# Relativistic magnetotransport in graphene, at quantum criticality and in black holes

Markus Müller

in collaboration with
Sean Hartnoll (Harvard)
Pavel Kovtun (Victoria)
Subir Sachdev (Harvard)

Lars Fritz (Harvard)
Jörg Schmalian (Iowa)

The Abdus Salam
ICTP Trieste

AdS/CFT: Strongly coupled systems and exact results - Paris 26 Nov, 2009

# Outline

- Relativistic physics in graphene, quantum critical systems and conformal field theories
- Strong coupling features in collision-dominated transport – as inspired by AdS-CFT results
- Comparison with strongly coupled fluids (via AdS-CFT)
- Graphene: an almost perfect quantum liquid

# Quantum critical systems in condensed matter

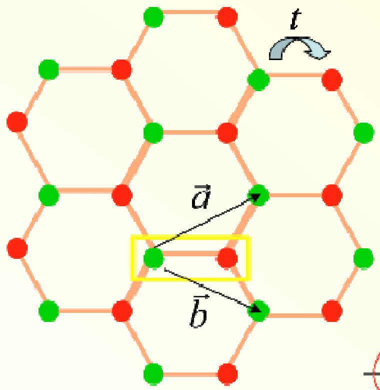
# A few examples

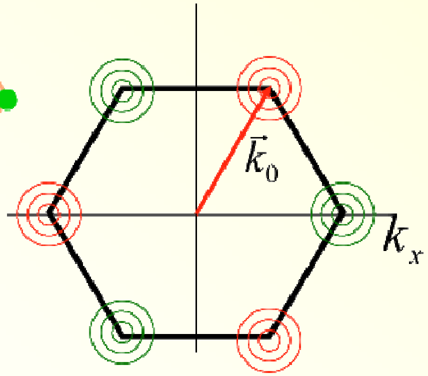
- Graphene
- High Tc
- Superconductor-to-insulator transition (interaction driven)

# Dirac fermions in graphene

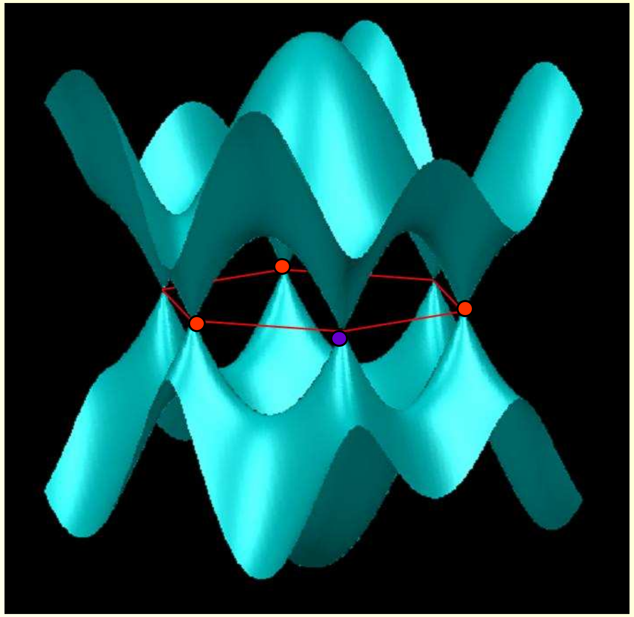
*(Semenoff '84, Haldane '88)*

Honeycomb lattice of C atoms




Tight binding dispersion



2 massless Dirac cones in the Brillouin zone:
(Sublattice degree of freedom ↔ pseudospin)

Close to the two Fermi points **K**, **K'**:

$$H \approx \mathrm{v}_F \left(\vec{\mathbf{p}} - \vec{\mathbf{K}}\right) \cdot \vec{\sigma}_{\text{sublattice}}$$
$$\rightarrow \quad E_{\mathbf{p}} = \mathrm{v}_F \left|\vec{\mathbf{p}} - \mathbf{K}\right|$$

Fermi velocity (speed of light")

$$\mathrm{v}_F \approx 1.1 \cdot 10^6\ \mathrm{m/s} \approx \frac{c}{300}$$

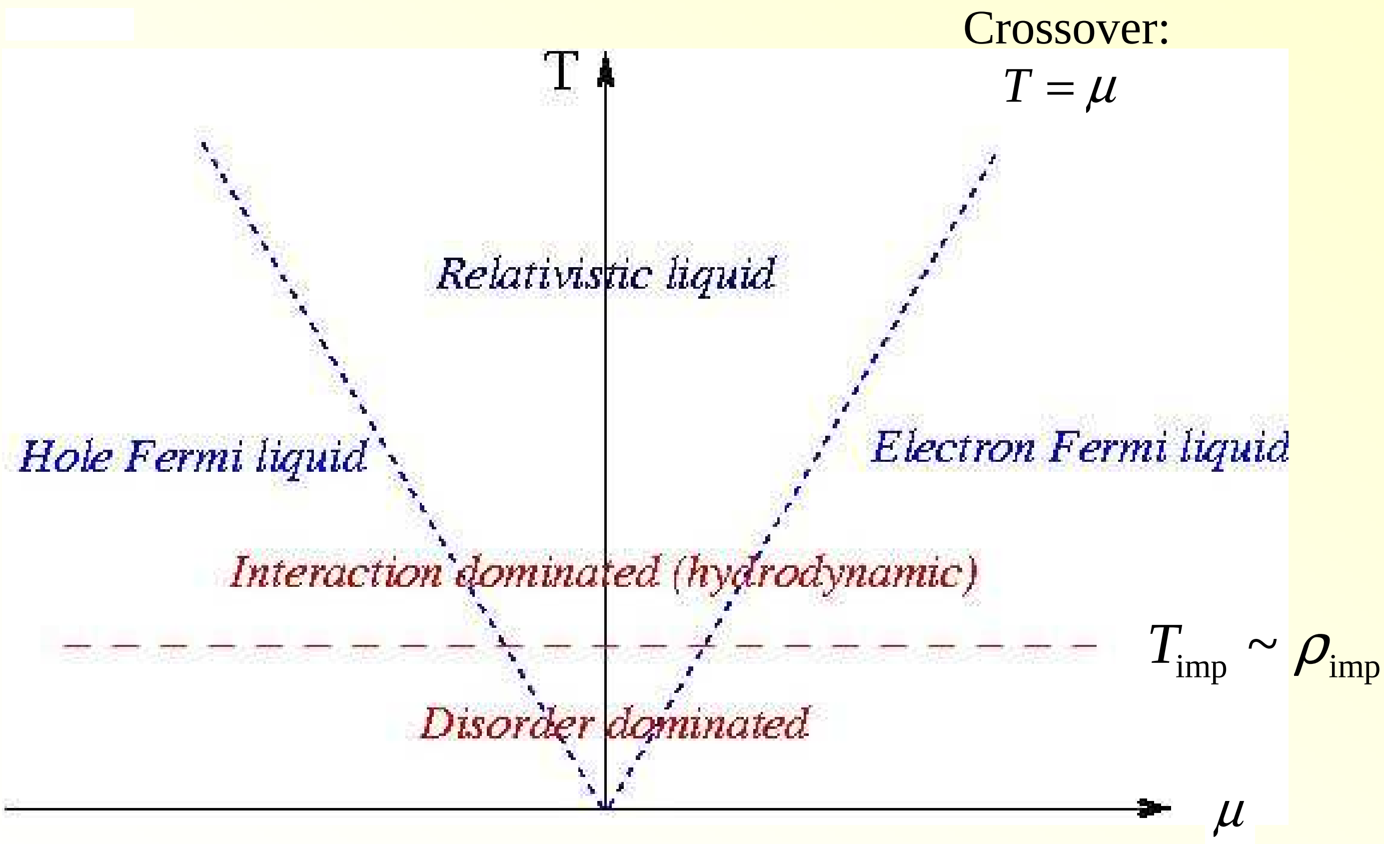
Coulomb interactions: Fine structure constant

