Relativistic transport at strong coupling: graphene, quantum criticality and black holes

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in collaboration with

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Outline

- Philosophy and basic recipes
- Strong coupling features in collision-dominated transport in AdS-CFT
- Strong coupling features at quantum criticality, especially in graphene:
  Graphene as an almost perfect quantum liquid

The challenge of strong coupling in condensed matter theory

- Electrons have strong bare interactions (Coulomb)
- But: non-interacting quasiparticle picture (Landau-Fermi liquid) works very well for most metals

  Reason: RG irrelevance of interactions,
  ↔ screening and dressing of quasiparticles
- Opposite extreme: Interactions much stronger than the Fermi energy → Mott insulators with localized e's
- Biggest challenge: strong coupling physics close to quantum phase transitions.

  Maximal competition between wave and particle character (e.g.: high Tc superconductors, heavy fermions, cold atoms, graphene)

The challenge of strong coupling in condensed matter theory

Idea and Philosophy:

Study [certain] strongly coupled CFTs (= QFT's for quantum critical systems) by the AdS-CFT correspondence

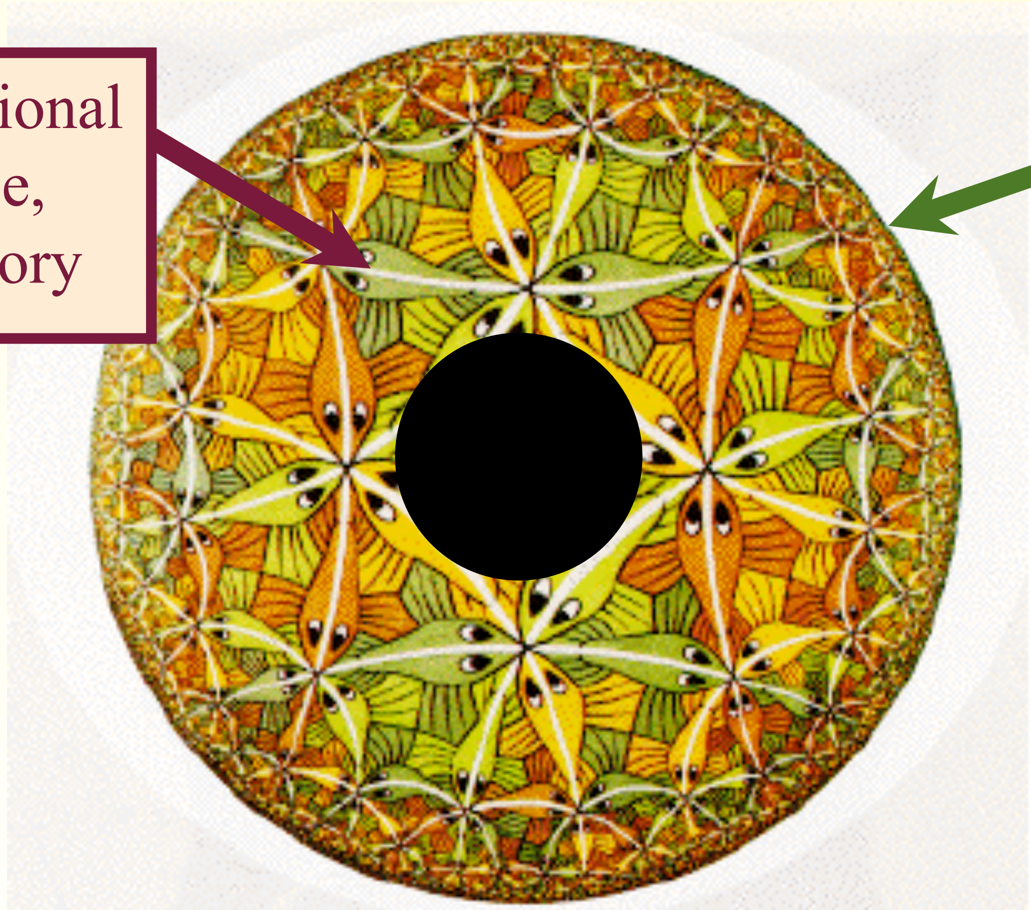
→ Learn about physical properties of strongly coupled theories (beyond ε- and $1/N$ expansions)

→ Exotic matter? (Bose fluids, strong coupling superconductivity, etc)

→ Extract the general/universal physics from the particular examples to make the lessons useful for condensed matter theory.

Holographic duality

D+1 dimensional AdS space, gravity theory



A D=d+1 dimensional system at its quantum critical point, (conformal) QFT

Maldacena, Gubser, Klebanov, Polyakov, Witten

Gravity side: Anti de Sitter space

Gravity action (Hilbert-Einstein) – if curvature small compared to string scale

$$S_{grav}[g] = \frac{1}{16\pi G_N} \int d^{D+1}x \sqrt{g} \left(R - 2\Lambda + \ldots \right)$$

R Ricci scalar, Λ cosmological constant

$$2\Lambda = -\frac{D(D-1)}{L^2}$$

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Saddlepoints of exp[-S] $\leftrightarrow$ solutions of Einsteins equations:

$$R_{ab} = -\frac{d}{L^2} g_{ab}$$

Symmetric solution: Anti-de Sitter space (space of constant negative curvature 1/L^2)

$$g_{ab} : ds^2 = L^2 \left(\frac{-dt^2 + \sum_{i=1}^{d} dx_i^2}{z^2} + \frac{dz^2}{z^2} \right)$$

$z = 0$: boundary; $z = \infty$: horizon

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$$ds^2 = \frac{u^2}{L^2}\left(-dt^2 + \sum_{i=1}^{d} dx_i^2\right) + L^2\frac{du^2}{u^2}$$

$u \equiv \frac{L^2}{z}$; $u = \infty:$ UV (boundary); $u = 0:$ infrared

Anti de Sitter space AdS_{D+1}

Extra dimension: the RG scale of the boundary theory

Extra dimension z: length scale

$$g_{ab} : ds^2 = L^2 \left(\frac{-dt^2 + \sum_{i=1}^{d} dx_i^2}{z^2} + \frac{dz^2}{z^2} \right)$$

$z = \infty :$ horizon; $\quad z = 0 :$ boundary

Extra dimension $u = L^2/z$: energy scale

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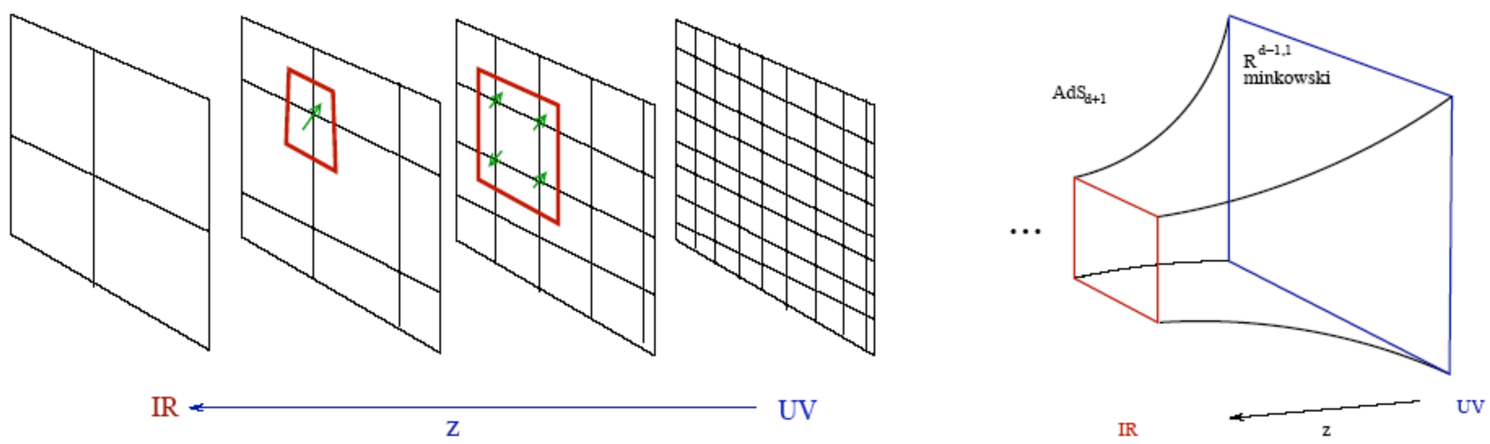
$u = 0 :$ IR (horizon); $\quad u = \infty :$ UV (boundary)



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$u = 0$: IR (horizon); $u = \infty$: UV (boundary)

Remarks:

- Metric on the boundary (t, x_i, $z = 0$): Minkowski
- Symmetry of the metric: SO(D,2) [AdS can be embedded in R_{D+2} as symmetric hyperboloid]
- Dilation symmetry (part of conformal symmery) : $t, x_i, z \to \lambda t, \lambda x_i, \lambda z \quad u \to u/\lambda$
- SO(D,2) is also the conformal group in D dimensions! Strong hint that AdS_{D+1} is the space to be related with conformal QFT's in D dimension

Correspondence

Quantum gravity (bulk)	QFT (boundary)
Gravity+extra matter	SU(N) gauge theory
Bulk fields	Operators of the QFT
D+1=d+2 space time dimensions	D=d+1 space time dimensions
Semi-Classical limit (saddle point)	Non-trivial strong coupling limit ('t Hooft limit) $\lambda = g^2 N, \quad N \to \infty$
Extra dimension: u z=L^2/u	Energy scale (RG) Length scale

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Central duality conjecture (taken for granted):

$$Z_{bulk}\left[\phi(z,x) \to z^{d-\Delta}\delta\phi_{(0)}(x)\right] = \left\langle \exp\left(i\int d^D x\, \delta\phi_{(0)}(x) O(x)\right)\right\rangle_{QFT}$$

Large N limit: easy (classical saddle point, ODE) $\leftrightarrow$ hard (non-trivial strong coupling)

The classical limit (“large N”)

Classical limit (saddle point approximation, “large N limit”)

Gravity: AdS radius (radius of curvature) much larger than the Planck scale

Indeed:

$$S_{grav}[g] = \frac{1}{16\pi G_N}\int d^{D+1}x\sqrt{g}\left(R - 2\Lambda + \ldots\right) \sim \frac{L^{D-1}}{\ell_{Pl}^{D-1}} \gg 1$$

Holographic principle: (Area of boundary)/$G_N \sim N^2$

$$\frac{L^{D-1}}{G_N} \approx \frac{L^{D-1}}{\ell_{Pl}^{D-1}} \approx N^2 \gg 1$$

