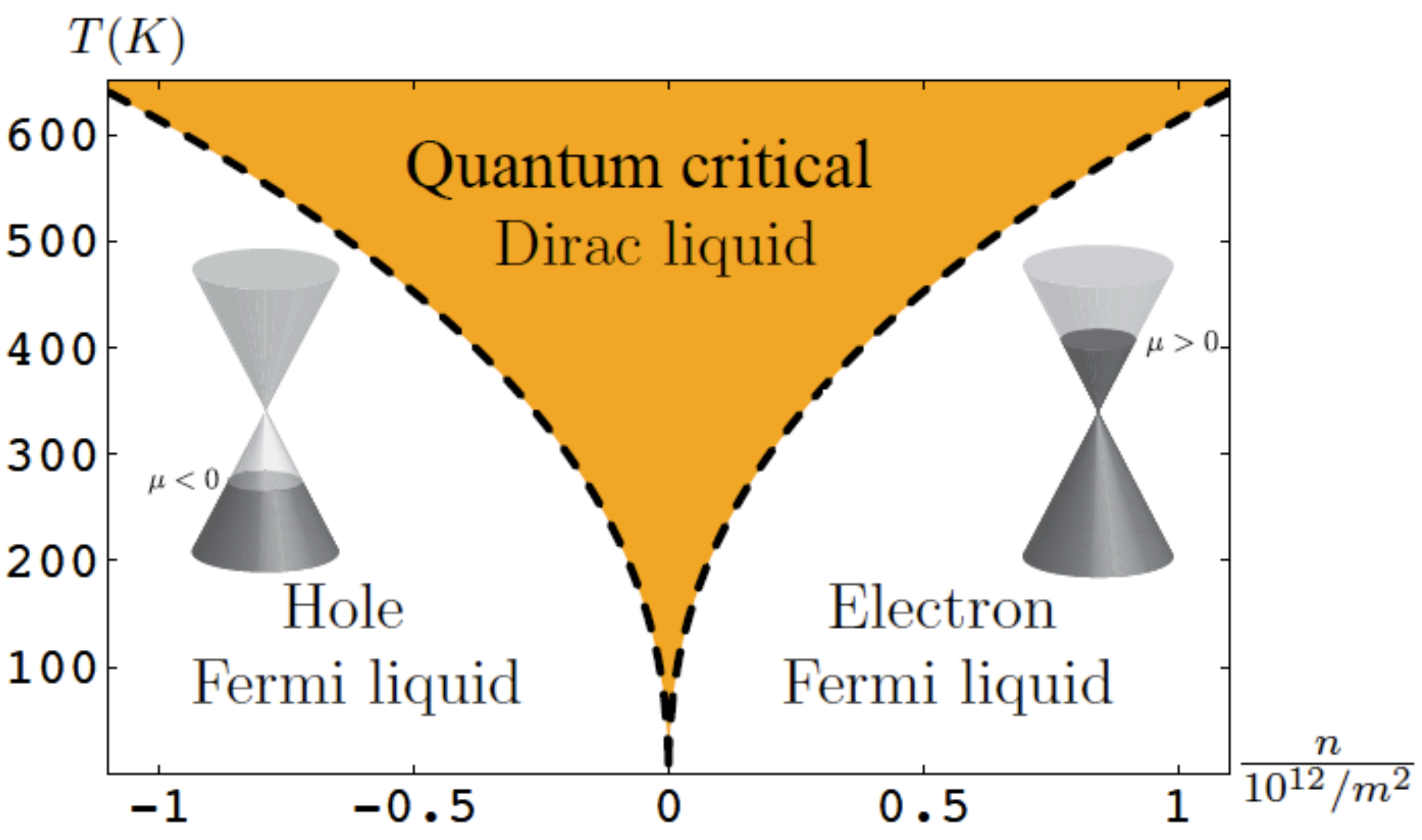
Coulomb interactions: Fine structure constant

$$\alpha \equiv \frac{e^2}{\varepsilon \hbar \mathrm{v}_F} = O(1)$$

Relativistic fluid at the Dirac point

D. Sheehy, J. Schmalian, Phys. Rev. Lett. ***99****, 226803 (2007).*

- Relativistic plasma physics of interacting particles and holes!



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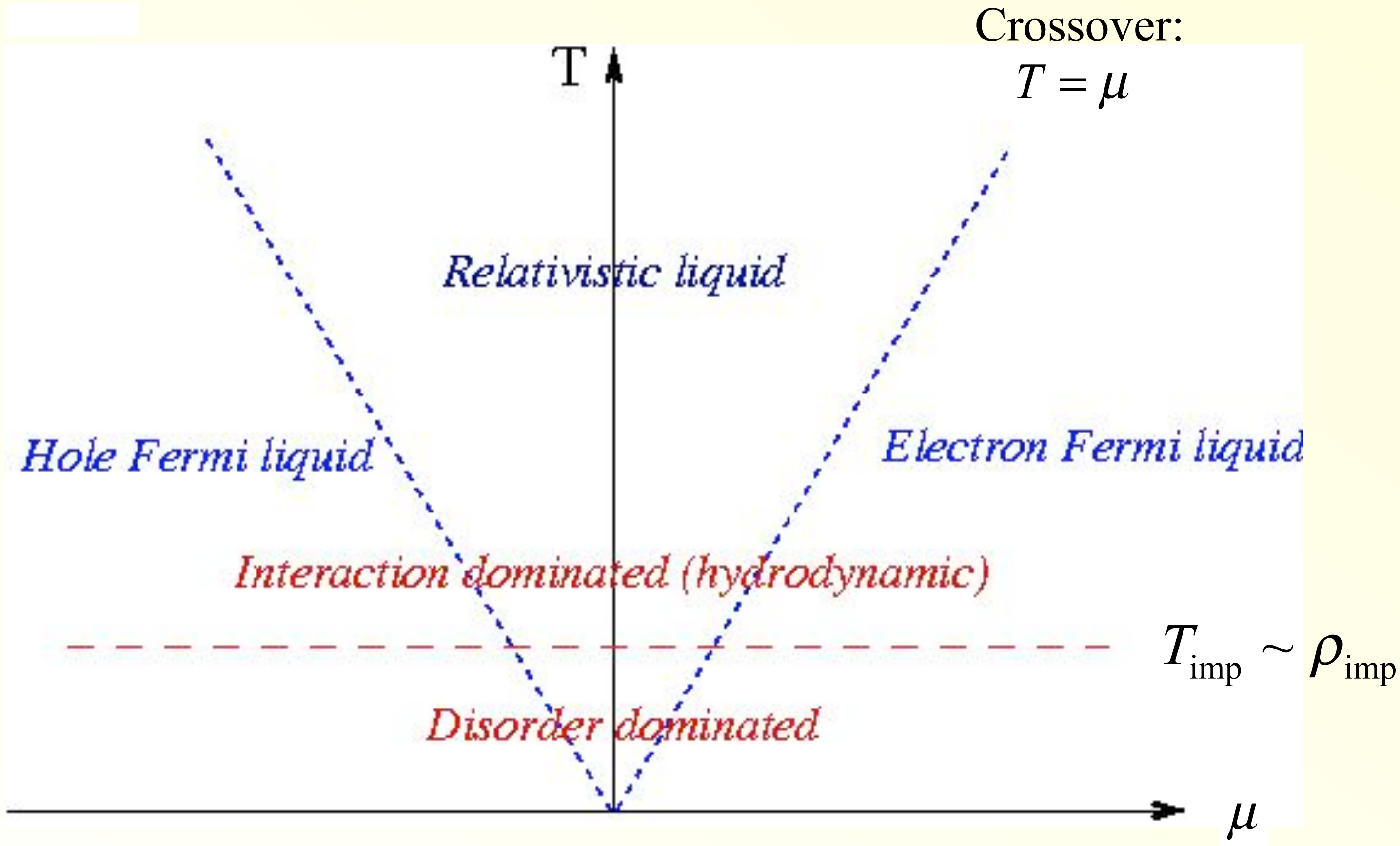