$$\alpha \equiv \frac{e^2}{\varepsilon \hbar \mathrm{v}_F} = O(1)$$

# Relativistic fluid at the Dirac point

*D. Sheehy, J. Schmalian, Phys. Rev. Lett.* ***99****, 226803 (2007).*

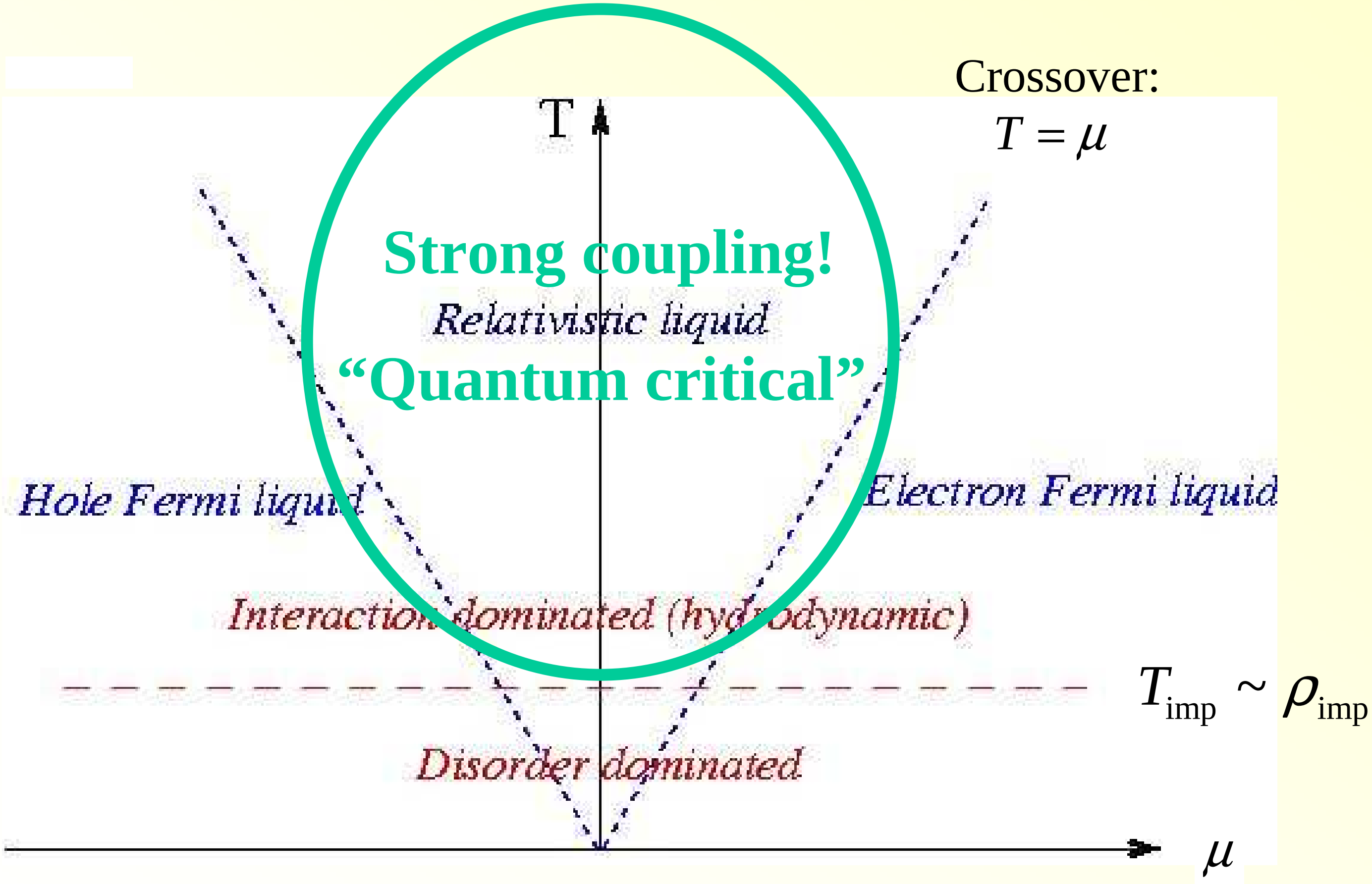
- Relativistic plasma physics of interacting particles and holes!