$N^2 >> 1$

↔ (QFT): Number of degrees of freedom per site >> 1

↔ central charge c of the CFT c >> 1

AdS_{D+1}-CFT_D dictionary

Quantum gravity (bulk)	QFT (boundary)
Gravity+extra matter	SU(N) gauge theory
Bulk fields	Operators of the QFT
Graviton	Energy momentum tensor
Global current J	Maxwell field A
Scalar/fermionic operator	Scalar/fermionic field

Best established examples

- $\mathcal{N} = 4$ super Yang-Mills (SU(N)) in D=3+1:
Content: gauge field, scalars and fermions in the adjoint representation (conformal, $\beta(g)=0$)

- $\mathcal{N} = 8$ super Yang-Mills in D=2+1
(asymptoticaly conformal strong coupling IR fixed point)

Finite T

• T breaks scale invariance by introducing an IR scale in the CFT

$\leftrightarrow$

IR modification of the AdS metric: horizon at z ~ 1/T! $\rightarrow$ Black hole solution:

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + \sum_{i=1}^{d} dx_i^2 + dz^2\right) \rightarrow ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \sum_{i=1}^{d} dx_i^2 + \frac{dz^2}{f(z)}\right)$$

$$f(z) = 1 - \left(\frac{z}{z_H}\right)^D \quad \text{with} \quad z_H \propto \frac{1}{T}$$

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• Asymptotic AdS metric is conserved
• Event horizon when $f(z_H)=0$;
• Boundary condition at z_H: only infalling waves!
• Precise connection with temperature:
In Euclidean time: space time is non-singular only if $\tau=it$ is periodic with period

$$\Delta\tau \equiv \frac{1}{T} = \frac{4\pi}{|f'(z_H)|} = \frac{4\pi z_H}{d} = \frac{1}{T_{Hawking}}$$

AdS CFT in practice

Use the correspondence:

$$Z_{bulk}\left[\phi(z,x)\to z^{d-\Delta}\delta\phi_{(0)}(x)\right] = \left\langle \exp\left(i\int d^D x\,\delta\phi_{(0)}(x)\mathrm{O}(x)\right)\right\rangle_{QFT}$$

To compute correlation functions in the CFT:

$$Z_{bulk}\left[\phi(z,x)\to z^{d-\Delta}\delta\phi_{(0)}(x)\right] \approx \exp\left[-S_{cl}\left(\delta\phi_{(0)}(x)\right)\right]$$

$$\left\langle \mathrm{O}(x)\mathrm{O}(y)\right\rangle_{QFT} = \left.\frac{\delta^2 S_{cl}\left(\delta\phi_{(0)}(x)\right)}{\delta\phi_{(0)}(x)\,\delta\phi_{(0)}(y)}\right|_{\delta\phi_{(0)}=0}$$

Transport coefficients (thermo-electric conductivities, viscosity) from Kubo formula (retarded Greens functions)

Application:

Thermoelectric transport in D=2+1

Supersymmetric SU(N)
dual to
Einstein + Maxwell (dual to the current)

SU(N) transport from AdS/CFT

Gravitational dual to SUSY SU(N)-CFT$_{2+1}$: Einstein-Maxwell theory

$$I = \frac{1}{g^2}\int d^4x\sqrt{-g}\left[-\frac{1}{4}R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{3}{2}\right] .$$

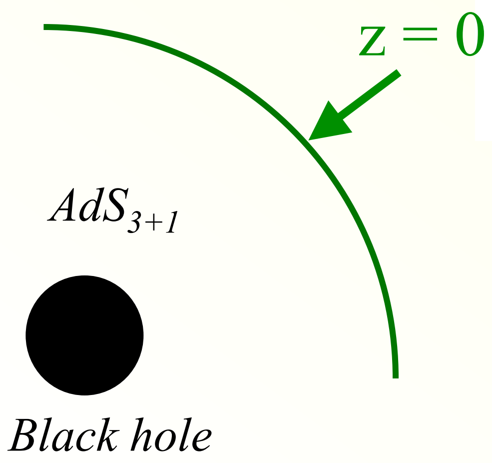
(embedded in M theory as $AdS_4 \times S^7$: $1/g^2 \sim N^{3/2}$)

It has a black hole solution (with electric and magnetic charge):

$$ds^2 = \frac{\alpha^2}{z^2}\left[-f(z)dt^2 + dx^2 + dy^2\right] + \frac{1}{z^2}\frac{dz^2}{f(z)} ,$$

$$F = h\alpha^2 dx \wedge dy + q\alpha dz \wedge dt ,$$

$$f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3 .$$



Electric charge q and magnetic charge h

$\leftrightarrow$ μ and B for the CFT

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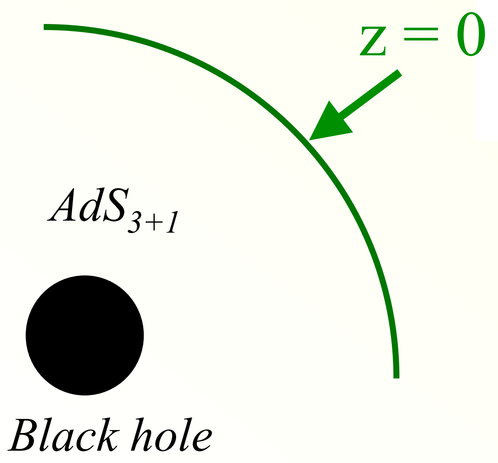

Background $\leftrightarrow$ Equilibrium

Transport $\leftrightarrow$ Perturbations in $g_{tx,ty}, A_{x,y}$.

Response via Kubo formula from $\delta^2 I / \delta(g, A)^2$.

Compare graphene to Strongly coupled relativistic liquids

S. Hartnoll, P. Kovtun, MM, S. Sachdev (2007)

Obtain exact results via string theoretical AdS–CFT correspondence

→

- Thermoelectric response functions $\sigma(\omega)$, resonances: relat. hydrodynamics
- Calculate the transport coefficients for a strongly coupled theory!

SUSY - SU(N): $\sigma(\mu = 0) = \sqrt{\frac{2}{9}} N^{3/2} \frac{e^2}{h}$

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SUSY - SU(N): $\sigma(\mu=0)=\sqrt{\frac{2}{9}}N^{3/2}\frac{e^2}{h}$; $\frac{\eta_{shear}}{s}(\mu=0)=\frac{1}{4\pi}\frac{\hbar}{k_B}$

Interpretation: $N^{3/2}$ effective degrees of freedom, strongly coupled: $\tau T = O(1)$

Anomalously low viscosity (like quark-gluon plasma)

"Heisenberg" $\boxed{\frac{\eta}{s} \sim E_{qp}\tau \geq 1}$ → Measure of strong coupling!

Quantum critical systems in condensed matter

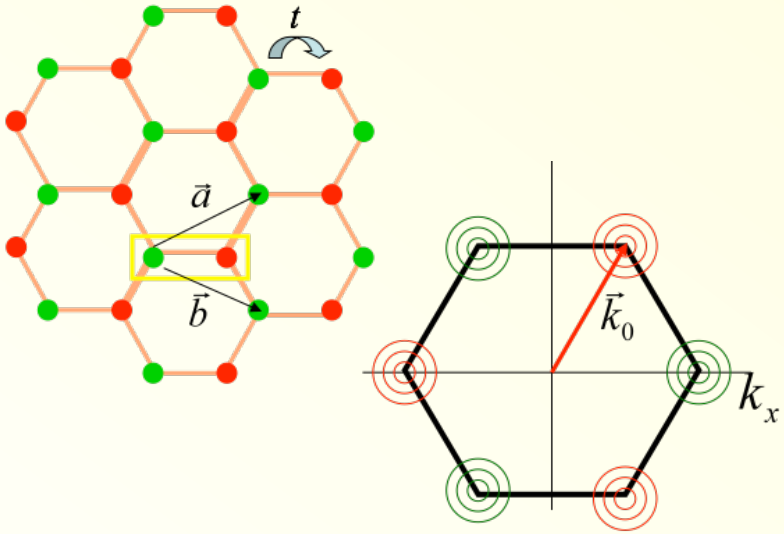
A few examples

- Graphene
- High Tc
- Superconductor-to-insulator transition (interaction driven)

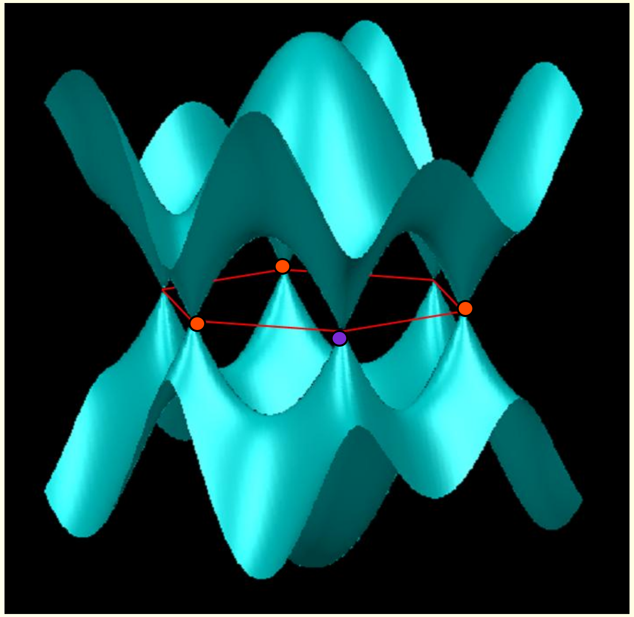
Dirac fermions in graphene

(Semenoff '84, Haldane '88)

Honeycomb lattice of C atoms



Tight binding dispersion



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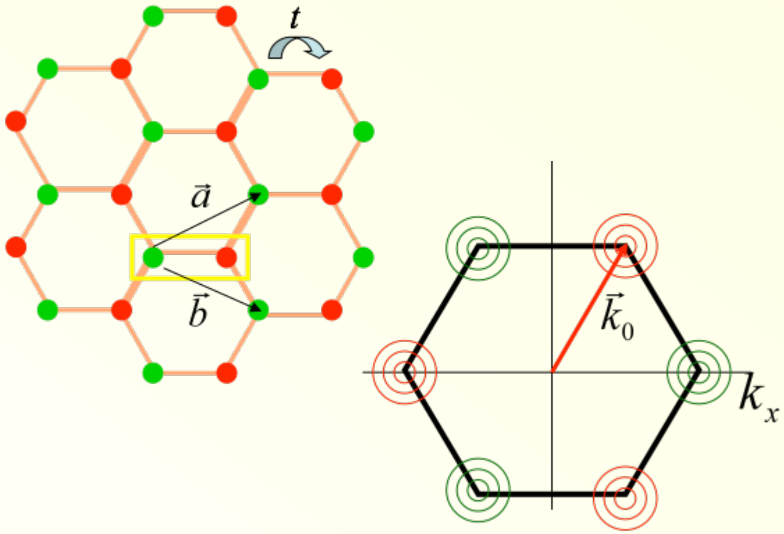

