Relativistic transport at strong coupling: graphene, quantum criticality and black holes

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in collaboration with
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Outline

• Philosophy and basic recipes

• Strong coupling features in collision-dominated transport in AdS-CFT

• Strong coupling features at quantum criticality, especially in graphene: Graphene as an almost perfect quantum liquid
The challenge of strong coupling in condensed matter theory

• Electrons have strong bare interactions (Coulomb)

• But: non-interacting quasiparticle picture (Landau-Fermi liquid) works very well for most metals
  Reason: RG irrelevance of interactions,
  ↔ screening and dressing of quasiparticles

• Opposite extreme: Interactions much stronger than the Fermi energy ➔ Mott insulators with localized e’s

• Biggest challenge: strong coupling physics close to quantum phase transitions.
  Maximal competition between wave and particle character (e.g.: high Tc superconductors, heavy fermions, cold atoms, graphene)
The challenge of strong coupling in condensed matter theory

Idea and Philosophy:

Study [certain] strongly coupled CFTs (= QFT’s for quantum critical systems) by the AdS-CFT correspondence

→ Learn about physical properties of strongly coupled theories (beyond $\varepsilon$- and $1/N$ expansions)

→ Exotic matter? (Bose fluids, strong coupling superconductivity, etc)

→ Extract the general/universal physics from the particular examples to make the lessons useful for condensed matter theory.
Holographic duality

D+1 dimensional AdS space, gravity theory

A D=d+1 dimensional system at its quantum critical point, (conformal) QFT

Maldacena, Gubser, Klebanov, Polyakov, Witten
Gravity side: Anti de Sitter space

Gravity action (Hilbert-Einstein) – if curvature small compared to string scale

\[ S_{\text{grav}}[g] = \frac{1}{16\pi G_N} \int d^{D+1}x \sqrt{g} \left( R - 2\Lambda + \ldots \right) \]

\( R \) Ricci scalar, \( \Lambda \) cosmological constant

\[ 2\Lambda = -\frac{D(D-1)}{L^2} \]
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Saddlepoints of \( \exp[-S] \leftrightarrow \) solutions of Einsteins equations:

\[ R_{ab} = -\frac{d}{L^2} g_{ab} \]

Symmetric solution: Anti-de Sitter space (space of constant negative curvature \( 1/L^2 \))

\[ g_{ab} : ds^2 = L^2 \left( -dt^2 + \sum_{i=1}^{d} dx_i^2 \right) = \frac{dz^2}{z^2} + \frac{dz^2}{z^2} \]

\( z = 0 : \) boundary; \( z = \infty : \) horizon
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\[ g_{ab} : ds^2 = L^2 \left( -dt^2 + \sum_{i=1}^{d} dx_i^2 \frac{dz^2}{z^2} \right) \]

\[ ds^2 = \frac{u^2}{L^2} \left( -dt^2 + \sum_{i=1}^{d} dx_i^2 \right) + L^2 \frac{du^2}{u^2} \]

\( z = 0 \) : boundary; \( z = \infty \) : horizon

\( u \equiv \frac{L^2}{z} \); \( u = \infty \) : UV (boundary); \( u = 0 \) : infrared
Anti de Sitter space $\text{AdS}_{D+1}$

Extra dimension: the RG scale of the boundary theory

Extra dimension $z$: length scale

$$g_{ab} : ds^2 = L^2 \left( -dt^2 + \sum_{i=1}^{d} dx_i^2 + \frac{dz^2}{z^2} \right)$$

$z = \infty$: horizon; $z = 0$: boundary

Extra dimension $u = L^2/z$: energy scale

$$ds^2 = \frac{u^2}{L^2} \left( -dt^2 + \sum_{i=1}^{d} dx_i^2 \right) + L^2 \frac{du^2}{u^2}$$

$u = 0$: IR (horizon); $u = \infty$: UV (boundary)
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Extra dimension: the RG scale of the boundary theory

Extra dimension $z$: length scale

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$ds^2 = \frac{u^2}{L^2} \left( -dt^2 + \sum_{i=1}^{d} dx_i^2 \right) + L^2 \frac{du^2}{u^2}$