# Relativistic fluid at the Dirac point

*D. Sheehy, J. Schmalian, Phys. Rev. Lett.* ***99****, 226803 (2007).*

- Relativistic plasma physics of interacting particles and holes!
- Strongly coupled, nearly quantum critical fluid at $\mu = 0$



# Other relativistic fluids:

- Systems close to quantum criticality (with z = 1)

Example: Superconductor-insulator transition (Bose-Hubbard model)

$$\tau_{rel}^{-1} \approx \frac{k_B T}{\hbar}$$

Maximal possible relaxation rate!

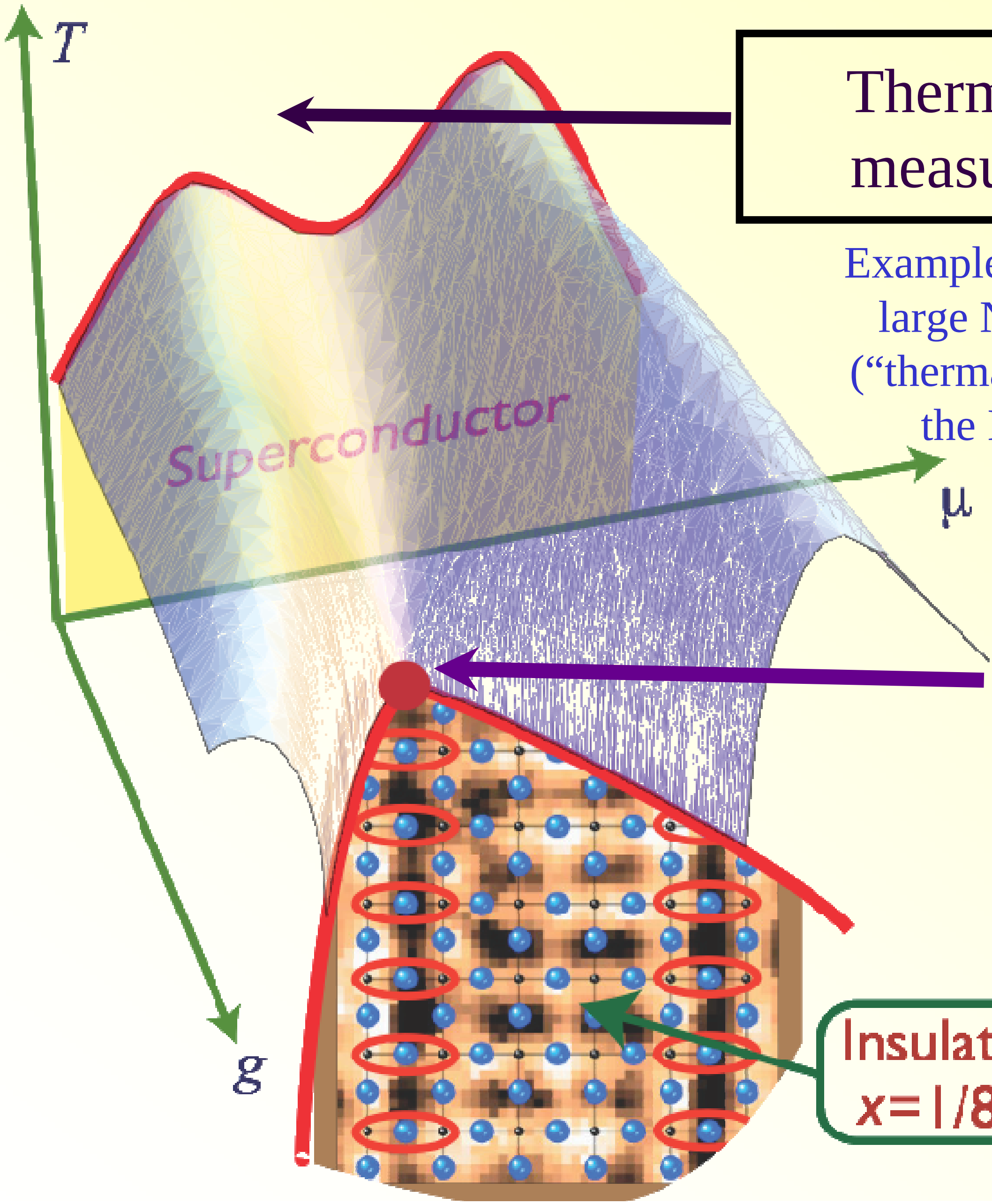
*Damle, Sachdev (1996, 1997)*
*Bhaseen, Green, Sondhi (2007).*
*Hartnoll, Kovtun, MM, Sachdev (2007)*

- Conformal field theories

E.g.: strongly coupled Yang-Mills theories
$\rightarrow$ Exact treatment via AdS-CFT correspondence

*C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son (2007)*
*Hartnoll, Kovtun, MM, Sachdev (2007)*

# Strongly correlated electrons: Cuprate high $T_c$




Thermoelectric measurements.

Example: Anomalously large Nernst Effect! ("thermal analogue" of the Hall effect)

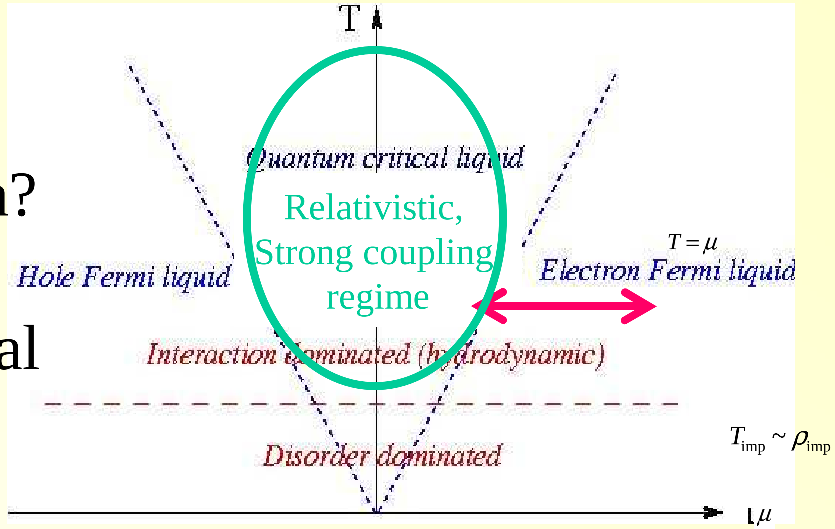
Conformal field theory?