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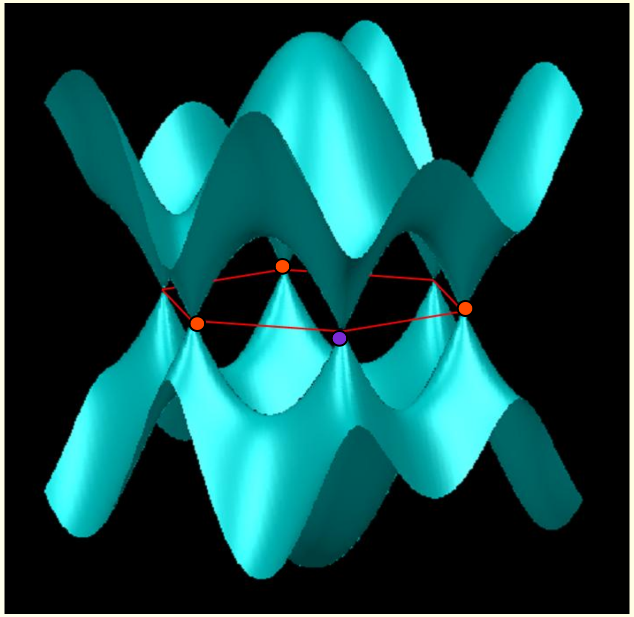

2 massless Dirac cones in the Brillouin zone:
(Sublattice degree of freedom ↔ pseudospin)

Close to the two Fermi points **K**, **K'**:

$$H \approx v_F \left(\vec{\mathbf{p}} - \vec{\mathbf{K}}\right) \cdot \vec{\sigma}_{\text{sublattice}}$$

$$\rightarrow \quad E_{\mathbf{p}} = v_F \left|\vec{\mathbf{p}} - \mathbf{K}\right|$$

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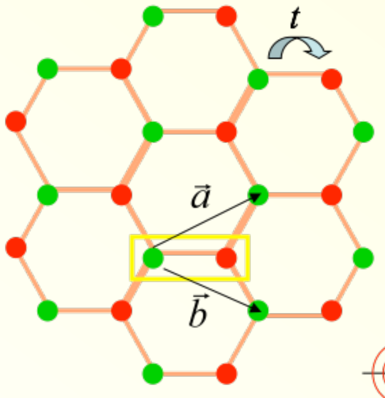

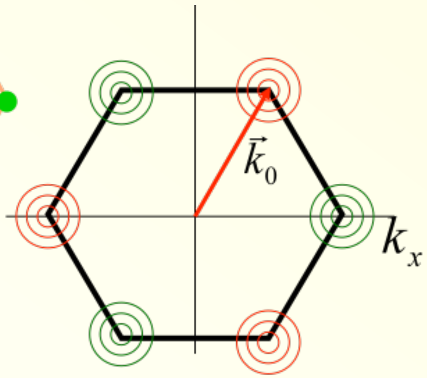

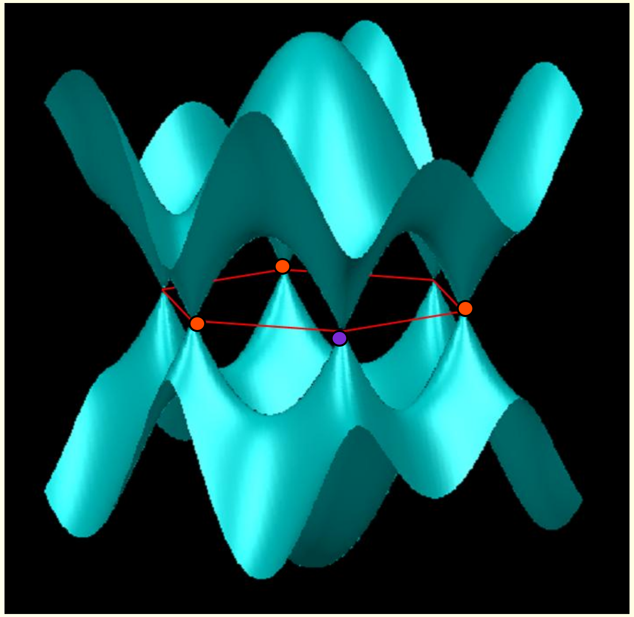

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Fermi velocity (speed of light")

$$\boxed{\mathrm{v}_F \approx 1.1 \cdot 10^6\ \mathrm{m/s} \approx \frac{c}{300}}$$

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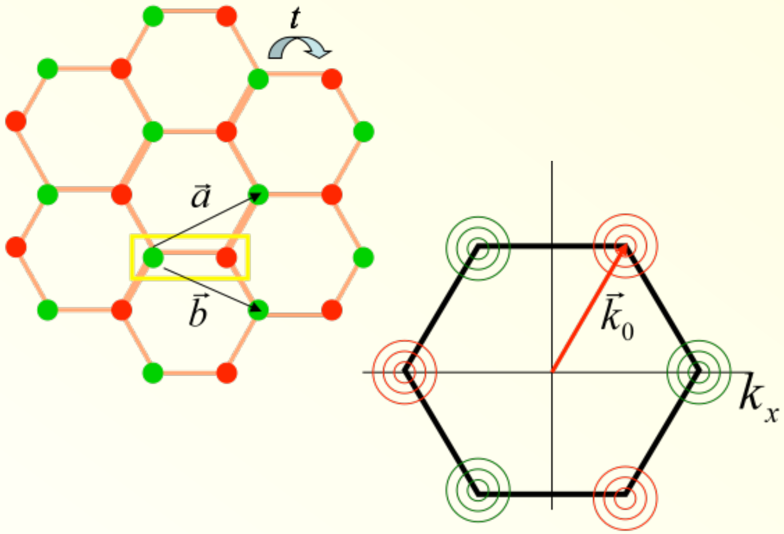

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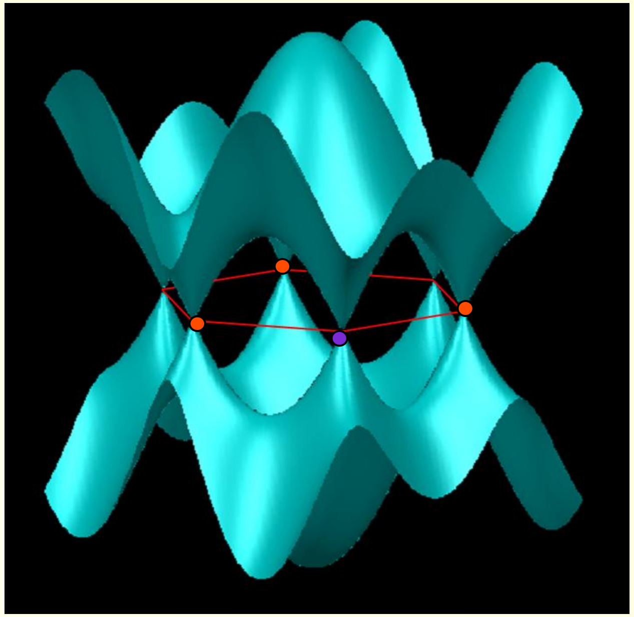

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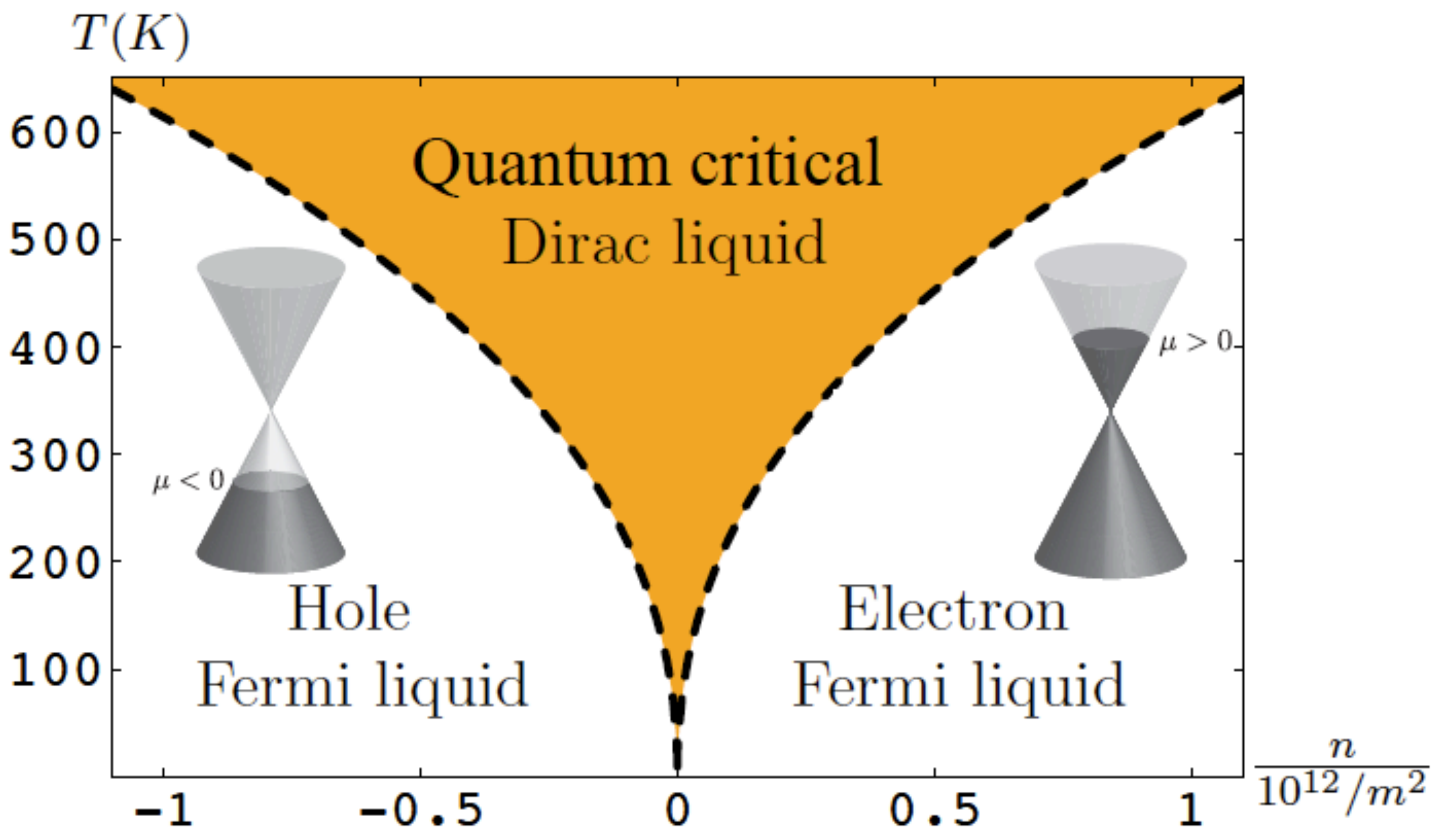
Coulomb interactions: Fine structure constant

$$\alpha \equiv \frac{e^2}{\varepsilon \hbar v_F} = O(1)$$

Relativistic fluid at the Dirac point

D. Sheehy, J. Schmalian, Phys. Rev. Lett. ***99****, 226803 (2007).*

- Relativistic plasma physics of interacting particles and holes!