$u = 0$: IR (horizon); $u = \infty$: UV (boundary)

Remarks:

• Metric on the boundary $(t,x_i,z = 0)$: Minkowski

• Symmetry of the metric: $\text{SO}(D,2)$ [$\text{AdS}$ can be embedded in $\text{R}_{D+2}$ as symmetric hyperboloid]

• Dilation symmetry (part of conformal symmetry): $t,x_i,z \rightarrow \lambda t, \lambda x_i, \lambda z$ $u \rightarrow u/\lambda$

• $\text{SO}(D,2)$ is also the conformal group in $D$ dimensions! Strong hint that $\text{AdS}_{D+1}$ is the space to be related with conformal QFT’s in $D$ dimension
## Correspondence

<table>
<thead>
<tr>
<th>Quantum gravity (bulk)</th>
<th>QFT (boundary)</th>
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<td>Semi-Classical limit (saddle point)</td>
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Central duality conjecture (taken for granted):

$$Z_{bulk} \left[ \phi(z, x) \to z^{d-\Delta} \delta \phi_0(x) \right] = \exp \left( i \int d^D x \delta \phi_0(x) O(x) \right)_{QFT}$$

Large $N$ limit: easy (classical saddle point, ODE) $\leftrightarrow$ hard (non-trivial strong coupling)
The classical limit ("large N")

Classical limit (saddle point approximation, "large N limit")

Gravity: AdS radius (radius of curvature) much larger than the Planck scale

Indeed:

\[
S_{\text{grav}}[g] = \frac{1}{16\pi G_N} \int d^{D+1}x \sqrt{g} \left( R - 2\Lambda + ... \right) \sim \frac{L^{D-1}}{\ell_{Pl}^{D-1}} \gg 1
\]

Holographic principle: (Area of boundary)/\(G_N\) \(\sim\) \(N^2\)

\[
\frac{L^{D-1}}{G_N} \approx \frac{L^{D-1}}{\ell_{Pl}^{D-1}} \approx N^2 \gg 1
\]

\(N^2 \gg 1\)

\(\leftrightarrow\) (QFT): Number of degrees of freedom per site \(\gg 1\)

\(\leftrightarrow\) central charge \(c\) of the CFT \(c \gg 1\)
AdS\textsubscript{D+1}-CFT\textsubscript{D} dictionary

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<td>Global current J</td>
<td>Maxwell field A</td>
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<td>Scalar/fermionic operator</td>
<td>Scalar/fermionic field</td>
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Best established examples

• $\mathcal{N} = 4$ super Yang-Mills (SU(N)) in D=3+1:
  Content: gauge field, scalars and fermions in the adjoint representation
  (conformal, $\beta(g)=0$)

• $\mathcal{N} = 8$ super Yang-Mills in D=2+1
  (asymptotically conformal strong coupling IR fixed point)
Finite T

• $T$ breaks scale invariance by introducing an IR scale in the CFT

$\leftrightarrow$

IR modification of the AdS metric: horizon at $z \sim 1/T$! $\rightarrow$ Black hole solution:

$$
\begin{align*}
    ds^2 &= \frac{L^2}{z^2} \left( -dt^2 + \sum_{i=1}^{d} dx_i^2 + dz^2 \right) \\
    \rightarrow \quad ds^2 &= \frac{L^2}{z^2} \left( -f(z) dt^2 + \sum_{i=1}^{d} dx_i^2 + \frac{dz^2}{f(z)} \right) \\
    f(z) &= 1 - \left( \frac{z}{z_H} \right)^D \\
    \quad &\text{with} \quad z_H \propto \frac{1}{T}
\end{align*}
$$
Finite T

• T breaks scale invariance by introducing an IR scale in the CFT

IR modification of the AdS metric: horizon at $z \sim 1/T! \rightarrow$ Black hole solution:

$$ds^2 = \frac{L^2}{z^2} \left(-dt^2 + \sum_{i=1}^{d} dx_i^2 + dz^2\right) \rightarrow ds^2 = \frac{L^2}{z^2} \left(-f(z)dt^2 + \sum_{i=1}^{d} dx_i^2 + \frac{dz^2}{f(z)}\right)$$

$$f(z) = 1 - \left(\frac{z}{z_H}\right)^D \quad \text{with} \quad z_H \propto \frac{1}{T}$$