Simplest example exhibiting "quantum critical" features:

Graphene

# Questions

- Transport characteristics in the strongly coupled relativistic plasma?
- Response functions, nearly universal transport coefficients
- Graphene as a nearly perfect and possibly turbulent quantum fluid (like the quark-gluon plasma)



# Graphene – Fermi liquid?

1. Tight binding kinetic energy
→ massless Dirac quasiparticles

$$H_0 = \sum_{\lambda=\pm} \sum_{a=1}^{N} \int \frac{d^2k}{(2\pi)^2} \lambda v_F k \, \gamma^\dagger_{\lambda a}(\mathbf{k}) \gamma_{\lambda a}(\mathbf{k})$$

2. Coulomb interactions:
Unexpectedly strong!
→ nearly quantum critical!

$$V(\mathbf{q}) = \frac{2\pi e^2}{\varepsilon |\mathbf{q}|}$$

$$H_1 = \frac{1}{2} \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \Psi^\dagger_a(\mathbf{k}_2 - \mathbf{q}) \Psi_a(\mathbf{k}_2) V(\mathbf{q}) \Psi^\dagger_b(\mathbf{k}_1 + \mathbf{q}) \Psi_b(\mathbf{k}_1)$$

Coulomb **only marginally** irrelevant for μ = 0!

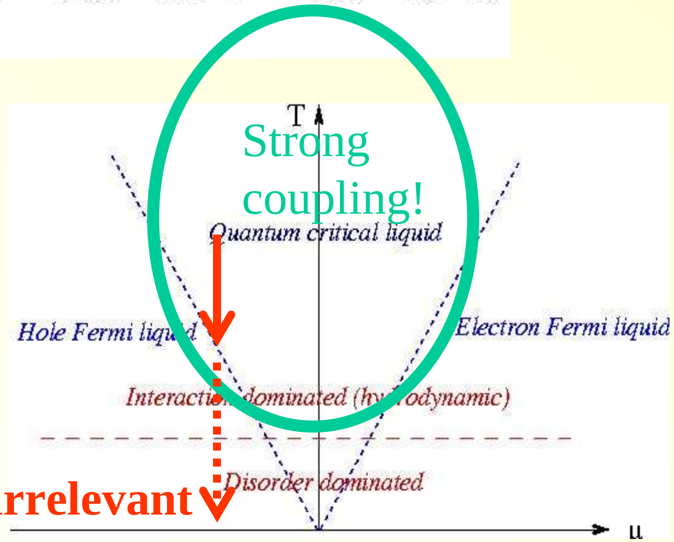
RG:
(μ = 0)

$$\frac{d\alpha}{d\ell} = -\frac{\alpha^2}{4} + \mathcal{O}(\alpha^3)$$

$$\alpha \equiv \frac{e^2}{\varepsilon \hbar v_F} = O(1)$$

$$\alpha(T) = \frac{\alpha^0}{1 + (\alpha^0/4)\ln(\Lambda/T)} \overset{T\to 0}{\sim} \frac{4}{\ln(\Lambda/T)}$$

(μ > 0) $T < \mu$: **Screening kicks in, short ranged Cb irrelevant**



# Strong coupling in undoped graphene

*MM, L. Fritz, and S. Sachdev, PRB '08.*

Inelastic scattering rate
(Electron-electron interactions)

μ >> T: standard 2d Fermi liquid

$$\tau_{\mathrm{ee}}^{-1} = C \frac{(k_B T)^2}{\hbar \mu}$$

Relaxation rate ~ T,
like in quantum critical systems!
Fastest possible rate!

μ < T: strongly coupled relativistic liquid

$$\tau_{\mathrm{ee}}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar}$$

"Heisenberg uncertainty principle for well-defined quasiparticles"

$$E_{qp}(\sim k_B T) \geq \Delta E_{\mathrm{int}} = \hbar \tau_{\mathrm{ee}}^{-1} \sim \alpha^2 k_B T$$

As long as α(T) ~ 1, energy uncertainty is saturated, scattering is maximal
→ Nearly universal strong coupling features in transport,
similarly as at the 2d superfluid-insulator transition *[Damle, Sachdev (1996, 1997)]*

# Consequences for transport

1. -Collisionlimited conductivity σ in clean undoped graphene;
   -Collisionlimited spin diffusion $D_s$ at any doping

2. Graphene - a perfect quantum liquid: very small viscosity η!

3. Emergent relativistic invariance at low frequencies!
   Despite the breaking of relativistic invariance by
   - finite T,
   - finite μ,
   - instantaneous 1/r Coulomb interactions

Collision-dominated transport → relativistic hydrodynamics:
a) Response fully determined by covariance, thermodynamics, and σ, η
b) Collective cyclotron resonance in small magnetic field (low frequency)

Hydrodynamic regime:
(collision-dominated)

$$\tau_{ee}^{-1} >> \tau_{imp}^{-1}, \omega_c^{typ}, \omega_{AC}$$

# Collisionlimited conductivities

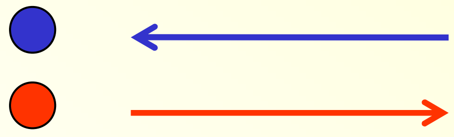
*Damle, Sachdev, (1996).*
*Fritz et al. (2008), Kashuba (2008)*

Finite charge or spin conductivity in a pure system (for μ = 0 or B = 0) !