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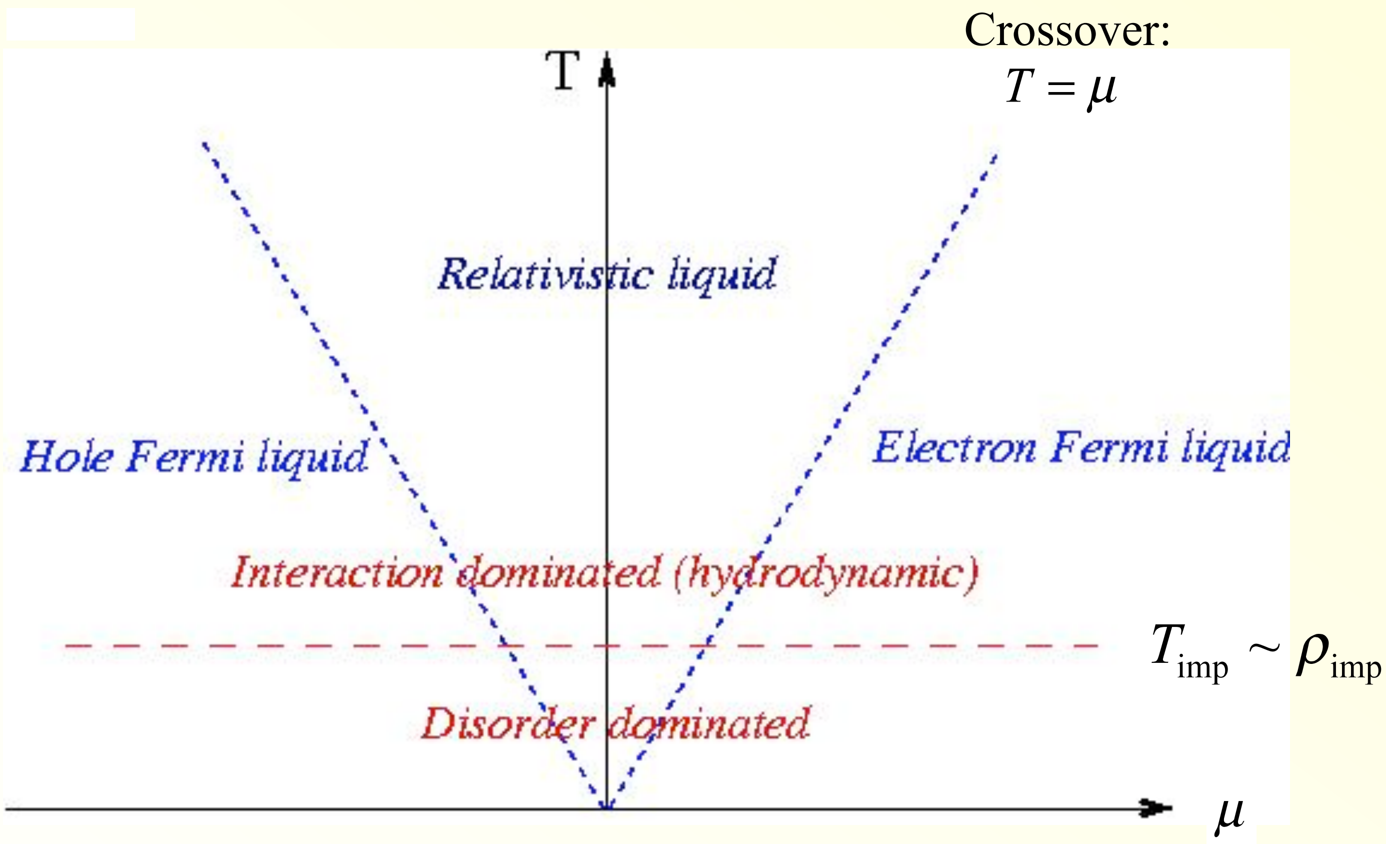
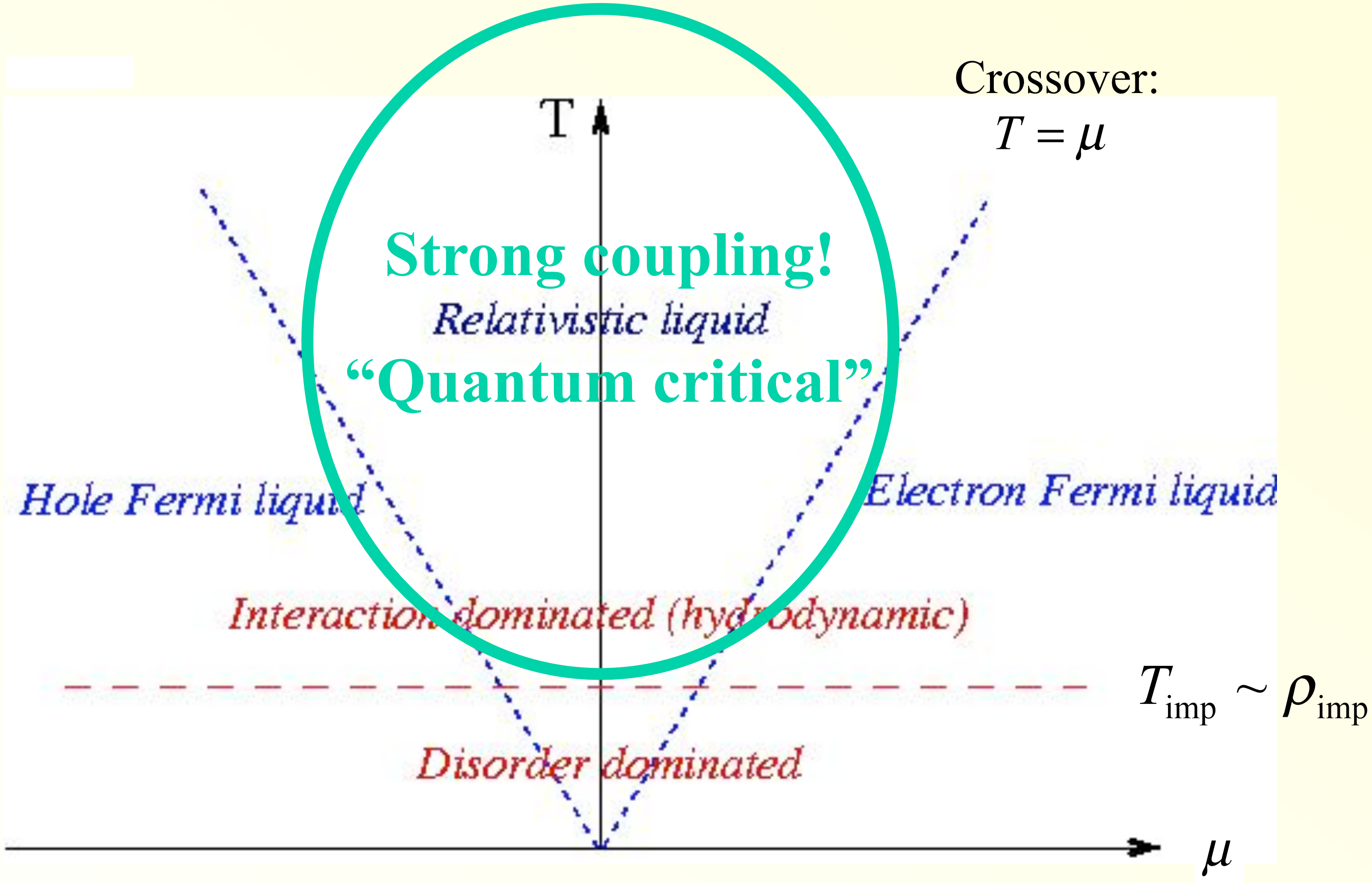


Relativistic fluid at the Dirac point

*D. Sheehy, J. Schmalian, Phys. Rev. Lett. **99**, 226803 (2007).*

- Relativistic plasma physics of interacting particles and holes!
- Strongly coupled, nearly quantum critical fluid at $\mu = 0$



Other relativistic fluids:

- Systems close to quantum criticality (with z = 1)

Example: Superconductor-insulator transition (Bose-Hubbard model)

$$\boxed{\tau_{rel}^{-1} \approx \frac{k_B T}{\hbar}}$$

Maximal possible relaxation rate!