• Asymptotic AdS metric is conserved
• Event horizon when $f(z_H)=0$;
• Boundary condition at $z_H$: only infalling waves!
• Precise connection with temperature:

In Euclidean time: space time is non-singular only if $\tau$=it is periodic with period

$$\Delta \tau \equiv \frac{1}{T} = \frac{4\pi}{|f'(z_H)|} = \frac{4\pi z_H}{d} = \frac{1}{T_{\text{Hawking}}}$$
Use the correspondence:

\[
Z_{\text{bulk}} \left[ \phi(z, x) \rightarrow z^{d-\Delta} \delta \phi(0)(x) \right] = \left\langle \exp \left( i \int d^D x \delta \phi(0)(x) O(x) \right) \right\rangle_{QFT} 
\]

To compute correlation functions in the CFT:

\[
Z_{\text{bulk}} \left[ \phi(z, x) \rightarrow z^{d-\Delta} \delta \phi(0)(x) \right] \approx \exp \left[ -S_{\text{cl}} \left( \delta \phi(0)(x) \right) \right]
\]

\[
\left\langle O(x)O(y) \right\rangle_{QFT} = \left. \frac{\delta^2 S_{\text{cl}} \left( \delta \phi(0)(x) \delta \phi(0)(y) \right)}{\delta \phi(0)(x) \delta \phi(0)(y)} \right|_{\delta \phi(0) = 0}
\]

Transport coefficients (thermo-electric conductivities, viscosity) from Kubo formula (retarded Greens functions)
Application:
Thermoelectric transport in D=2+1
Supersymmetric SU(N)
dual to
Einstein + Maxwell (dual to the current)
SU(N) transport from AdS/CFT

Gravitational dual to SUSY SU(N)-CFT\(_{2+1}\): Einstein-Maxwell theory

\[
I = \frac{1}{g^2} \int d^4x \sqrt{-g} \left[ -\frac{1}{4} R + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{3}{2} \right].
\]

(embedded in M theory as \(AdS_4 \times S^7\): \(1/g^2 \sim N^{3/2}\))

It has a black hole solution (with electric and magnetic charge):

\[
ds^2 = \frac{\alpha^2}{z^2} \left[ -f(z) dt^2 + dx^2 + dy^2 \right] + \frac{1}{z^2} \frac{dz^2}{f(z)},
\]

\[
F = h\alpha^2 dx \wedge dy + q\alpha dz \wedge dt,
\]

\[
f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3.
\]

Electric charge \(q\) and magnetic charge \(h\)

\[\leftrightarrow \mu\] and \(B\) for the CFT
SU(N) transport from AdS/CFT

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\[ f(z) = 1 + (h^2 + q^2)z^4 - (1 + h^2 + q^2)z^3. \]

Background ↔ Equilibrium

Transport ↔ Perturbations in \( g_{ix,ty}, A_{x,y} \).

Response via Kubo formula from \( \delta^2 I / \delta (g, A)^2 \).
Compare graphene to Strongly coupled relativistic liquids


Obtain exact results via string theoretical AdS–CFT correspondence

- Thermoelectric response functions \( \sigma(\omega) \), resonances: relat. hydrodynamics
- Calculate the transport coefficients for a strongly coupled theory!

\[
\text{SUSY - SU(N): } \sigma(\mu = 0) = \sqrt{\frac{2}{9}} N^{3/2} \frac{e^2}{h}
\]
Compare graphene to Strongly coupled relativistic liquids


Obtain exact results via string theoretical AdS–CFT correspondence

• Thermoelectric response functions $\sigma(\omega)$, resonances: relat. hydrodynamics
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SUSY - SU(N):

$$\sigma(\mu = 0) = \sqrt{\frac{2}{9}} N^{3/2} \frac{e^2}{\hbar}$$

$$\eta_{\text{shear}}(\mu = 0) = \frac{1}{4\pi} \frac{\hbar}{k_B}$$

Interpretation: $N^{3/2}$ effective degrees of freedom, strongly coupled: $\tau T = O(1)$