- Key: Charge or spin current without momentum

(particle/spin up)

(hole/spin down)



$$\vec{J}_{s,c} \neq 0, \text{ but } \vec{P} = 0$$

Pair creation/annihilation leads to current decay

- Finite collision-limited conductivity! $\sigma(\mu=0) < \infty \quad ; \quad \sigma(\mu \neq 0) = \infty$
- Finite collision-limited spin diffusivity! $D_s(\mu; B=0) \propto v_F^2 \tau_{ee} < \infty,$
- Only marginal irrelevance of Coulomb: Maximal possible relaxation rate ~ T

$$\tau_{ee}^{-1} \approx \alpha^2 \frac{k_B T}{\hbar}$$

→ Nearly universal conductivity at strong coupling

$$\sigma_{Drude} = \frac{e}{m}\rho\tau \to \sigma(\mu=0) \sim \frac{e}{k_B T / \mathrm{v}^2}\left(e\frac{(k_B T)^2}{(\hbar \mathrm{v})^2}\right)\frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2}\frac{e^2}{h}$$

**Expect saturation as α →1, and eventually phase transition to insulator**

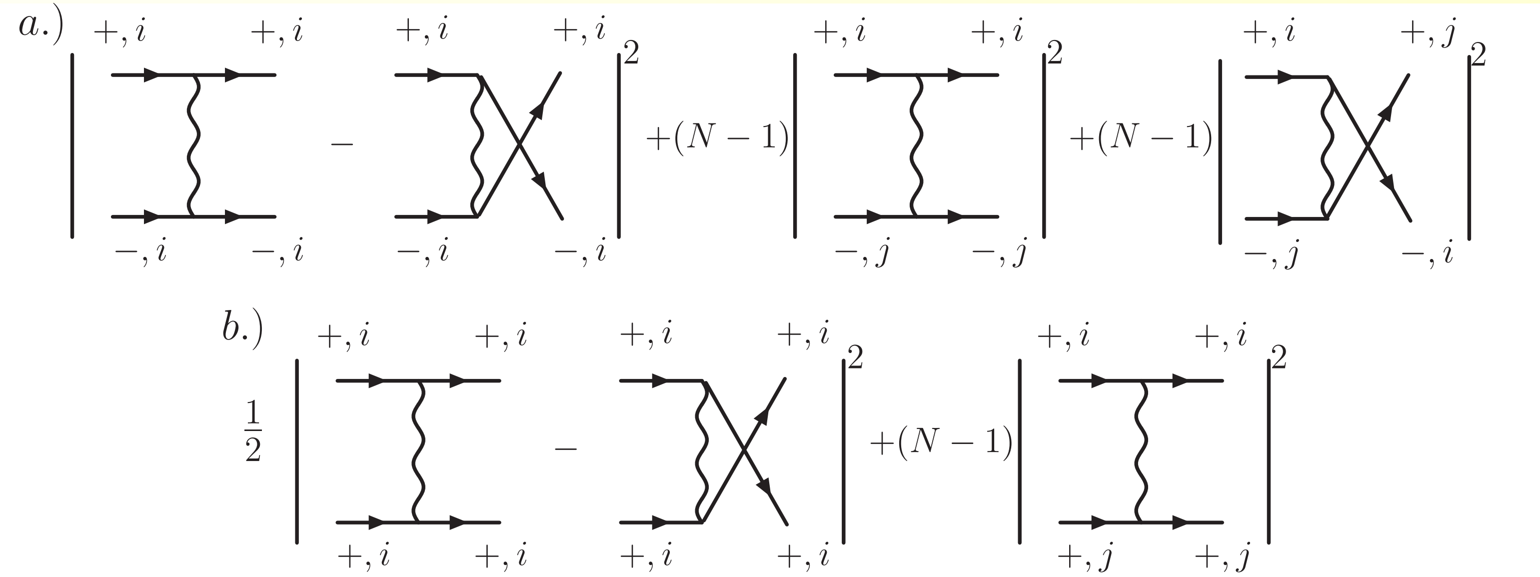
Marginal irrelevance of Coulomb: $\alpha \approx \frac{4}{\log(\Lambda/T)} < 1$

# Boltzmann approach

*L. Fritz, J. Schmalian, MM, and S. Sachdev, PRB 2008*

Boltzmann equation in Born approximation

$$\left(\partial_t + e\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{k}}\right) f_\pm(\mathbf{k},t) = I_{\text{coll}}^{Cb}\left[\mathbf{k},t \mid \{f_\pm(\mathbf{k}',t)\}\right] \propto \alpha^2(T)$$




→ Collision-limited conductivity:

$$\sigma(\mu=0) = \frac{0.76}{\alpha^2(T)}\frac{e^2}{h}$$

# Transport and thermoelectric response at low frequencies?

Hydrodynamic regime:
(collision-dominated)

$$\tau_{\mathrm{ee}}^{-1} >> \omega, \tau_{\mathrm{imp}}^{-1}, \omega_{\mathrm{cyclo}}^{\mathrm{th}}$$

# Hydrodynamics

Hydrodynamic collision-dominated regime

Long times,
Large scales

$t >> \tau_{ee}$

- Local equilibrium established: $T_{\text{loc}}(r)\ ,\ \mu_{\text{loc}}(r)\ ;\ \vec{u}_{\text{loc}}(r)$

- Study relaxation towards global equilibrium

- Slow modes: Diffusion of the density of conserved quantities:
  - Charge
  - Momentum
  - Energy

# Relativistic Hydrodynamics

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B **76**, 144502 (2007).*

$$J^\mu = \rho u^\mu + \nu^\mu$$

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}$$

Irrelevant at small $k$

Conservation laws (equations of motion):

$$\partial_\mu J^\mu = 0$$ Charge conservation

Energy/momentum conservation

$$\partial_\nu T^{\mu\nu} = F^{\mu\nu} J_\nu + \frac{1}{\tau_{\text{imp}}} T^{0\nu}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B \\ -E_y & -B & 0 \end{pmatrix}$$

1. Construct entropy current $S^\mu = Q^\mu / T$
2. Second law of thermodynamics $\partial_\mu S^\mu \geq 0$
3. Covariance →

$$\nu^\mu = \sigma_Q (g^{\mu\nu} + u^\mu u^\nu)\left[(-\partial_\nu \mu + F_{\nu\lambda} u^\lambda) + \mu \frac{\partial_\mu T}{T}\right]$$

# Thermoelectric response

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B **76**, 144502 (2007).*

Thermo-electric response

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \hat{\sigma} & \hat{\alpha} \\ T\hat{\alpha} & \hat{\bar{\kappa}} \end{pmatrix} \begin{pmatrix} \vec{E} \\ -\vec{\nabla} T \end{pmatrix}$$

$$\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$ etc.