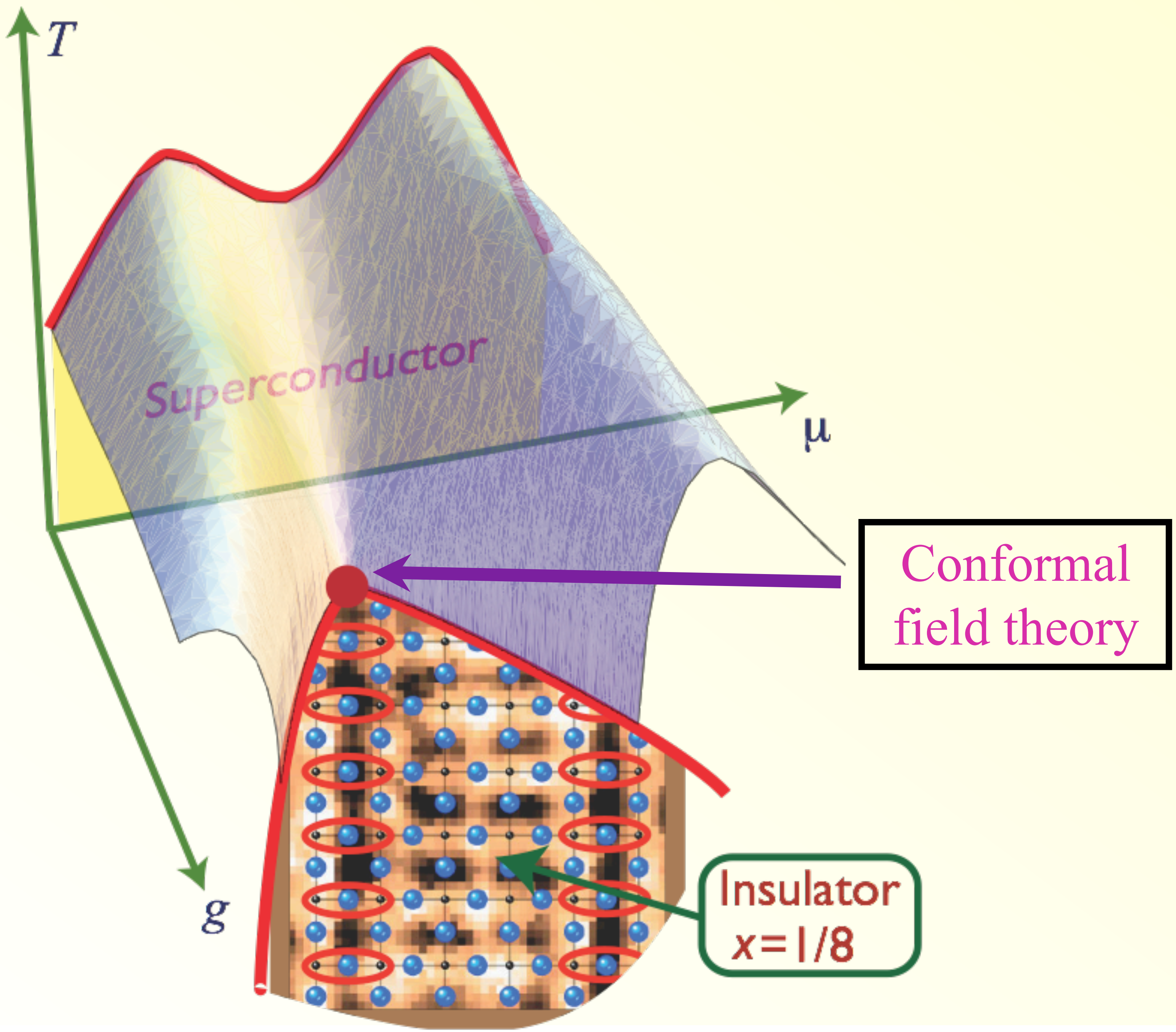
Damle, Sachdev (1996, 1997)
Bhaseen, Green, Sondhi (2007).
Hartnoll, Kovtun, MM, Sachdev (2007)

- Conformal field theories (QFTs for quantum criticality)

E.g.: strongly coupled Yang-Mills theories
$\rightarrow$ Exact treatment via AdS-CFT correspondence

C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son (2007)
Hartnoll, Kovtun, MM, Sachdev (2007)

Quantum criticality in cuprate high T_c's



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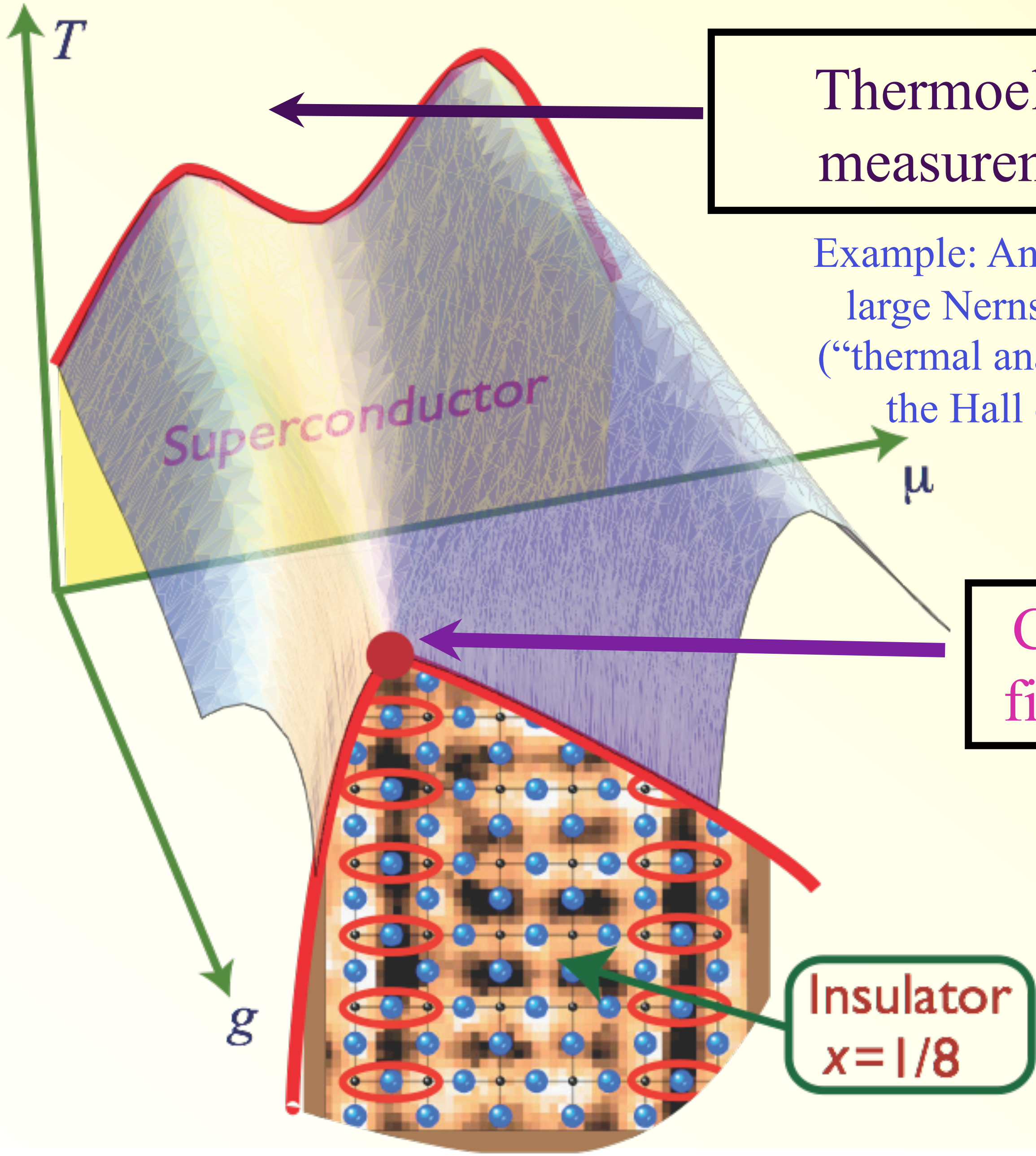


Thermoelectric measurements.

Example: Anomalously large Nernst Effect! ("thermal analogue" of the Hall effect)

Conformal field theory

Simplest example exhibiting "quantum critical" features:

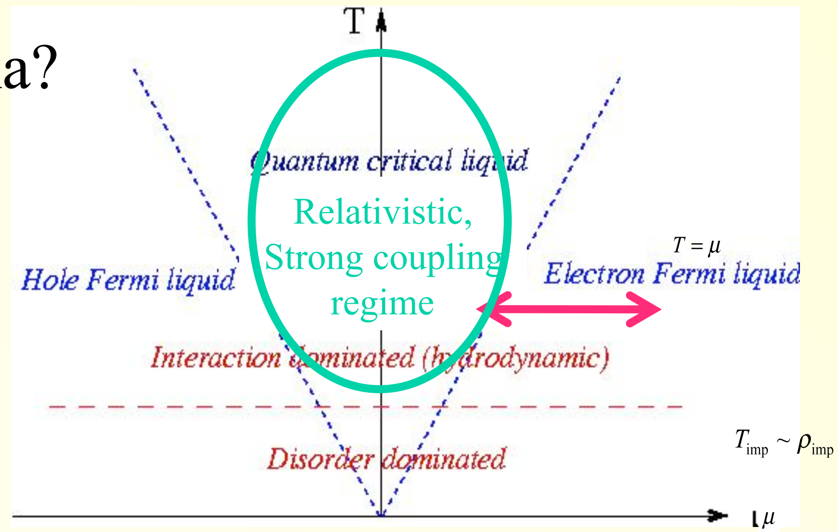
Graphene

Questions

- Transport characteristics in the strongly coupled relativistic plasma?

- Response functions and transport coefficients at strong coupling?

- Graphene as a nearly perfect and possibly turbulent quantum fluid (like the quark-gluon plasma)?



Graphene – Fermi liquid?

1. Tight binding kinetic energy
   $\longrightarrow$ massless Dirac quasiparticles

$$H_0 = \sum_{\lambda=\pm} \sum_{a=1}^{N} \int \frac{d^2k}{(2\pi)^2} \lambda v_F k \, \gamma^\dagger_{\lambda a}(\mathbf{k}) \gamma_{\lambda a}(\mathbf{k})$$

2. Coulomb interactions:
Unexpectedly strong!
$\longrightarrow$ nearly quantum critical!

$$\boxed{V(\mathbf{q}) = \frac{2\pi e^2}{\varepsilon |\mathbf{q}|}}$$

$$H_1 = \frac{1}{2} \int \frac{d^2k_1}{(2\pi)^2} \frac{d^2k_2}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \Psi^\dagger_a(\mathbf{k}_2 - \mathbf{q}) \Psi_a(\mathbf{k}_2) V(\mathbf{q}) \Psi^\dagger_b(\mathbf{k}_1 + \mathbf{q}) \Psi_b(\mathbf{k}_1)$$

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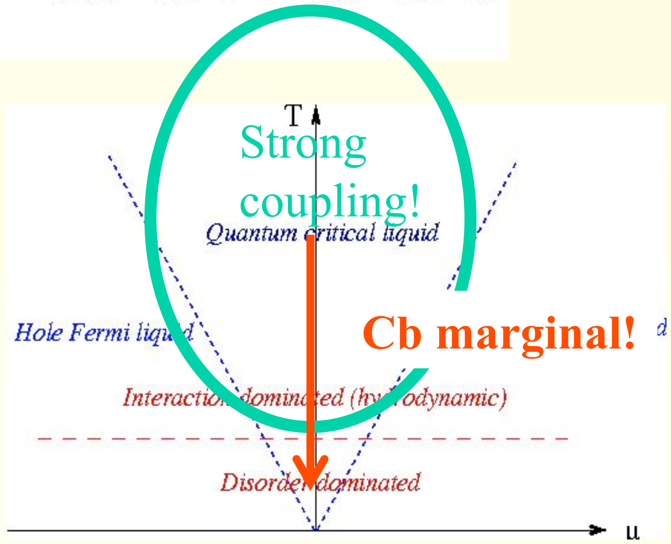
Coulomb **only marginally** irrelevant for $\mu = 0$!