Anomalously low viscosity (like quark-gluon plasma)

“Heisenberg”

$$\frac{\eta}{s} \sim E_{qp} \tau \geq 1$$

Measure of strong coupling!
Quantum critical systems in condensed matter

A few examples

• Graphene
• High Tc
• Superconductor-to-insulator transition (interaction driven)
Dirac fermions in graphene

(Semenoff '84, Haldane '88)

Honeycomb lattice of C atoms

Tight binding dispersion
Dirac fermions in graphene

Honeycomb lattice of C atoms

Tight binding dispersion

2 massless Dirac cones in the Brillouin zone:
(Sublattice degree of freedom ↔ pseudospin)

Close to the two Fermi points $\mathbf{K}$, $\mathbf{K}'$:

\[ H \approx v_F \left( \mathbf{p} - \mathbf{K} \right) \cdot \tilde{\mathbf{\sigma}}_{\text{sublattice}} \]

\[ \rightarrow E_p = v_F \left| \mathbf{p} - \mathbf{K} \right| \]
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\[
v_F \approx 1.1 \cdot 10^6 \text{ m/s} \approx \frac{c}{300}
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2 massless Dirac cones in the Brillouin zone:
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Fermi velocity (speed of light”)

Coulomb interactions: Fine structure constant

$$H \approx v_F \left( \vec{p} - \vec{K} \right) \cdot \vec{\sigma}_{\text{sublattice}}$$

$$\rightarrow \quad E_p = v_F |\vec{p} - \vec{K}|$$

$$v_F \approx 1.1 \cdot 10^6 \text{ m/s} \approx \frac{c}{300}$$

$$\alpha \equiv \frac{e^2}{\varepsilon \hbar v_F} = O(1)$$
Relativistic fluid at the Dirac point


- Relativistic plasma physics of interacting particles and holes!
Relativistic fluid at the Dirac point


- Relativistic plasma physics of interacting particles and holes!

![Diagram of crossover in relativistic fluid at the Dirac point.](image)
Relativistic fluid at the Dirac point


- Relativistic plasma physics of interacting particles and holes!
- Strongly coupled, nearly quantum critical fluid at $\mu = 0$
Other relativistic fluids:

• Systems close to quantum criticality (with \( z = 1 \))
  
  Example: Superconductor-insulator transition (Bose-Hubbard model)

\[ \tau^{-1}_{rel} \approx \frac{k_B T}{\hbar} \]

Maximal possible relaxation rate!

• Conformal field theories (QFTs for quantum criticality)

  E.g.: strongly coupled Yang-Mills theories
  
  → Exact treatment via AdS-CFT correspondence

  \textit{Damle, Sachdev (1996, 1997)}
  \textit{Bhaseen, Green, Sondhi (2007).}
  \textit{Hartnoll, Kovtun, MM, Sachdev (2007)}
Quantum criticality in cuprate high $T_c$'s
Quantum criticality in cuprate high $T_c$'s

Thermoelectric measurements.

Example: Anomalously large Nernst Effect! ("thermal analogue" of the Hall effect)

Conformal field theory

$T$

$\mu$

Superconductor

Insulator $x=1/8$
Simplest example exhibiting “quantum critical” features:

Graphene
Questions

• **Transport characteristics** in the strongly coupled relativistic plasma?

• Response functions and transport coefficients at strong coupling?

• Graphene as a nearly perfect and possibly turbulent quantum fluid (like the quark-gluon plasma)?
Graphene – Fermi liquid?