Transverse thermoelectric response (Nernst)

Charge and heat current:

$$J^{\mu} = \rho u^{\mu} - \nu^{\mu}$$

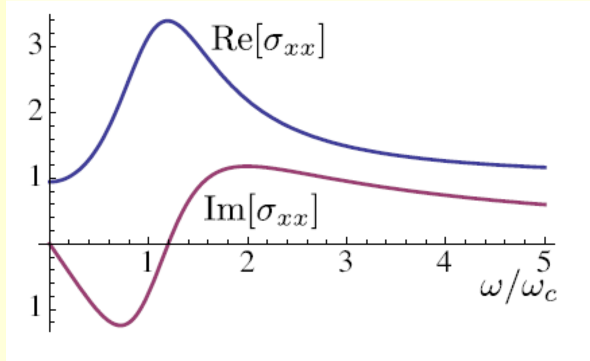
$$Q^{\mu} = (\varepsilon + P) u^{\mu} - \mu J^{\mu}$$

Recipe: i) Solve linearized hydrodynamic equations
ii) Read off the response functions *(Kadanoff & Martin 1960)*

# Collective cyclotron resonance

*S. Hartnoll and C Herzog, 2007;   MM, and S. Sachdev, 2008*

Relativistic magnetohydrodynamics: pole in AC response



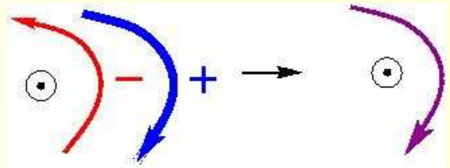

$$\sigma_{xx}(\omega) = \sigma_Q \frac{\omega\left(\omega + i\gamma + i\omega_c^2/\gamma\right)}{(\omega + i\gamma)^2 - \omega_c^2}$$

Pole in the response

$$\omega^* = \pm\omega_c - i\gamma$$

Collective cyclotron frequency of the relativistic plasma

$$\omega_c = \frac{\rho\, B/c}{(\varepsilon + P)/\mathrm{v}_F^2} \leftrightarrow \omega_c^{FL} = \frac{e\, B/c}{m}$$




Broadening of resonance: $$\gamma = \sigma_Q \frac{(B/c)^2}{(\varepsilon + P)/\mathrm{v}_F^2}$$

**Observable at room temperature in the GHz regime!**

# Relativistic hydrodynamics from microscopics

## Does relativistic hydro really apply to graphene even though Coulomb interactions break relativistic invariance?

Yes!

Key point: There is a zero (“momentum”) mode of the collision integral due to translational invariance of the interactions

The dynamics of the zero mode under an AC driving field (coupling it to other modes) reproduces relativistic hydrodynamics at low frequencies exactly.

Condition: Relativistic, weak-coupling quasiparticle picture applies

Beyond weak coupling approximation:

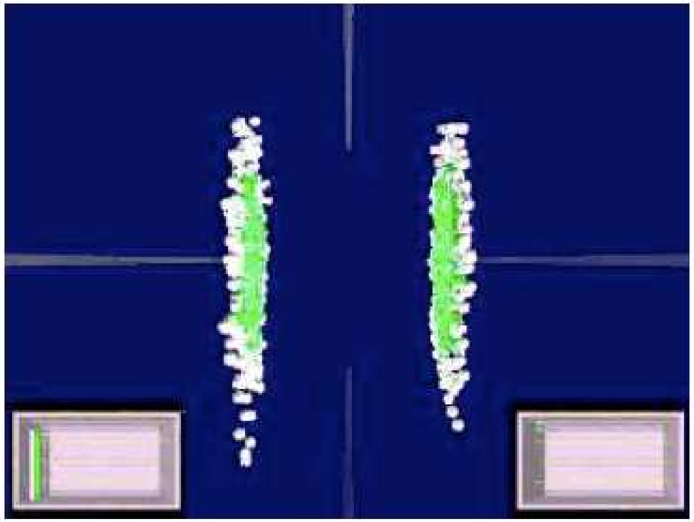
Graphene

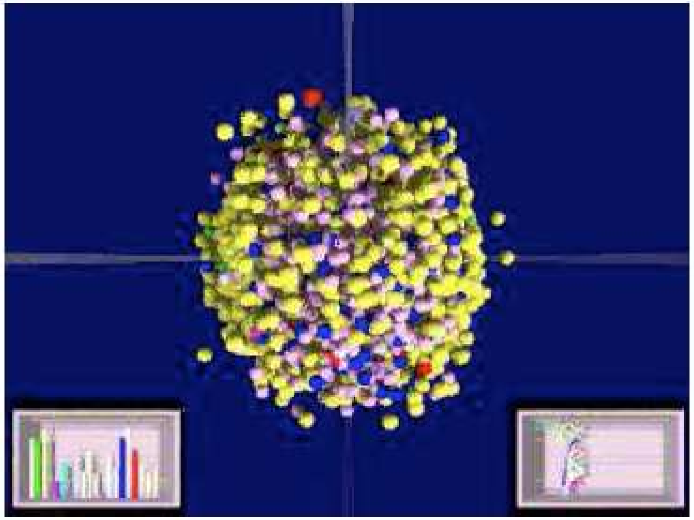
↔

Very strongly coupled, critical relativistic liquids?

AdS – CFT !

# Au+Au collisions at RHIC





Quark-gluon plasma is described by QCD (nearly conformal, critical theory)

–

Low viscosity fluid!