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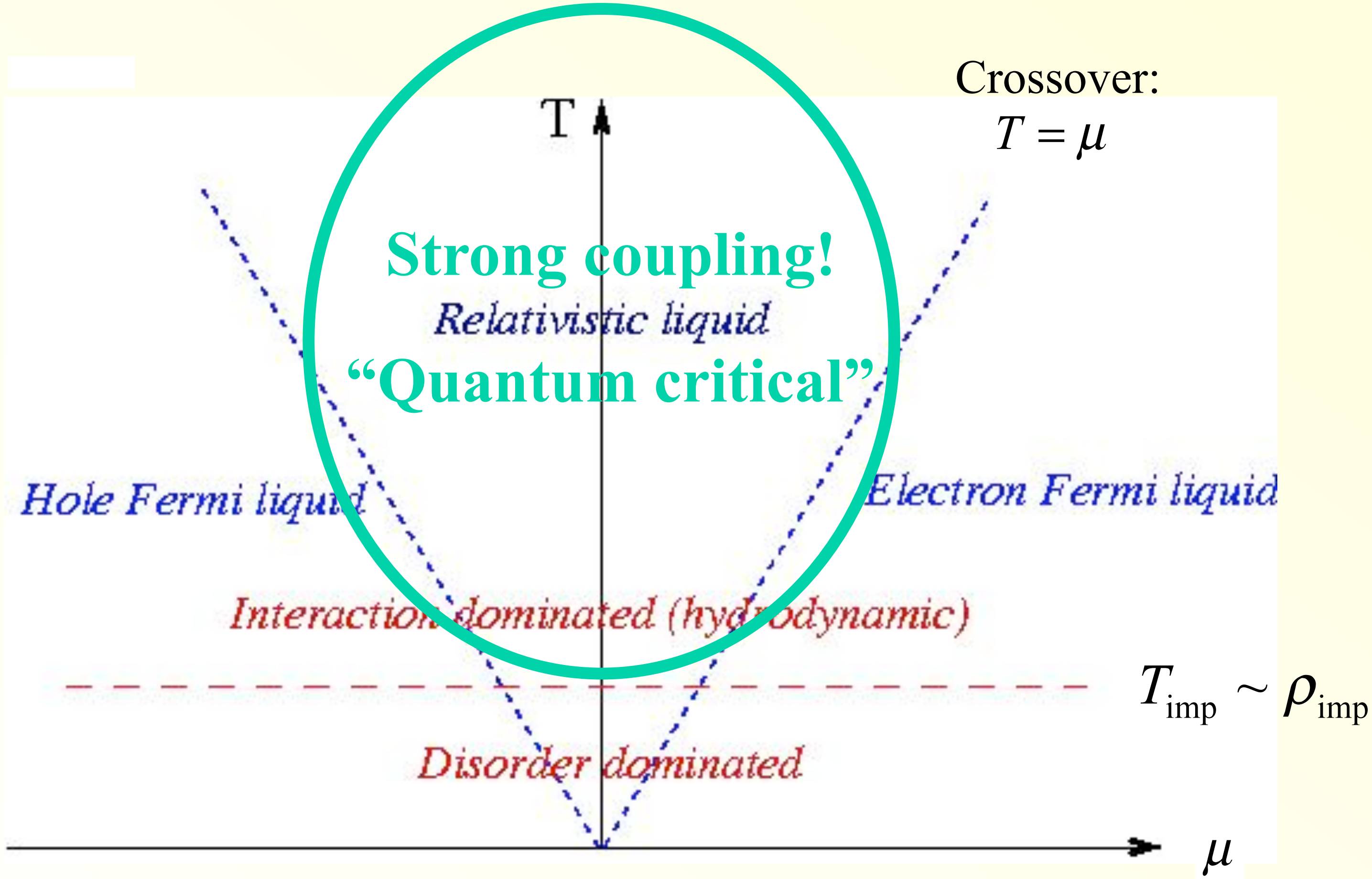
- Relativistic plasma physics of interacting particles and holes!
- Strongly coupled, nearly quantum critical fluid at $\mu = 0$



Other relativistic fluids:

- Systems close to quantum criticality (with z = 1)

Example: Superconductor-insulator transition (Bose-Hubbard model)

$$\boxed{\tau_{rel}^{-1} \approx \frac{k_B T}{\hbar}}$$

Maximal possible relaxation rate!

Damle, Sachdev (1996, 1997)
Bhaseen, Green, Sondhi (2007).
Hartnoll, Kovtun, MM, Sachdev (2007)

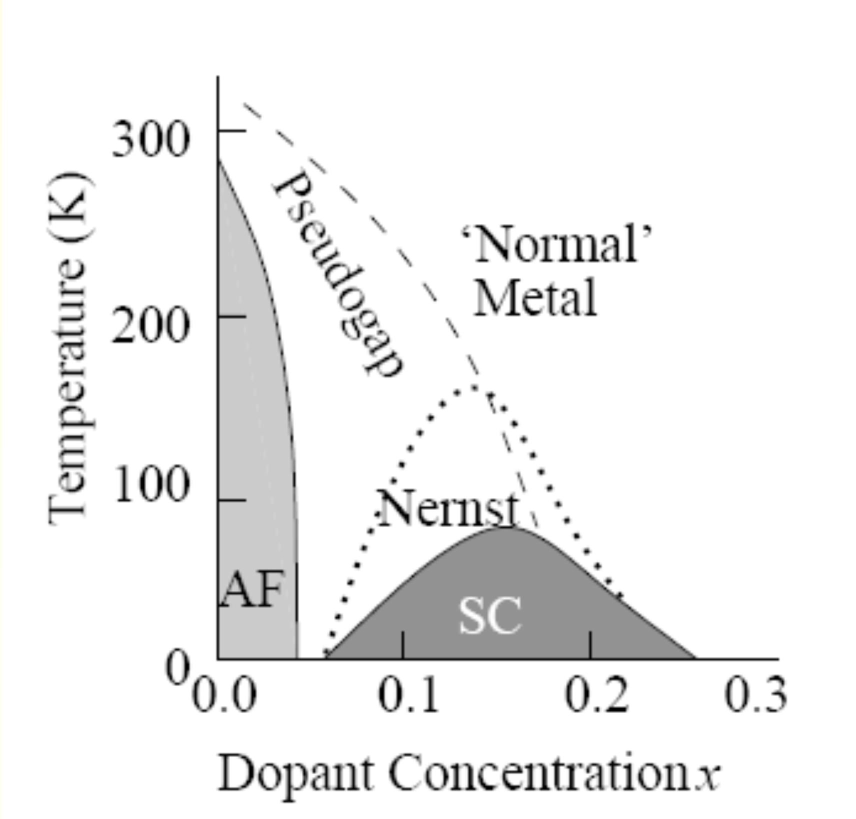
- Conformal field theories (QFTs for quantum criticality)

E.g.: strongly coupled Yang-Mills theories

→ Exact treatment via AdS-CFT correspondence

C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son (2007)
Hartnoll, Kovtun, MM, Sachdev (2007)

Nernst effect in High T_c's



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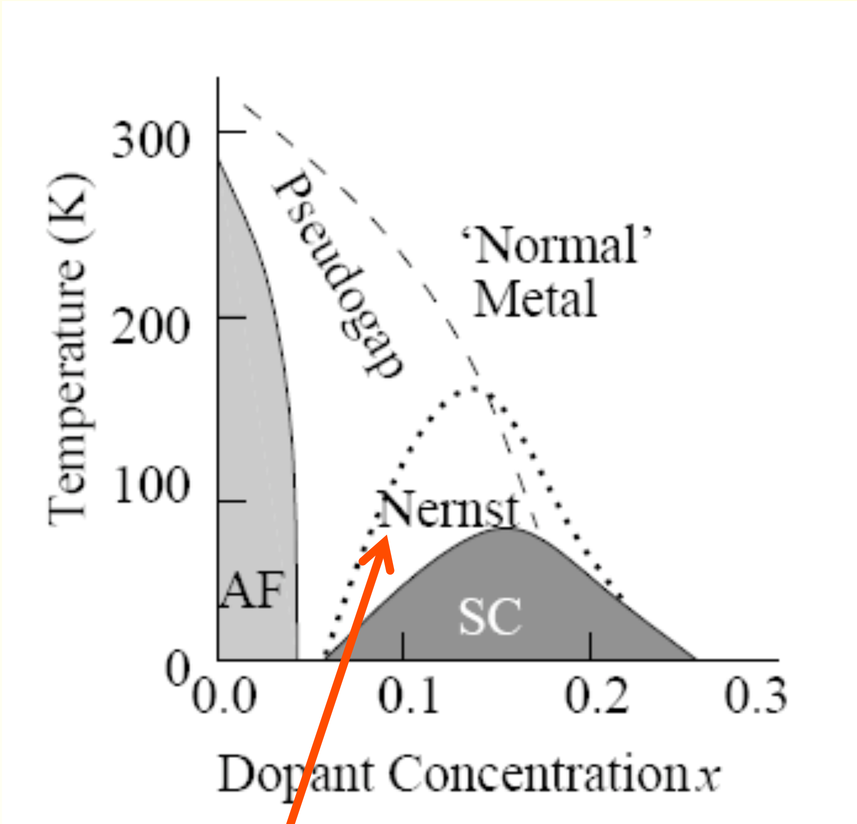
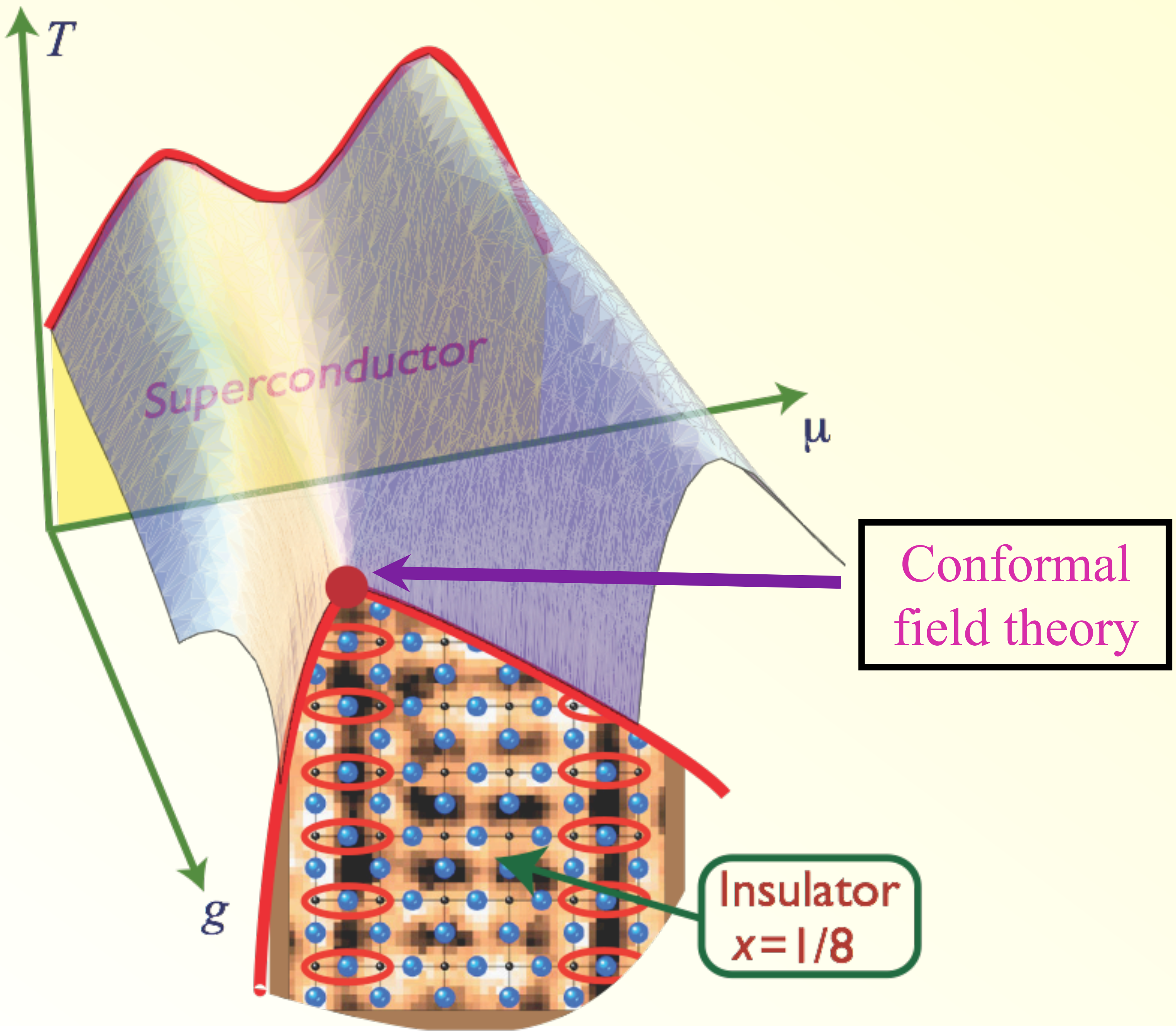