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$$H_0 = \sum_{\lambda=\pm}\sum_{a=1}^{N}\int \frac{d^2k}{(2\pi)^2}\lambda v_F k\, \gamma^\dagger_{\lambda a}(\mathbf{k})\gamma_{\lambda a}(\mathbf{k})$$

2. Coulomb interactions:
Unexpectedly strong!
→ nearly quantum critical!

$$\boxed{V(\mathbf{q}) = \frac{2\pi e^2}{\varepsilon|\mathbf{q}|}}$$

$$H_1 = \frac{1}{2}\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\Psi^\dagger_a(\mathbf{k}_2-\mathbf{q})\Psi_a(\mathbf{k}_2)V(\mathbf{q})\Psi^\dagger_b(\mathbf{k}_1+\mathbf{q})\Psi_b(\mathbf{k}_1)$$

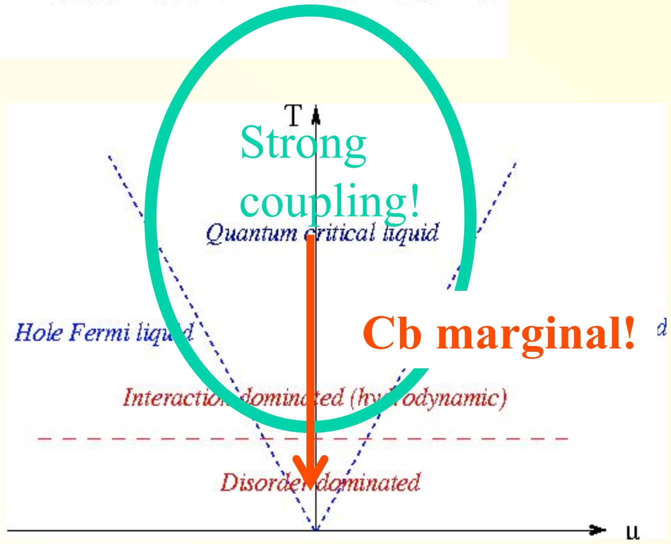
Coulomb **only marginally** irrelevant for $\mu = 0$!

RG:
$(\mu = 0)$

$$\frac{d\alpha}{d\ell} = -\frac{\alpha^2}{4} + \mathcal{O}(\alpha^3)$$

$$\boxed{\alpha \equiv \frac{e^2}{\varepsilon\hbar v_F} = O(1)}$$

$$\alpha(T) = \frac{\alpha^0}{1+(\alpha^0/4)\ln(\Lambda/T)} \overset{T\to 0}{\sim} \frac{4}{\ln(\Lambda/T)}$$



Graphene – Fermi liquid?

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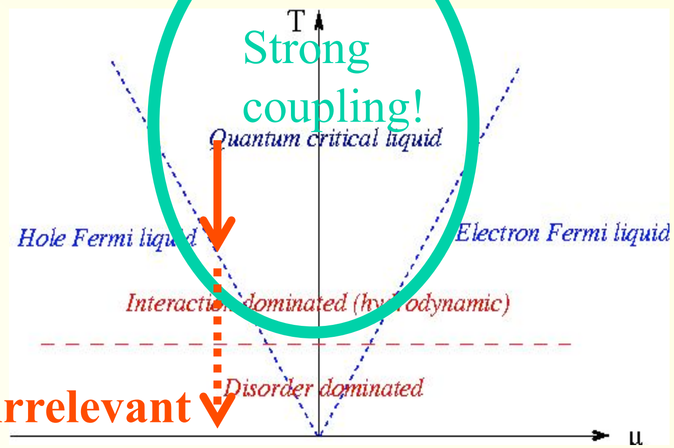
($\mu > 0$) $T < \mu$: **Screening kicks in, short ranged Cb irrelevant**



Strong coupling in undoped graphene

MM, L. Fritz, and S. Sachdev, PRB '08.

Inelastic scattering rate
(Electron-electron interactions)

μ >> T: standard 2d Fermi liquid

$$\boxed{\tau_{\text{ee}}^{-1} = C \frac{(k_B T)^2}{\hbar\mu}}$$

C: Independent of the Coulomb coupling strength!

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Fastest possible rate!

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"Heisenberg uncertainty principle for well-defined quasiparticles"

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As long as $\alpha(T) \sim 1$, energy uncertainty is saturated, scattering is maximal
→ Nearly universal strong coupling features in transport,
similarly as at the 2d superfluid-insulator transition *[Damle, Sachdev (1996, 1997)]*

Consequences for transport

1. -Collisionlimited conductivity σ in clean undoped graphene;
   -Collisionlimited spin diffusion D_s at any doping
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Collision-dominated transport → relativistic hydrodynamics:
a) Response fully determined by covariance, thermodynamics, and σ, η
b) Collective cyclotron resonance in small magnetic field (low frequency)

Hydrodynamic regime:
(collision-dominated)

$$\tau_{\mathrm{ee}}^{-1} >> \tau_{\mathrm{imp}}^{-1}, \omega_{\mathrm{c}}^{\mathrm{typ}}, \omega_{\mathrm{AC}}$$

Collisionlimited conductivities

Damle, Sachdev, (1996).
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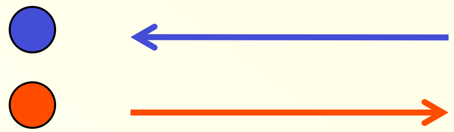

$\vec{J}_{s,c} \neq 0,$ but $\vec{P} = 0$

Pair creation/annihilation leads to current decay

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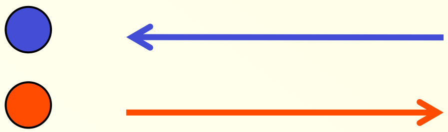

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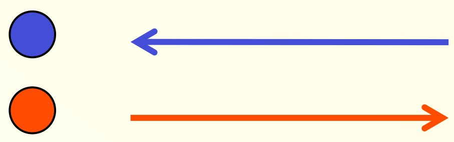

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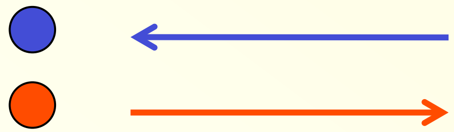

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Saturation as $\alpha \to 1$, (finally: phase transition to insulator

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Analog in SU(N):

$$\boxed{\sigma(\mu = 0) = \sqrt{\frac{2}{9}} N^{3/2} \frac{e^2}{h}}$$

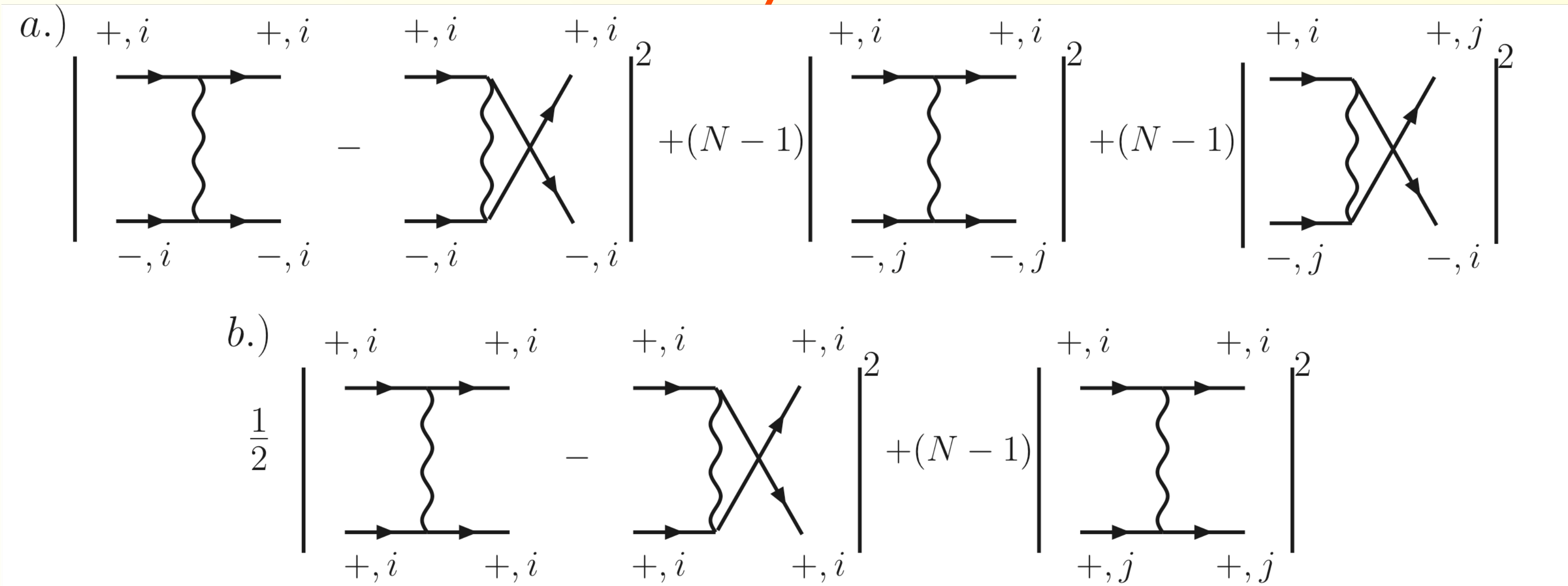
Marginal irrelevance of Coulomb: $\boxed{\alpha \approx \frac{4}{\log(\Lambda / T)} < 1}$

Boltzmann approach

L. Fritz, J. Schmalian, MM, and S. Sachdev, PRB 2008

Boltzmann equation in Born approximation

$$\left(\partial_t + e\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{k}}\right) f_{\pm}(\mathbf{k},t) = I^{Cb}_{\text{coll}}\left[\mathbf{k},t \mid \{f_{\pm}(\mathbf{k}',t)\}\right] \propto \alpha^2(T)$$



→ Collision-limited conductivity in weak coupling!

$$\boxed{\sigma(\mu=0) = \frac{0.76}{\alpha^2(T)}\frac{e^2}{h}}$$

Transport and thermoelectric response at low frequencies?