# Compare graphene to Strongly coupled relativistic liquids

*S. Hartnoll, P. Kovtun, MM, S. Sachdev (2007)*

Response for special strongly coupled relativistic fluids
(maximally supersymmetric SU(N) Yang Mills theory with $N \rightarrow \infty$ colors)
By mapping to weakly coupled gravity problem:

AdS (gravity) $\leftrightarrow$ $CFT_{2+1}$ [SU(N>>1)]

weak coupling $\leftrightarrow$ strong coupling

Obtain exact results for transport via the AdS–CFT correspondence

# SU(N) transport from AdS/CFT

Gravitational dual to SUSY SU(N)-CFT$_{2+1}$: Einstein-Maxwell theory

$$I = \frac{1}{g^2}\int d^4x\sqrt{-g}\left[-\frac{1}{4}R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{3}{2}\right].$$

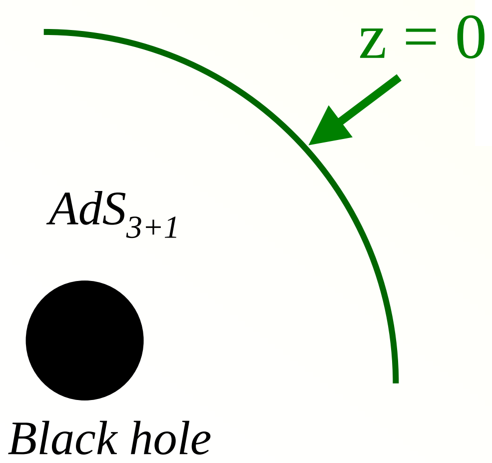
(embedded in M theory as $AdS_4 \times S^7$: $1/g^2 \sim N^{3/2}$)

It has a black hole solution (with electric and magnetic charge):

$$ds^2 = \frac{\alpha^2}{z^2}\left[-f(z)dt^2 + dx^2 + dy^2\right] + \frac{1}{z^2}\frac{dz^2}{f(z)},$$

$$F = h\alpha^2 dx \wedge dy + q\alpha dz \wedge dt,$$

$$f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3.$$




Electric charge $q$ and magnetic charge $h$
$\leftrightarrow$ $\mu$ and $B$ for the CFT

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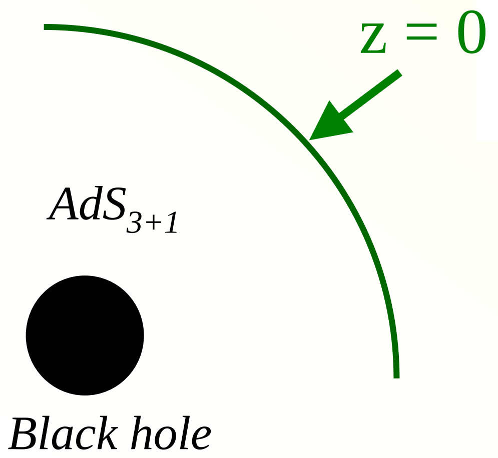



Background $\leftrightarrow$ Equilibrium

Transport $\leftrightarrow$ Perturbations in $g_{tx,ty}, A_{x,y}$.

Response via Kubo formula from $\delta^2 I/\delta(g,A)^2$.

# Compare graphene to
# Strongly coupled relativistic liquids

*S. Hartnoll, P. Kovtun, MM, S. Sachdev (2007)*

Obtain exact results via string theoretical AdS–CFT correspondence

→

- Confirm the results of hydrodynamics: response functions σ(ω), resonances
- Calculate the transport coefficients for a strongly coupled theory!

SUSY - SU(N): $\sigma(\mu=0)=\sqrt{\frac{2}{9}}N^{3/2}\frac{e^2}{h}$ ; $\frac{\eta_{shear}}{s}(\mu=0)=\frac{1}{4\pi}\frac{\hbar}{k_B}$

Interpretation: $N^{3/2}$ effective degrees of freedom, strongly coupled: $\tau T = O(1)$

# Graphene – a nearly perfect liquid!

*MM, J. Schmalian, and L. Fritz, (PRL 2009)*

↔

Anomalously low viscosity (like quark-gluon plasma)

**"Heisenberg"** $\frac{\eta}{s} \sim E_{qp}\tau \geq 1$ → Measure of strong coupling:

Conjecture from AdS-CFT:

$$\frac{\text{shear viscosity}}{\text{entropy density}} = \frac{\eta}{s} > \frac{\hbar}{k_B}\frac{1}{4\pi}$$

Doped Graphene & Fermi liquids: *(Khalatnikov etc)*

$$\eta \propto n \cdot mv^2 \cdot \tau \;\rightarrow\; n \cdot E_F \cdot \frac{\hbar E_F}{(k_B T)^2}$$