Underdoped high Tc superconductors:
Anomalously strong Nernst signal
up to $T=(2\text{-}3)T_c$

Quantum criticality in cuprate high T_c's



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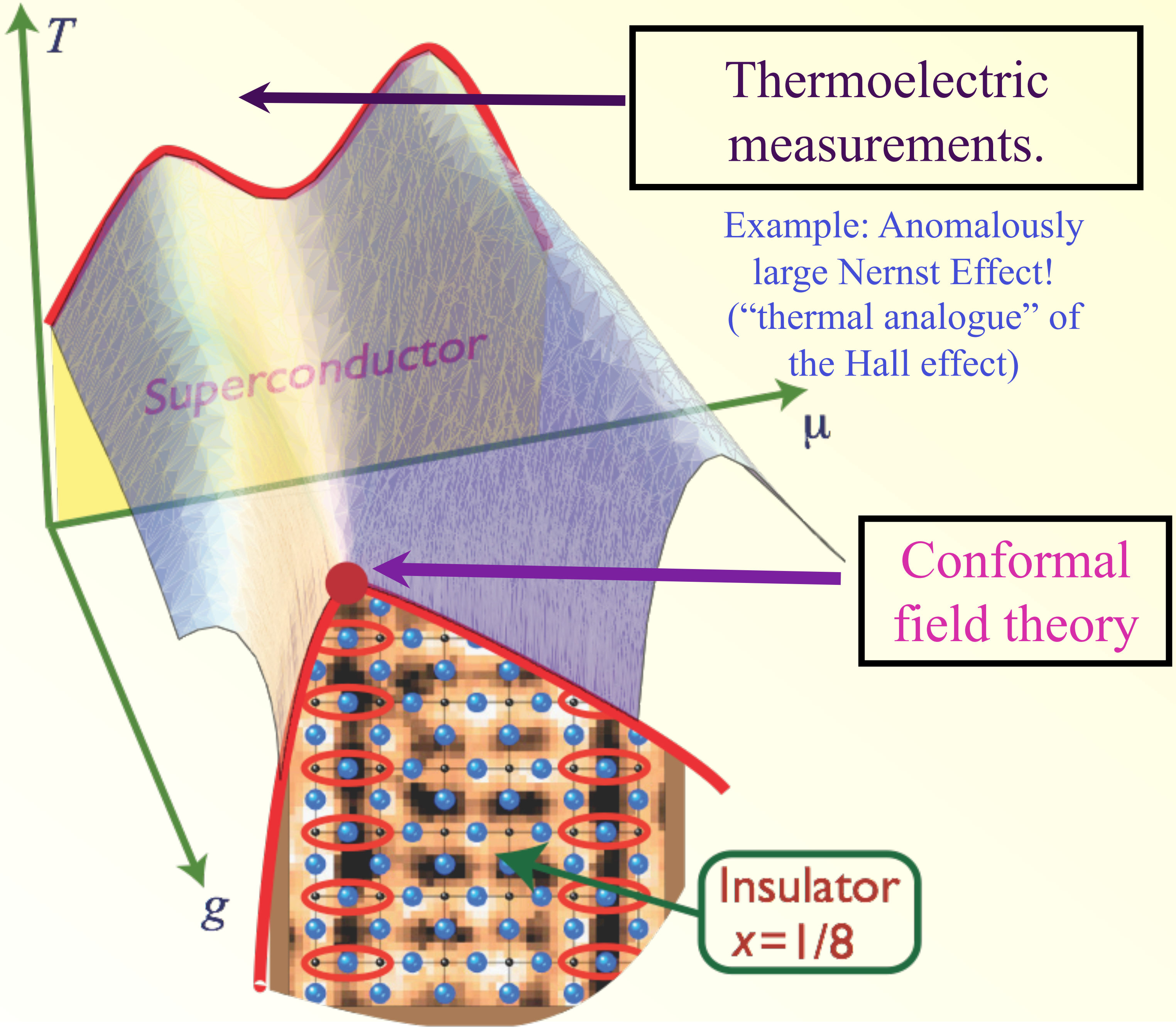
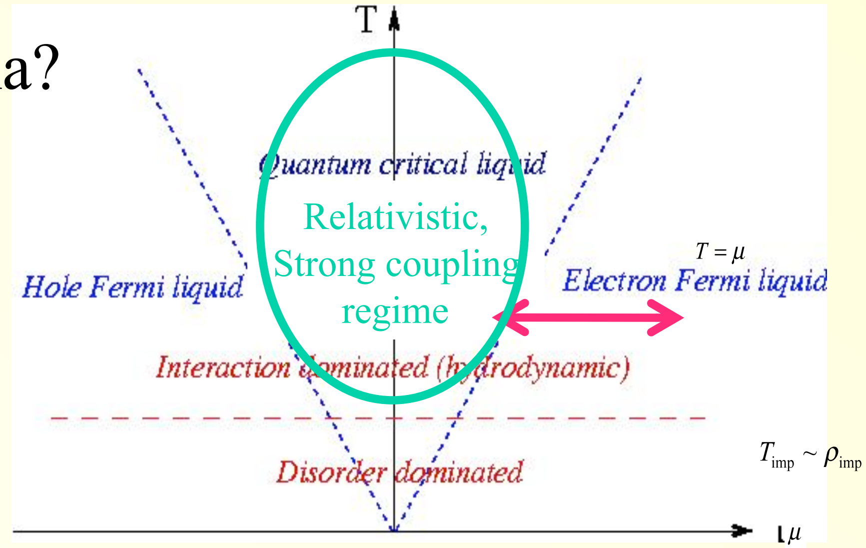


Simplest example exhibiting “quantum critical” features:

Graphene

Questions

- Transport characteristics in the strongly coupled relativistic plasma?
- Response functions and transport coefficients at strong coupling?
- Graphene as a nearly perfect and possibly turbulent quantum fluid (like the quark-gluon plasma)?



Graphene – Fermi liquid?