Hydrodynamic regime:
(collision-dominated)

$$\tau_{\mathrm{ee}}^{-1} >> \omega, \tau_{\mathrm{imp}}^{-1}, \omega_{\mathrm{cyclo}}^{\mathrm{th}}$$

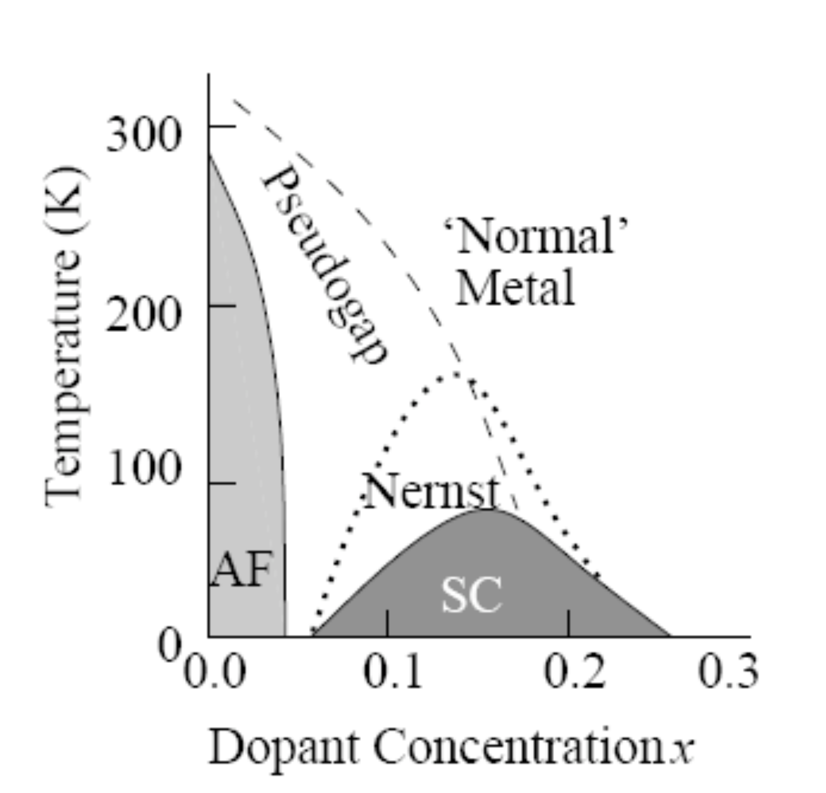
Three complementary approaches:

- AdS-CFT (strong coupling)
- Relativistic hydrodynamics (without fixing transport coefficients)
- Boltzmann theory (weak coupling)

They all agree at the level of the relativistic hydrodynamic structure, but have different microscopics.

Application: thermoelectric close to transport at quantum criticality

Nernst effect in High T_c's



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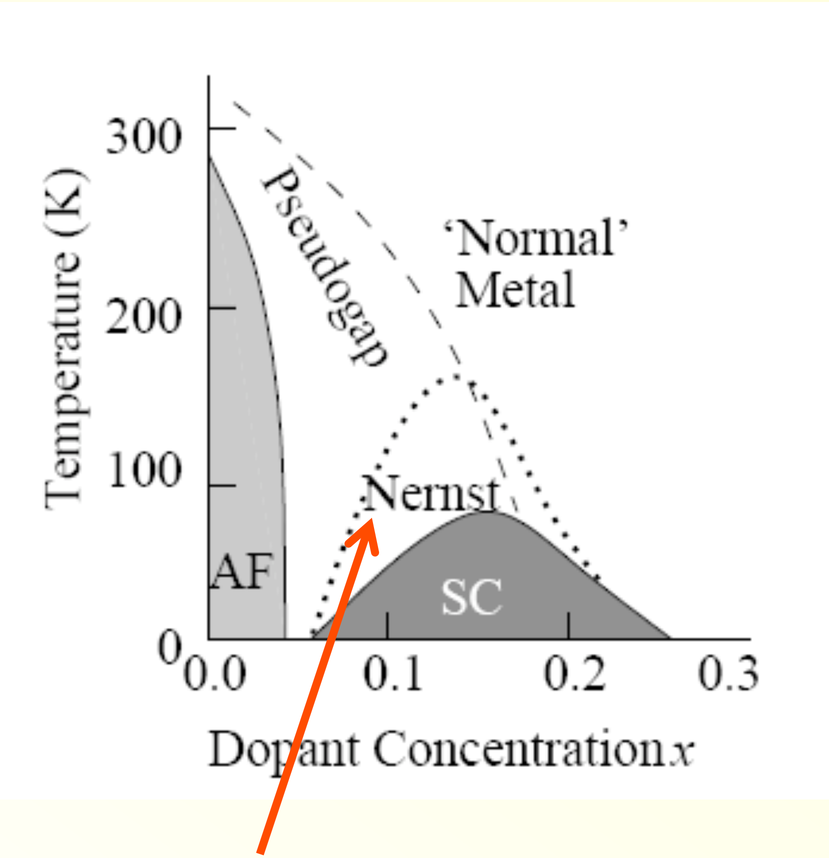
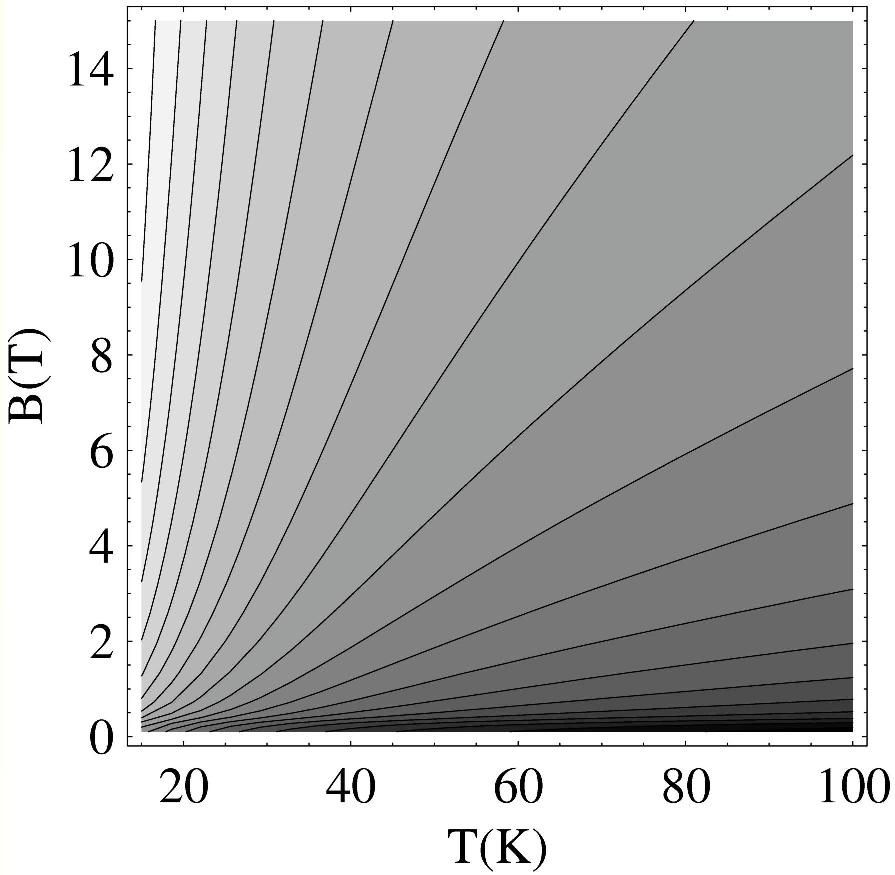


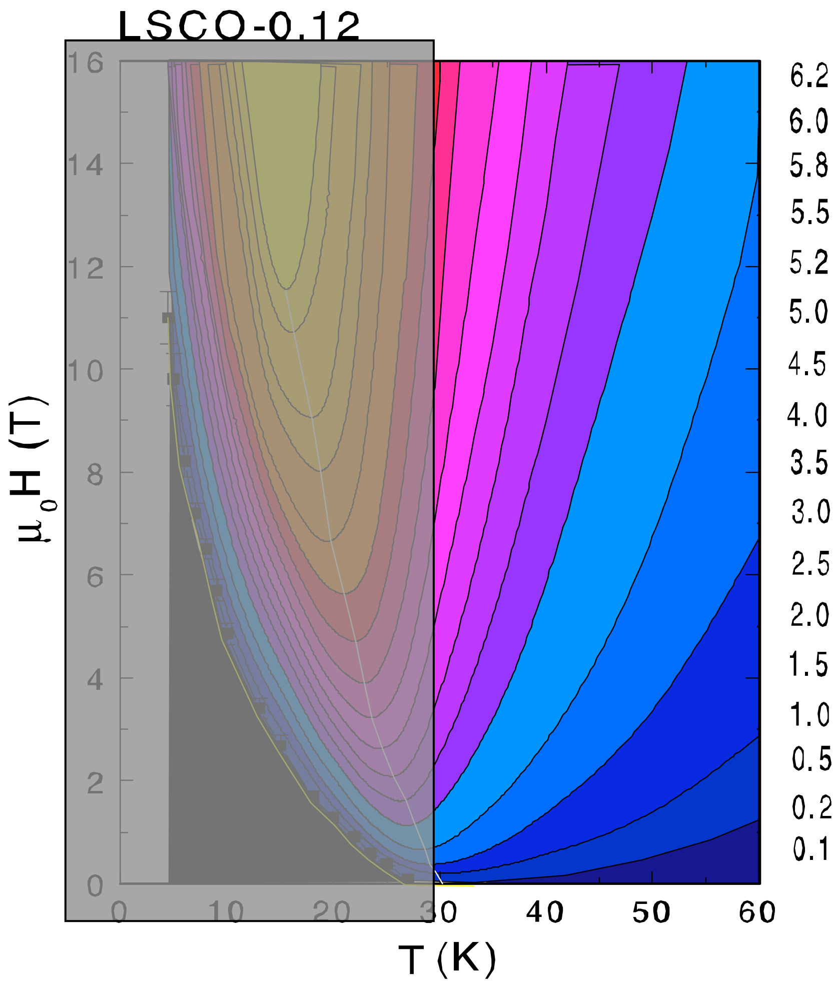
Underdoped high Tc superconductors:
Anomalously strong Nernst signal
up to $T=(2\text{-}3)T_c$