1. Tight binding kinetic energy
   $\rightarrow$ massless Dirac quasiparticles

   \[ H_0 = \sum_{\lambda = \pm} \sum_{a = 1}^{N} \int \frac{d^2 k}{(2\pi)^2} \lambda v_F k \gamma^\dagger_\lambda(k) \gamma_{\lambda a}(k) \]

2. Coulomb interactions:
   Unexpectedly strong!
   $\rightarrow$ nearly quantum critical!

   \[ V(q) = \frac{2\pi e^2}{\varepsilon |q|} \]

   \[ H_1 = \frac{1}{2} \int \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \Psi^\dagger_a(k_2 - q) \Psi_a(k_2) V(q) \Psi^\dagger_b(k_1 + q) \Psi_b(k_1) \]
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   RG:
   \( (\mu = 0) \)

   \[ \frac{d\alpha}{d\ell} = -\frac{\alpha^2}{4} + O(\alpha^3) \]

   \[ \alpha = \frac{e^2}{\varepsilon \hbar v_F} = O(1) \]

   \[ \alpha(T) = \frac{\alpha^0}{1 + (\alpha^0/4) \ln(\Lambda/T)} \approx \frac{4}{\ln(\Lambda/T)} \]

   Strong coupling!
   Quantum critical liquid
   Cb marginal!

   Hole Fermi liquid
   Interaction dominated (hydrodynamic)
   Disorder dominated
Graphene – Fermi liquid?

1. Tight binding kinetic energy
   → massless Dirac quasiparticles

   \[ H_0 = \sum_{\lambda=\pm} \sum_{a=1}^N \int \frac{d^2 k}{(2\pi)^2} \lambda v_F k \gamma_{\lambda a}(k) \gamma_{\lambda a}^+(k) \]

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\[ \alpha \equiv \frac{e^2}{\varepsilon \hbar v_F} = O(1) \]

\[ T < \mu : \text{ Screening kicks in, short ranged Cb irrelevant} \]
Strong coupling in undoped graphene

MM, L. Fritz, and S. Sachdev, PRB ‘08.

Inelastic scattering rate (Electron-electron interactions)

\[ \tau_{ee}^{-1} = C \frac{(k_B T)^2}{\hbar \mu} \]

\( \mu >> T \): standard 2d Fermi liquid

C: Independent of the Coulomb coupling strength!
Strong coupling in undoped graphene

Inelastic scattering rate
(Electron-electron interactions)

Inelastic scattering rate \( \sim T \),
like in quantum critical systems!
Fastest possible rate!

Relaxation rate \( \sim T \),
like in quantum critical systems!

\( \mu >> T \): standard 2d Fermi liquid

\( \mu < T \): strongly coupled relativistic liquid

\[ \tau_\text{ee}^{-1} = C \frac{(k_B T)^2}{\hbar \mu} \]

\[ \tau_\text{ee}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar} \]
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**Inelastic scattering rate**
(Electron-electron interactions)

- Relaxation rate \( \sim T \)
  - like in quantum critical systems!
- Fastest possible rate!

\[ \mu >> T: \text{standard 2d Fermi liquid} \]

\[ \tau_{ee}^{-1} = C \frac{(k_B T)^2}{\hbar \mu} \]

\[ \mu < T: \text{strongly coupled relativistic liquid} \]

\[ \tau_{ee}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar} \]

“Heisenberg uncertainty principle for well-defined quasiparticles”

\[ E_{qp} \left( \sim k_B T \right) \geq \Delta E_{int} = \hbar \tau_{ee}^{-1} \sim \alpha^2 k_B T \]
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\[ \tau_{ee}^{-1} = C \frac{(k_B T)^2}{\hbar \mu} \]

\[ \tau_{ee}^{-1} \sim \alpha^2 \frac{k_B T}{\hbar} \]

Relaxation rate \(\sim T\), like in quantum critical systems!

Fastest possible rate!