1. Tight binding kinetic energy
→ massless Dirac quasiparticles

$$H_0 = \sum_{\lambda=\pm} \sum_{a=1}^{N} \int \frac{d^2k}{(2\pi)^2} \lambda v_F k \, \gamma_{\lambda a}^\dagger(\mathbf{k}) \gamma_{\lambda a}(\mathbf{k})$$

2. Coulomb interactions:
Unexpectedly strong!
→ nearly quantum critical!

$$\boxed{V(\mathbf{q}) = \frac{2\pi e^2}{\varepsilon |\mathbf{q}|}}$$

$$H_1 = \frac{1}{2} \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \Psi_a^\dagger(\mathbf{k}_2 - \mathbf{q}) \Psi_a(\mathbf{k}_2) V(\mathbf{q}) \Psi_b^\dagger(\mathbf{k}_1 + \mathbf{q}) \Psi_b(\mathbf{k}_1)$$

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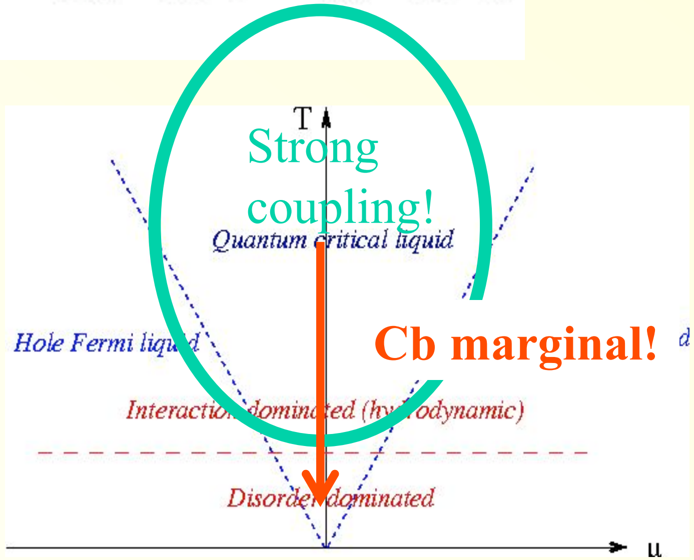
Coulomb **only marginally** irrelevant for μ = 0!



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RG:
($\mu = 0$)

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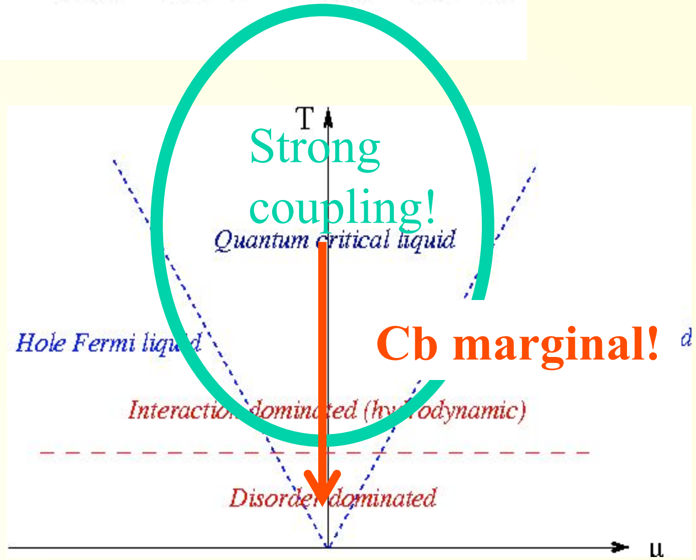

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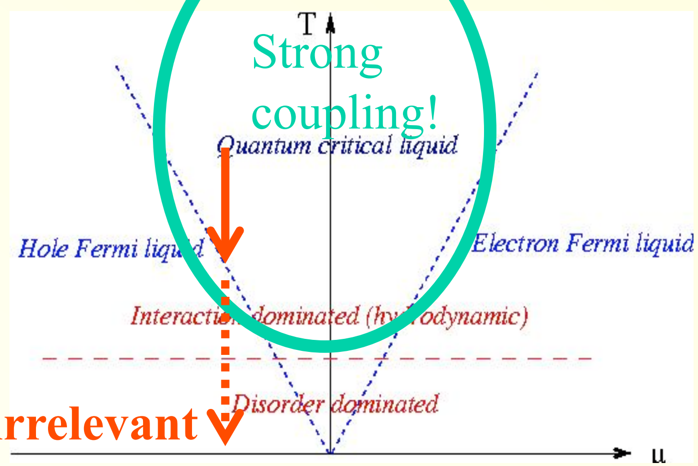
($\mu > 0$) $T < \mu$: **Screening kicks in, short ranged Cb irrelevant**



Strong coupling in undoped graphene

MM, L. Fritz, and S. Sachdev, PRB '08.

Inelastic scattering rate
(Electron-electron interactions)

μ>> T: standard 2d
Fermi liquid

$$\boxed{\tau_{\mathrm{ee}}^{-1} = C\,\frac{(k_B T)^2}{\hbar\mu}}$$

C: Independent of the Coulomb coupling strength!

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Fastest possible rate!

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$$\boxed{E_{qp}(\sim k_B T) \geq \Delta E_{\text{int}} = \hbar\tau_{\text{ee}}^{-1} \sim \alpha^2 k_B T}$$

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As long as α(T) ~ 1, energy uncertainty is saturated, scattering is maximal
→ Nearly universal strong coupling features in transport,
similarly as at the 2d superfluid-insulator transition *[Damle, Sachdev (1996, 1997)]*

Consequences for transport

1. -Collisionlimited conductivity σ in clean undoped graphene;
   -Collisionlimited spin diffusion D_s at any doping
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Collision-dominated transport → relativistic hydrodynamics:
a) Response fully determined by covariance, thermodynamics, and σ, η
b) Collective cyclotron resonance in small magnetic field (low frequency)

Hydrodynamic regime:
(collision-dominated)

$$\tau_{\mathrm{ee}}^{-1} >> \tau_{\mathrm{imp}}^{-1}, \omega_{\mathrm{c}}^{\mathrm{typ}}, \omega_{\mathrm{AC}}$$

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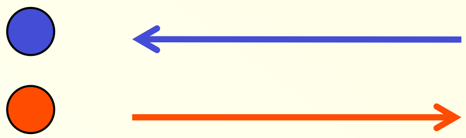

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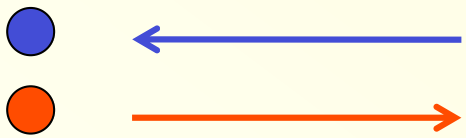

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→ Nearly universal conductivity at strong coupling

$$\boxed{\sigma_{Drude} = \frac{e}{m}\rho\tau \;\rightarrow\; \sigma(\mu=0) \sim \frac{e}{k_B T / \mathrm{v}^2}\left(e\frac{(k_B T)^2}{(\hbar \mathrm{v})^2}\right)\frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2}\frac{e^2}{h}}$$

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Marginal irrelevance of Coulomb: $\boxed{\alpha \approx \frac{4}{\log(\Lambda / T)} < 1}$

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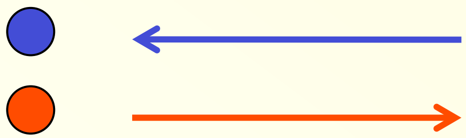

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→ Nearly universal conductivity at strong coupling

$$\sigma_{Drude} = \frac{e}{m}\rho\tau \rightarrow \sigma(\mu = 0) \sim \frac{e}{k_B T / \text{v}^2}\left(e\frac{(k_B T)^2}{(\hbar \text{v})^2}\right)\frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2}\frac{e^2}{h}$$

Expect saturation as $\alpha \rightarrow 1$, and eventually phase transition to insulator

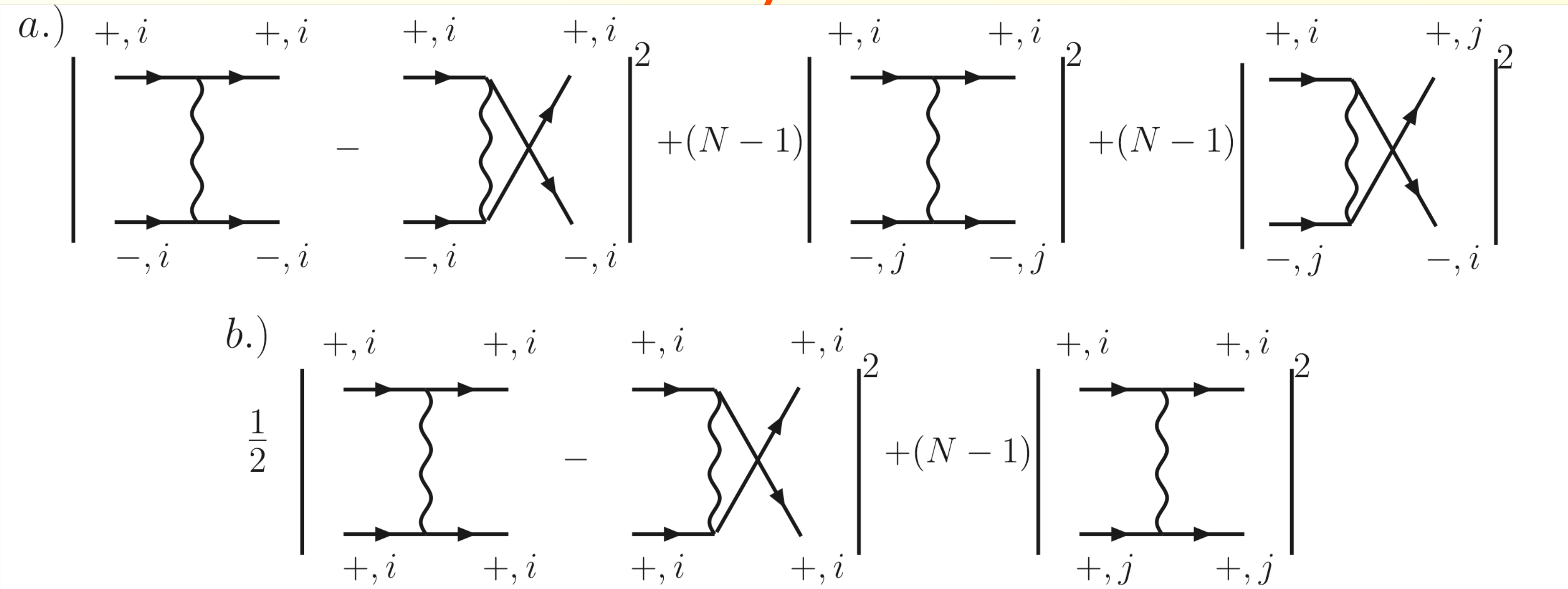
Marginal irrelevance of Coulomb: $\boxed{\alpha \approx \frac{4}{\log(\Lambda/T)} < 1}$