$$s \propto k_B n \frac{T}{E_F}$$

$$\frac{\eta}{s} \sim \frac{\hbar}{k_B}\left(\frac{E_F}{T}\right)^3$$

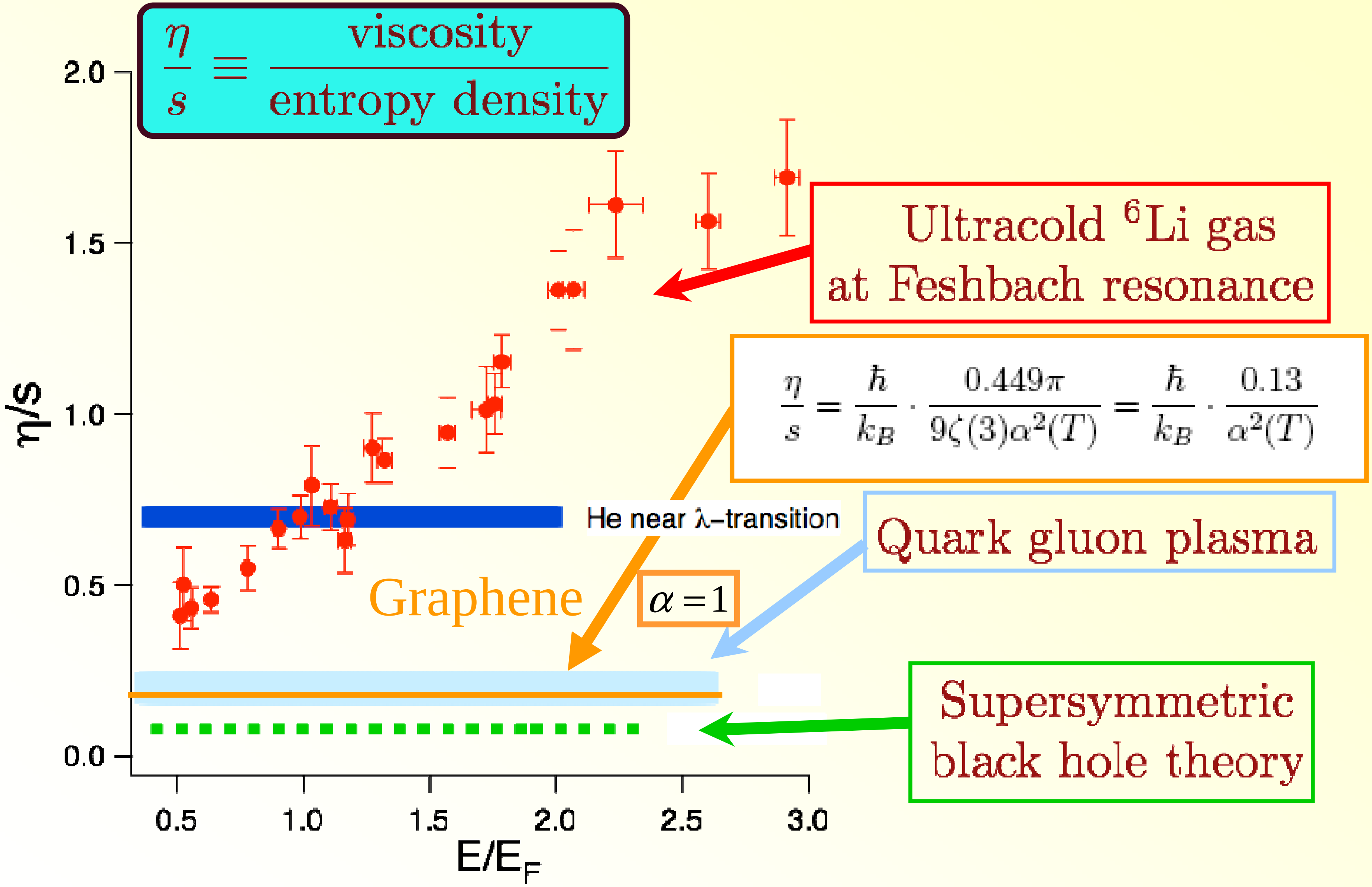
Undoped Graphene:

$$\eta \propto n \cdot mv^2 \cdot \tau \;\rightarrow\; n_{\text{th}} \cdot k_B T \cdot \frac{\hbar}{\alpha^2 k_B T} = \frac{\hbar}{\alpha^2} n_{\text{th}}$$

$$s \propto k_B n_{\text{th}}$$

Exact (Boltzmann-Born Approx):

$$\frac{\eta}{s} = \frac{\hbar}{k_B} \cdot \frac{0.449\pi}{9\zeta(3)\alpha^2(T)} = \frac{\hbar}{k_B} \cdot \frac{0.13}{\alpha^2(T)}$$




*T. Schäfer, Phys. Rev. A* ***76****, 063618 (2007).*

*A. Turlapov, J. Kinast, B. Clancy, Le Luo, J. Joseph, J. E. Thomas, J. Low Temp. Phys.* ***150****, 567 (2008)*

# Electronic consequences of low viscosity?

*MM, J. Schmalian, L. Fritz, (PRL 2009)*

Electronic turbulence in clean, strongly coupled graphene?
(or at quantum criticality!)
Reynolds number:

$$\mathrm{Re} = \frac{s/k_B}{\eta/\hbar} \times \frac{k_B T}{\hbar v/L} \times \frac{u_{\mathrm{typ}}}{v}$$

Strongly driven mesoscopic systems: (Kim's group [Columbia])

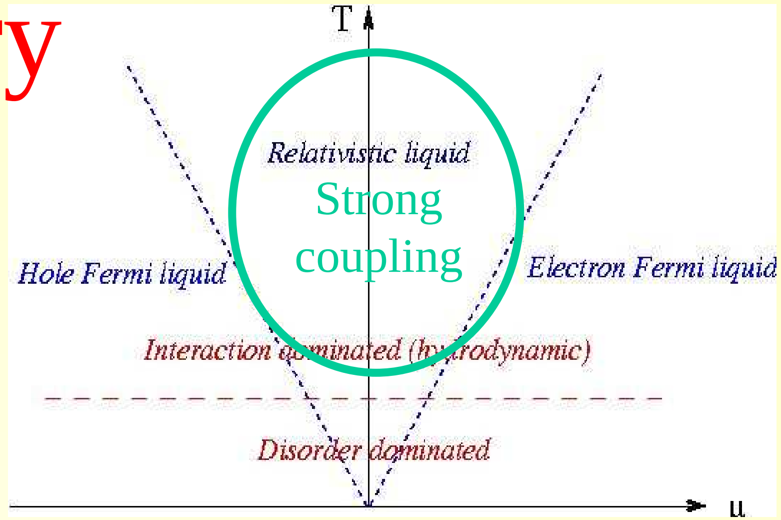
$L = 1\mu m$
$u_{typ} = 0.1\mathrm{v}$
$T = 100K$

→ $\mathrm{Re} \sim 10 - 100$

Complex fluid dynamics!
(pre-turbulent flow)

New phenomenon in an electronic system!

# Summary




- Undoped graphene is strongly coupled in a large temperature window!
- Nearly universal strong coupling features in transport; many similarities with strongly coupled critical fluids (described by AdS-CFT)
- Emergent relativistic hydrodynamics at low frequency
- Graphene: Nearly perfect quantum liquid!
  → Possibility of complex (turbulent?) current flow at high bias