Nernst Experiments in high Tc's

Transverse thermoelectric response: B, T - dependence

Theory for $\alpha_{xy} \approx \sigma_{xx} N$



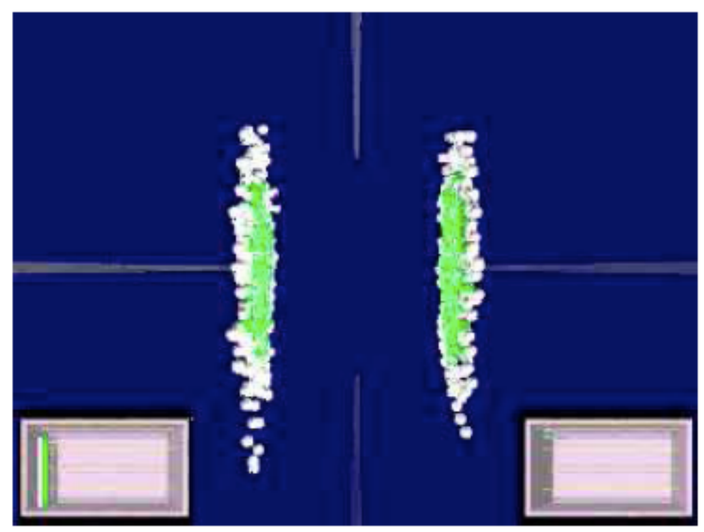


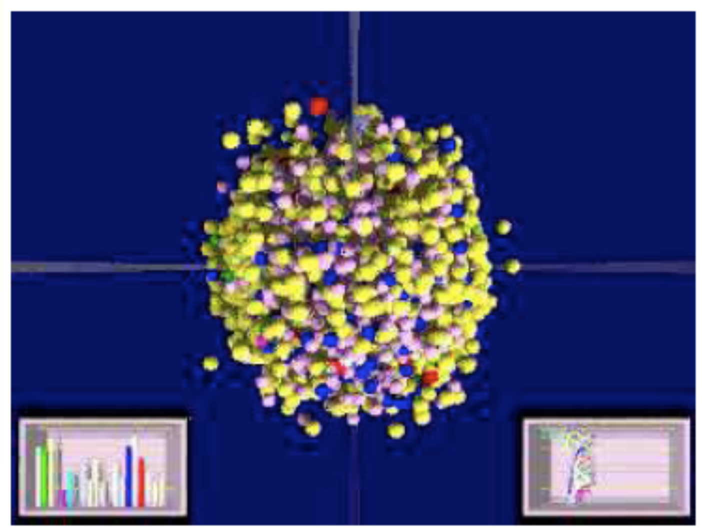
Y. Wang, L. Li, and N. P. Ong, Phys. Rev. B ***73****, 024510 (2006).*

Graphene versus
very strongly coupled, critical
relativistic liquids?

Are there further similarities?

Au+Au collisions at RHIC





Quark-gluon plasma is described by QCD (nearly conformal, critical theory)

–

Low viscosity fluid!

$$\boxed{\frac{\eta}{s} \sim 1}$$

Graphene – a nearly perfect liquid!

MM, J. Schmalian, and L. Fritz, (PRL 2009)

$\longleftrightarrow$

Anomalously low viscosity (like quark-gluon plasma)

"Heisenberg" $\boxed{\frac{\eta}{s} \sim E_{qp}\tau \geq 1}$ $\longrightarrow$ Measure of strong coupling:

(False) conjecture from AdS-CFT:

$$\boxed{\frac{\text{shear viscosity}}{\text{entropy density}} = \frac{\eta}{s} > \frac{\hbar}{k_B}\frac{1}{4\pi}}$$

What about graphene?

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Doped Graphene & Usual metals:
(Khalatnikov etc)

$$\boxed{\eta \propto n \cdot m\mathrm{v}^2 \cdot \tau \;\rightarrow\; n \cdot E_F \cdot \frac{\hbar E_F}{(k_B T)^2}} \quad \boxed{s \propto k_B n \frac{T}{E_F}} \quad \boxed{\frac{\eta}{s} \sim \frac{\hbar}{k_B}\left(\frac{E_F}{T}\right)^3}$$

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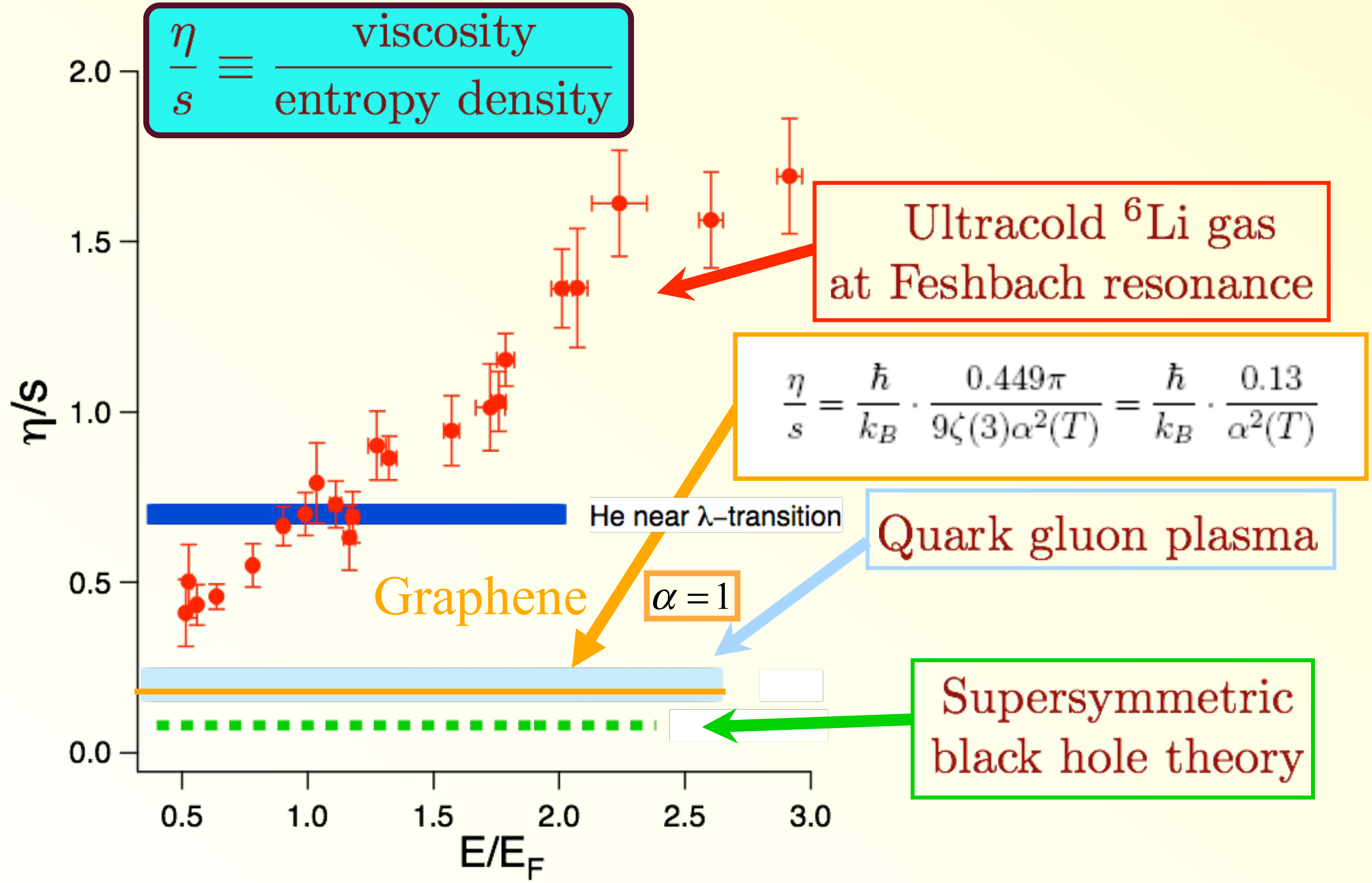
Undoped Graphene:

$$\boxed{\eta \propto n \cdot mv^2 \cdot \tau \;\rightarrow\; n_{\text{th}} \cdot k_B T \cdot \frac{\hbar}{\alpha^2 k_B T} = \frac{\hbar}{\alpha^2} n_{\text{th}}} \quad \boxed{s \propto k_B n_{\text{th}}}$$

Boltzmann-Born Approximation:

$$\boxed{\frac{\eta}{s} = \frac{\hbar}{k_B} \cdot \frac{0.449\pi}{9\zeta(3)\alpha^2(T)} = \frac{\hbar}{k_B} \cdot \frac{0.13}{\alpha^2(T)}}$$



*T. Schäfer, Phys. Rev. A **76**, 063618 (2007).*

*A. Turlapov, J. Kinast, B. Clancy, Le Luo, J. Joseph, J. E. Thomas, J. Low Temp. Phys. **150**, 567 (2008)*

Electronic consequences of low viscosity?

MM, J. Schmalian, L. Fritz, (PRL 2009)

Electronic turbulence in clean, strongly coupled graphene?
(or at quantum criticality!)
Reynolds number:

$$\boxed{\mathrm{Re} = \frac{s/k_B}{\eta/\hbar} \times \frac{k_B T}{\hbar v/L} \times \frac{u_{\mathrm{typ}}}{v}}$$

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Strongly driven mesoscopic systems: (Kim's group [Columbia])

$$L = 1\mu m$$
$$u_{typ} = 0.1\mathrm{v}$$
$$T = 100K$$

$\longrightarrow$ $\mathrm{Re} \sim 10 - 100$

Complex fluid dynamics!
(pre-turbulent flow)

New phenomenon in an
electronic system!

Summary

- AdS-CFT helped establish hydrodynamic structure (crossover ballistic to collision-dominated can be described, too, tuning ω/T)

- Interesting microscopic calculations of transport coefficients in strong coupling

- Guide for interesting strong coupling *phenomenology* in graphene:
  - Emergent relativistic hydrodynamics at low frequency
  - Nearly perfect quantum liquid with possible tendency to electronic turbulence!