Boltzmann approach

L. Fritz, J. Schmalian, MM, and S. Sachdev, PRB 2008

Boltzmann equation in Born approximation

$$\left(\partial_t + e\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{k}}\right) f_\pm(\mathbf{k},t) = I_{\rm coll}^{Cb}\left[\mathbf{k},t \mid \{f_\pm(\mathbf{k}',t)\}\right] \propto \alpha^2(T)$$



→ Collision-limited conductivity: $\boxed{\sigma(\mu=0) = \frac{0.76}{\alpha^2(T)}\frac{e^2}{h}}$

Transport and thermoelectric response at low frequencies?

Hydrodynamic regime:
(collision-dominated)

$$\tau_{\mathrm{ee}}^{-1} >> \omega, \tau_{\mathrm{imp}}^{-1}, \omega_{\mathrm{cyclo}}^{\mathrm{th}}$$

Hydrodynamics

Hydrodynamic collision-dominated regime

Long times,
Large scales

$\boxed{t >> \tau_{ee}}$

- Local equilibrium established: $\boxed{T_{\mathrm{loc}}(r)\ ,\ \mu_{\mathrm{loc}}(r)\ ;\ \vec{u}_{\mathrm{loc}}(r)}$

- Study relaxation towards global equilibrium

- Slow modes: Diffusion of the density of conserved quantities:
  - Charge
  - Momentum
  - Energy

Relativistic Hydrodynamics

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B **76**, 144502 (2007).*

Energy-momentum tensor $T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \cancel{\tau^{\mu\nu}}$ $\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$

Current 3-vector $J^\mu = \rho u^\mu + \nu^\mu$ $\begin{pmatrix} \rho \\ \rho u_x + \nu_x \\ \rho u_y + \nu_y \end{pmatrix}$

u^μ : 3-velocity: $u^\mu = (1,0,0) \rightarrow$ No energy current

ν^μ : Dissipative current (to be determined below)

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ν^μ : Dissipative current (to be determined below)

+ Thermodynamic relations

$$\varepsilon + P = Ts + \mu\rho, \quad d\varepsilon = Tds + \mu d\rho,$$

Gibbs-Duhamel 1st law of thermodynamics

Relativistic Hydrodynamics

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Conservation laws (equations of motion):

$\partial_\mu J^\mu = 0$ Charge conservation

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Energy/momentum conservation

$$\partial_\nu T^{\mu\nu} = F^{\mu\nu} J_\nu + \frac{1}{\tau_{\text{imp}}} T^{0\nu} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B \\ -E_y & -B & 0 \end{pmatrix}$$

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Dissipative current ν?

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1. Construct entropy current $S^\mu = Q^\mu / T$
2. Second law of thermodynamics $\partial_\mu S^\mu \geq 0$
3. Covariance

$$\rightarrow \boxed{\nu^\mu = \sigma_Q (g^{\mu\nu} + u^\mu u^\nu) \left[\left(-\partial_\nu \mu + F_{\nu\lambda} u^\lambda \right) + \mu \frac{\partial_\mu T}{T} \right]}$$

Thermoelectric response

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B **76**, 144502 (2007).*

Thermo-electric response

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \hat{\sigma} & \hat{\alpha} \\ T\hat{\alpha} & \hat{\bar{\kappa}} \end{pmatrix} \begin{pmatrix} \vec{E} \\ -\vec{\nabla} T \end{pmatrix}$$

$$\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

etc.

Transverse thermoelectric response (Nernst)

Charge and heat current:

$$J^\mu = \rho\, u^\mu - \nu^\mu$$

$$Q^\mu = (\varepsilon + P) u^\mu - \mu\, J^\mu$$

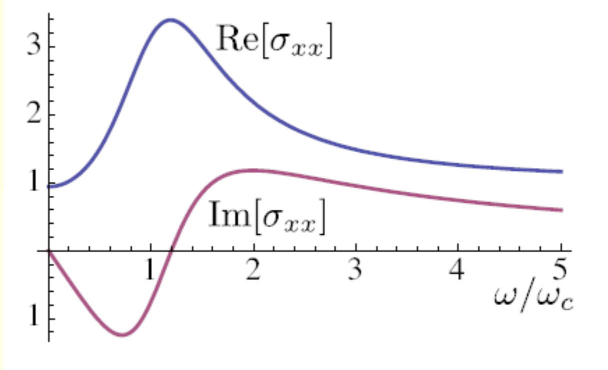
Recipe: i) Solve linearized hydrodynamic equations
ii) Read off the response functions *(Kadanoff & Martin 1960)*

Collective cyclotron resonance

S. Hartnoll and C Herzog, 2007;   MM, and S. Sachdev, 2008

Relativistic magnetohydrodynamics: pole in AC response

$$\sigma_{xx}(\omega) \;=\; \sigma_Q \frac{\omega\left(\omega + i\gamma + i\omega_c^2/\gamma\right)}{\left(\omega + i\gamma\right)^2 - \omega_c^2}$$



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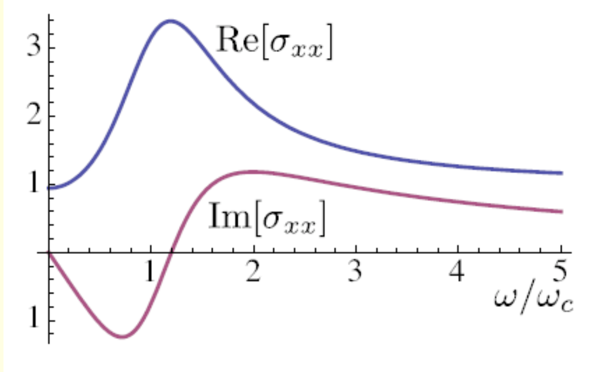
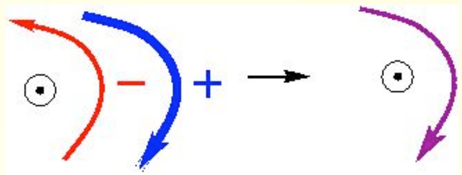


Pole in the response

$$\omega^* = \pm\omega_c - i\gamma$$

Collective cyclotron frequency of the relativistic plasma

$$\omega_c = \frac{\rho\, B/c}{(\varepsilon + P)/v_F^2} \leftrightarrow \omega_c^{FL} = \frac{e\, B/c}{m}$$



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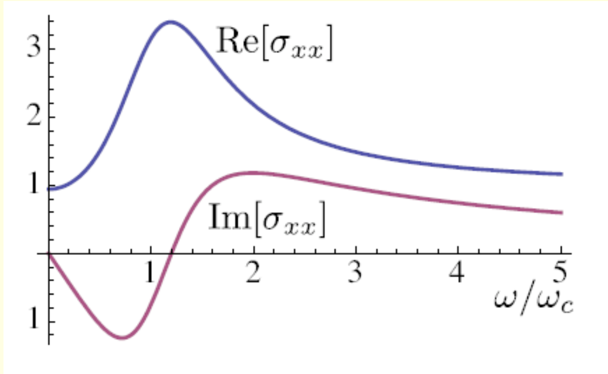

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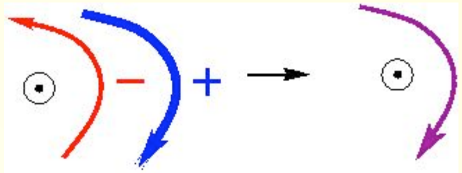


Broadening of resonance: $$\boxed{\gamma = \sigma_Q \frac{(B/c)^2}{(\varepsilon + P)/\mathrm{v}_F^2}}$$

Observable at room temperature in the GHz regime!

Relativistic hydrodynamics from microscopics

Does relativistic hydro really apply to graphene even though Coulomb interactions break relativistic invariance?

Yes! Within weak-coupling theory:

Key point: There is a zero (“momentum”) mode of the collision integral due to translational invariance of the interactions

The dynamics of the zero mode under an AC driving field reproduces relativistic hydrodynamics at low frequencies.

Application II:
thermoelectric close to transport at quantum criticality

Response functions at B=0

Lorentz symmetry ➝ plenty of relations between transport coefficients at quantum criticality!

Valid for weak AND strong coupling!

Longitudinal conductivity:

$$\sigma_{xx}(\omega, k; B=0) \;=\; \left(\sigma_Q + \frac{\rho^2}{P+\varepsilon}\frac{\tau}{1-i\omega\tau}\right)$$

Thermal conductivity:

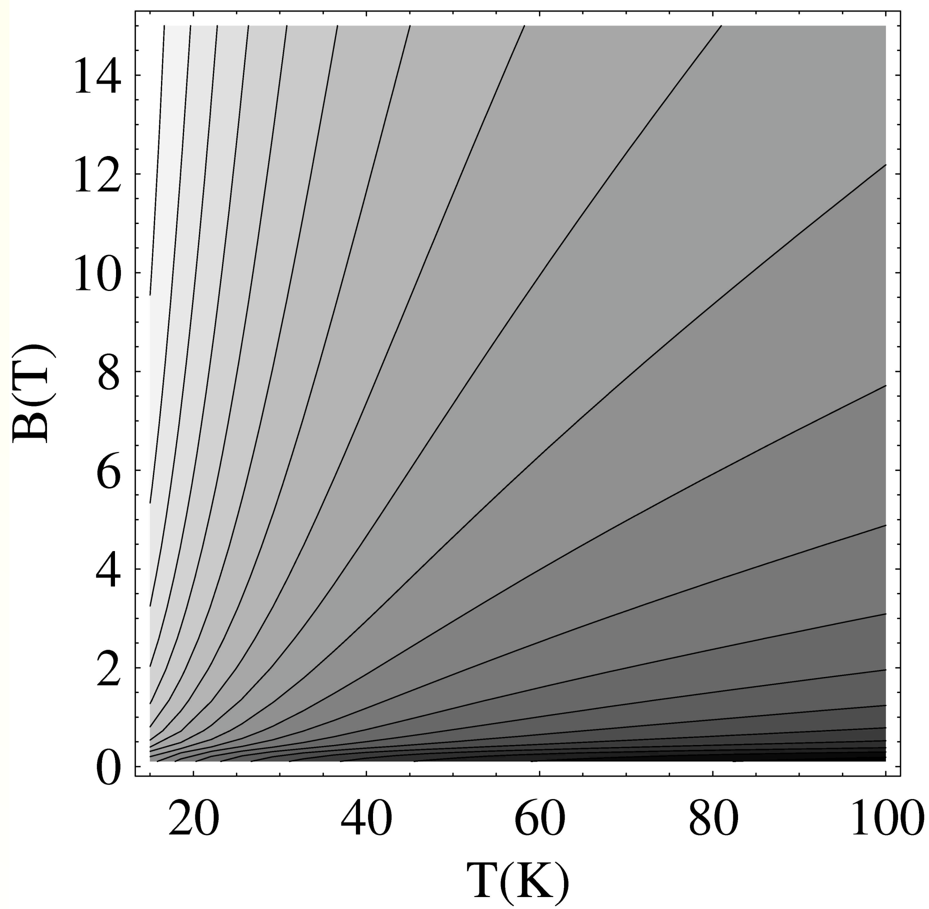
$$\kappa_{xx}(\omega, k; B=0) \;=\; \sigma_Q \frac{\mu^2}{T} + \frac{s^2\,T}{P+\varepsilon}\frac{\tau}{1-i\omega\tau} + \mathcal{O}(k^2).$$

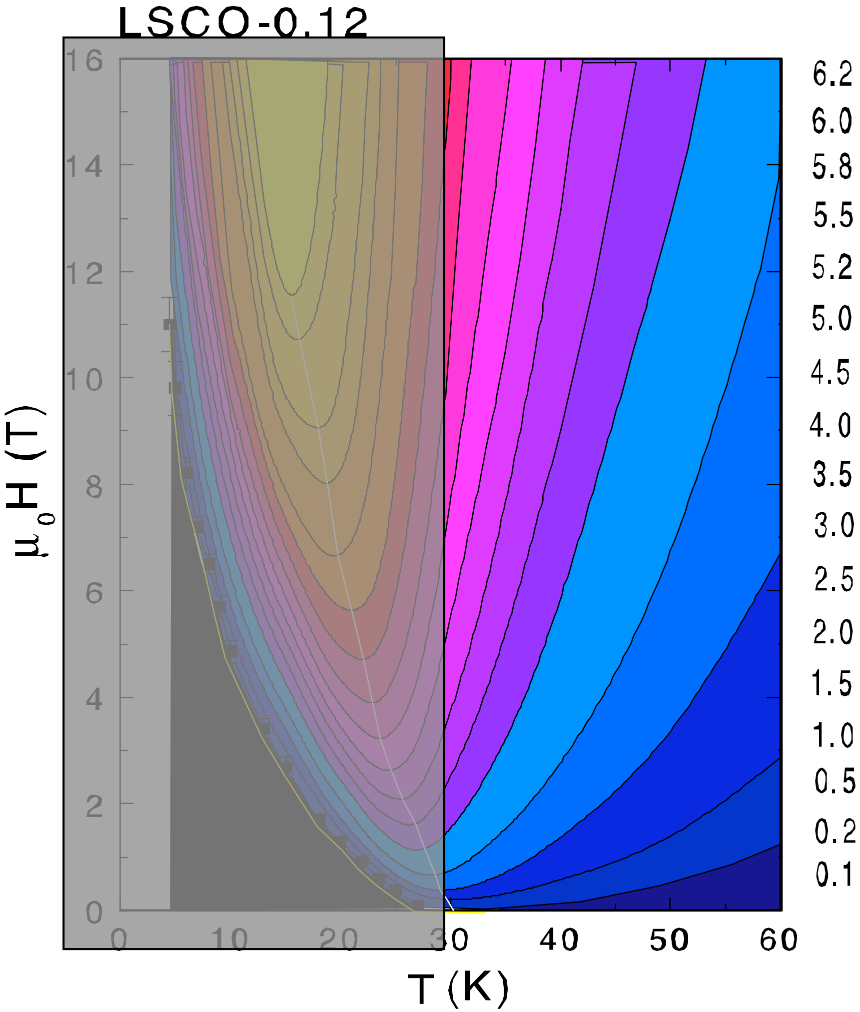
Relativistic Wiedemann-Franz-like relations between σ and κ!

Nernst Experiments in high Tc's

Transverse thermoelectric response: B, T - dependence

Theory for $\alpha_{xy} \approx \sigma_{xx} N$





Y. Wang, L. Li, and N. P. Ong, Phys. Rev. B ***73****, 024510 (2006).*

Boltzmann equation

MM, L. Fritz and S. Sachdev, PRB 2008

Boltzmann equation in Born approximation

$$\left(\partial_t + e\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right] \cdot \frac{\partial}{\partial \mathbf{k}}\right) f_\pm(\mathbf{k}, t) = I_{\text{coll}}^{Cb}\left[\mathbf{k}, t \mid \left\{f_\pm(\mathbf{k}', t)\right\}\right] + I_{\text{coll}}^{dis}\left[\mathbf{k}, t \mid \left\{f_\pm(\mathbf{k}', t)\right\}\right]$$

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Momentum conservation →
Exact zero mode of the Coulomb collision integral!

$$\delta f_\lambda^{(hydro)}(\mathbf{k}) = \frac{\partial f_\lambda^{eq}(\mathbf{k})}{\partial u_{\mathrm{cm}}^{\mathrm{i}}} = \lambda k^i \; f_\lambda^{eq}(\mathbf{k})\left[1 - f_\lambda^{eq}(\mathbf{k})\right]$$

Find relativistic hydrodynamics from the dynamics of this mode!

Boltzmann equation

MM, L. Fritz and S. Sachdev, PRB 2008

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Find relativistic hydrodynamics from the dynamics of this mode!

$$\boxed{\sigma_{xx}(\omega,B) = \sigma_{xx}^{\mathrm{MHD}}(\omega,B) + \mathcal{O}(b/\alpha^2,\omega/\alpha^2)}$$ **etc.**

Corrections to HD small if $\boxed{\tau_{\mathrm{ee}}^{-1} >> \tau_{\mathrm{imp}}^{-1}, \omega_{\mathrm{c}}^{\mathrm{typ}}, \omega_{\mathrm{AC}}}$

Beyond weak coupling approximation:

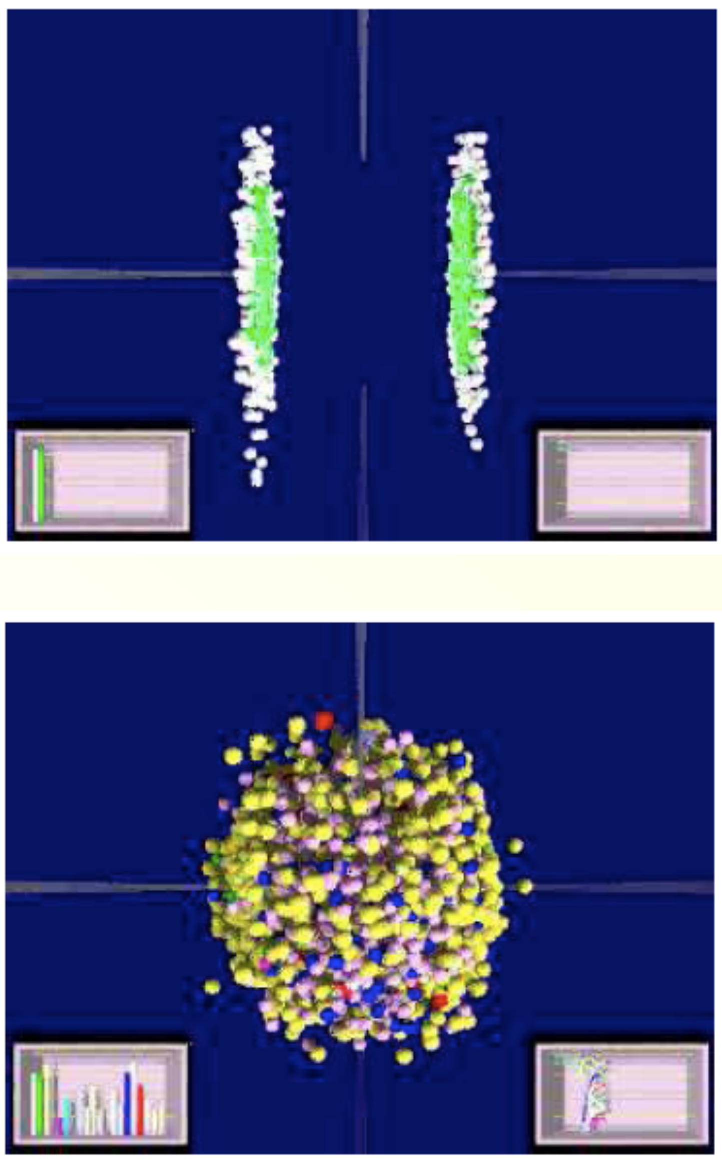
Graphene

$\leftrightarrow$

Very strongly coupled, critical relativistic liquids?

AdS – CFT !

Au+Au collisions at RHIC



Quark-gluon plasma is described by QCD (nearly conformal, critical theory)

–

Low viscosity fluid!

Compare graphene to Strongly coupled relativistic liquids

S. Hartnoll, P. Kovtun, MM, S. Sachdev (2007)

Response for special strongly coupled relativistic fluids
(maximally supersymmetric SU(N) Yang Mills theory with $N \to \infty$ colors)
By mapping to weakly coupled gravity problem:

AdS (gravity) $\leftrightarrow$ CFT_{2+1} [SU(N>>1)]

weak coupling $\leftrightarrow$ strong coupling

Obtain exact results for transport via the AdS–CFT correspondence

SU(N) transport from AdS/CFT

Gravitational dual to SUSY SU(N)-CFT$_{2+1}$: Einstein-Maxwell theory

$$I = \frac{1}{g^2}\int d^4x\sqrt{-g}\left[-\frac{1}{4}R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{3}{2}\right].$$

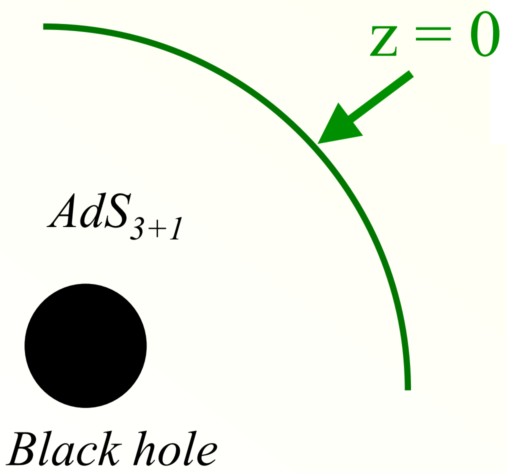
(embedded in M theory as $AdS_4 \times S^7$: $1/g^2 \sim N^{3/2}$)

It has a black hole solution (with electric and magnetic charge):

$$ds^2 = \frac{\alpha^2}{z^2}\left[-f(z)dt^2 + dx^2 + dy^2\right] + \frac{1}{z^2}\frac{dz^2}{f(z)},$$

$$F = h\alpha^2 dx \wedge dy + q\alpha dz \wedge dt,$$

$$f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3.$$



Electric charge *q* and magnetic charge *h*

↔ μ and *B* for the CFT

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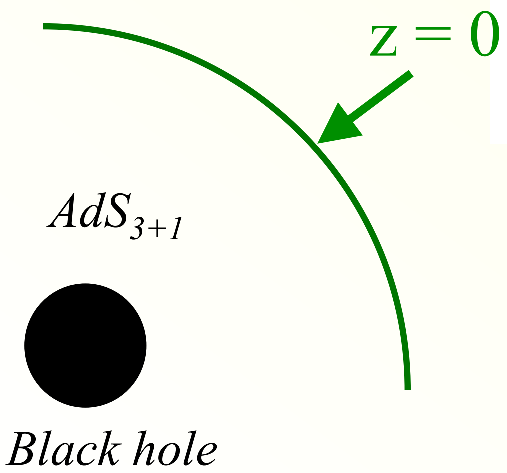

Background $\leftrightarrow$ Equilibrium

Transport $\leftrightarrow$ Perturbations in $g_{tx,ty}, A_{x,y}$.

Response via Kubo formula from $\delta^2 I/\delta(g,A)^2$.

Compare graphene to Strongly coupled relativistic liquids

S. Hartnoll, P. Kovtun, MM, S. Sachdev (2007)

Obtain exact results via string theoretical AdS–CFT correspondence

→

- Confirm the results of hydrodynamics: response functions $\sigma(\omega)$, resonances
- Calculate the transport coefficients for a strongly coupled theory!

SUSY - SU(N): $\sigma(\mu = 0) = \sqrt{\frac{2}{9}} N^{3/2} \frac{e^2}{h}$

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Interpretation: $N^{3/2}$ effective degrees of freedom, strongly coupled: $\tau T = O(1)$

Graphene – a nearly perfect liquid!

MM, J. Schmalian, and L. Fritz, (PRL 2009)

↔

Anomalously low viscosity (like quark-gluon plasma)

"Heisenberg" $\boxed{\frac{\eta}{s} \sim E_{qp}\tau \geq 1}$ → Measure of strong coupling:

Conjecture from AdS-CFT:

$$\boxed{\frac{\text{shear viscosity}}{\text{entropy density}} = \frac{\eta}{s} > \frac{\hbar}{k_B}\frac{1}{4\pi}}$$

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Doped Graphene & Fermi liquids:
(Khalatnikov etc)

$\boxed{\eta \propto n \cdot m\mathrm{v}^2 \cdot \tau \;\rightarrow\; n \cdot E_F \cdot \dfrac{\hbar E_F}{(k_B T)^2}}$ $\boxed{s \propto k_B n \dfrac{T}{E_F}}$ $\boxed{\dfrac{\eta}{s} \sim \dfrac{\hbar}{k_B}\left(\dfrac{E_F}{T}\right)^3}$

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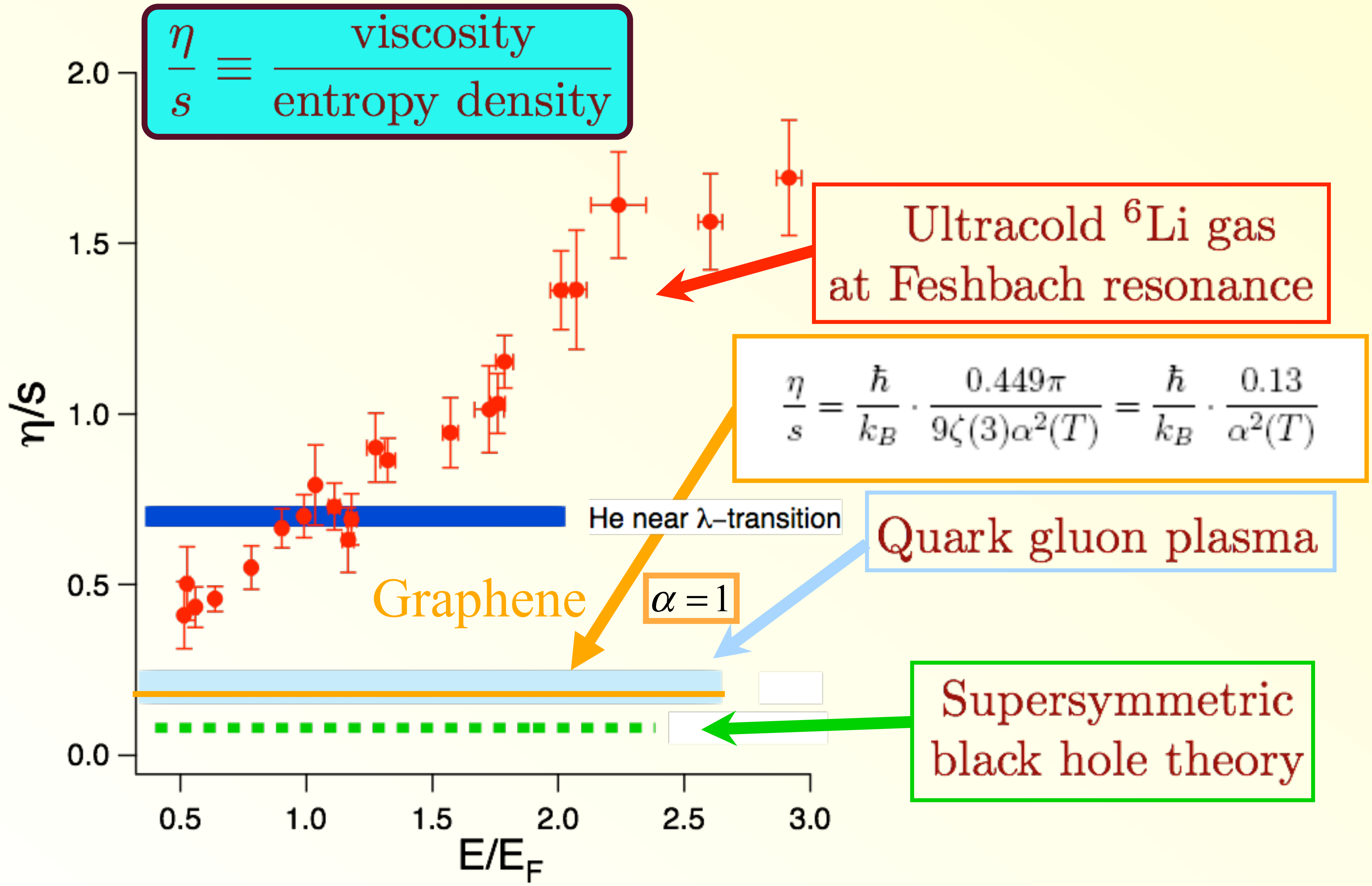
Undoped Graphene:

$$\boxed{\eta \propto n \cdot m\mathrm{v}^2 \cdot \tau \;\rightarrow\; n_{\mathrm{th}} \cdot k_B T \cdot \frac{\hbar}{\alpha^2 k_B T} = \frac{\hbar}{\alpha^2} n_{\mathrm{th}}} \quad \boxed{s \propto k_B n_{\mathrm{th}}}$$

Boltzmann-Born Approximation:

$$\boxed{\frac{\eta}{s} = \frac{\hbar}{k_B} \cdot \frac{0.449\pi}{9\zeta(3)\alpha^2(T)} = \frac{\hbar}{k_B} \cdot \frac{0.13}{\alpha^2(T)}}$$



*T. Schäfer, Phys. Rev. A **76**, 063618 (2007).*

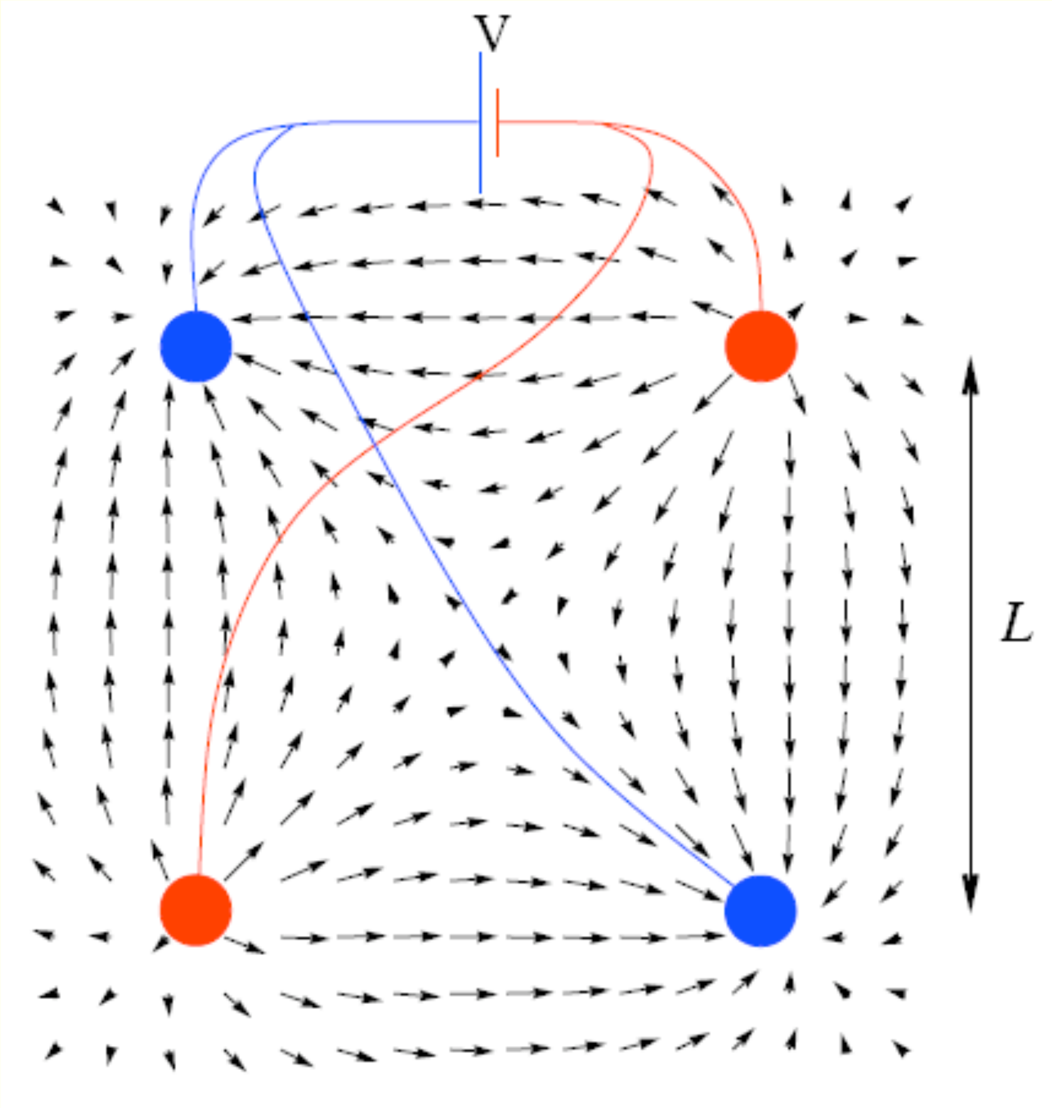
*A. Turlapov, J. Kinast, B. Clancy, Le Luo, J. Joseph, J. E. Thomas, J. Low Temp. Phys. **150**, 567 (2008)*

Electronic consequences of low viscosity?

MM, J. Schmalian, L. Fritz, (PRL 2009)

Expected viscous effects on conductance in non-uniform current flow:

Decrease of conductance with length scale L



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MM, J. Schmalian, L. Fritz, (PRL 2009)

Electronic turbulence in clean, strongly coupled graphene?
(or at quantum criticality!)
Reynolds number:

$$\boxed{\mathrm{Re} = \frac{s/k_B}{\eta/\hbar} \times \frac{k_B T}{\hbar v/L} \times \frac{u_{\mathrm{typ}}}{v}}$$

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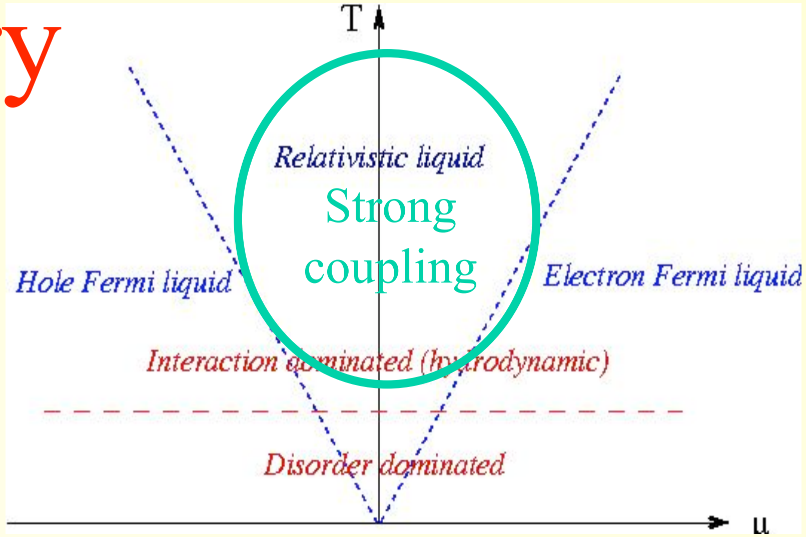
Strongly driven mesoscopic systems: (Kim's group [Columbia])

$$\boxed{\begin{array}{l} L = 1\mu m \\ u_{typ} = 0.1\mathrm{v} \\ T = 100K \end{array}} \longrightarrow \mathrm{Re} \sim 10 - 100$$

Complex fluid dynamics!
(pre-turbulent flow)

New phenomenon in an
electronic system!

Summary



- Undoped graphene is strongly coupled in a large temperature window!
- Nearly universal strong coupling features in transport; many similarities with strongly coupled critical fluids (described by AdS-CFT)
- Emergent relativistic hydrodynamics at low frequency
- Graphene: Nearly perfect quantum liquid!
  → Possibility of complex (turbulent?) current flow at high bias