As long as \(\alpha(T) \sim 1\), energy uncertainty is saturated, scattering is maximal

→ Nearly universal strong coupling features in transport, similarly as at the 2d superfluid-insulator transition [Damle, Sachdev (1996, 1997)]

“Heisenberg uncertainty principle for well-defined quasiparticles”

\[ E_{qp} (\sim k_B T) \geq \Delta E_{int} = \hbar \tau_{ee}^{-1} \sim \alpha^2 k_B T \]
Consequences for transport

1. Collisionlimited conductivity $\sigma$ in clean undoped graphene;
   Collisionlimited spin diffusion $D_s$ at any doping

2. Graphene - a perfect quantum liquid: very small viscosity $\eta$!
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   Despite the breaking of relativistic invariance by
   • finite T,
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Collision-dominated transport $\rightarrow$ relativistic hydrodynamics:
   a) Response fully determined by covariance, thermodynamics, and $\sigma, \eta$
   b) Collective cyclotron resonance in small magnetic field (low frequency)

Hydrodynamic regime:
  (collision-dominated)
  $\tau_{ee}^{-1} \gg \tau_{imp}^{-1}, \omega_{c}^{\text{typ}}, \omega_{AC}$
Collisionlimited conductivities

Fritz et al. (2008), Kashuba (2008)

Finite charge or spin conductivity in a pure system (for $\mu = 0$ or $B = 0$)!
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- Key: Charge or spin current without momentum

$J_{s,c} \neq 0$, but $\bar{P} = 0$

Pair creation/annihilation leads to current decay
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- Finite collision-limited conductivity! $\sigma(\mu = 0) < \infty$ ; $\sigma(\mu \neq 0) = \infty$
- Finite collision-limited spin diffusivity! $D_s(\mu; B = 0) \propto v_F^2 \tau_{ee} < \infty$, 

(pair/spin up) (hole/spin down)
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  Maximal possible relaxation rate $\sim T$

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\[ \tau_{ee}^{-1} \approx \alpha^2 \frac{k_B T}{\hbar} \]
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\[ \sigma_{Drude} = \frac{e}{m} \rho \tau \rightarrow \sigma(\mu = 0) \sim \frac{e}{k_B T / v^2} \left( \frac{e (k_B T)^2}{(\hbar v)^2} \right) \frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2} \frac{e^2}{\hbar} \]

→ Nearly universal conductivity at strong coupling

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Marginal irrelevance of Coulomb:

$\alpha \approx \frac{4}{\log(\Lambda/T)} < 1$
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Nearly universal conductivity at strong coupling

\[ \sigma_{Drude} = \frac{e}{m} \rho \tau \rightarrow \sigma(\mu = 0) \sim \frac{e}{k_B T / \nu^2} \left( \frac{k_B T}{\hbar \nu} \right)^2 \frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2} \frac{e^2}{\hbar} \]

Marginal irrelevance of Coulomb:

\[ \alpha \approx \frac{4}{\log(\Lambda/T)} < 1 \]

\[ \sigma(\mu = 0) = \sqrt{\frac{2}{9}} N^{3/2} \frac{e^2}{\hbar} \]

Analog in SU(N):

Saturation as $\alpha \rightarrow 1$, (finally: phase transition to insulator)
Boltzmann approach


Boltzmann equation in Born approximation

$$\left( \partial_t + eE \cdot \frac{\partial}{\partial k} \right) f_\pm(k, t) = I_{\text{coll}}^{cb}(k, t \mid \{ f_\pm(k', t) \}) \propto \alpha^2(T)$$

Collision-limited conductivity in weak coupling!

$$\sigma(\mu = 0) = \frac{0.76}{\alpha^2(T)} \frac{e^2}{h}$$
Transport and thermoelectric response at low frequencies?

Hydrodynamic regime:
(collision-dominated)

\[ \tau_{ee}^{-1} \gg \omega, \tau_{imp}^{-1}, \omega_{th}^{\text{cycl}} \]

Three complementary approaches:
• AdS-CFT (strong coupling)
• Relativistic hydrodynamics (without fixing transport coefficients)
• Boltzmann theory (weak coupling)

They all agree at the level of the relativistic hydrodynamic structure, but have different microscopics.
Application: thermoelectric close to transport at quantum criticality
Nernst effect in High $T_c$’s
Nernst effect in High $T_c$'s

Underdoped high Tc superconductors:
Anomalously strong Nernst signal
up to $T=(2-3)T_c$
Nernst Experiments in high Tc’s

Transverse thermoelectric response: $B, T$ - dependence

Theory for $\alpha_{xy} \approx \sigma_{xx} N$

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Graphene versus very strongly coupled, critical relativistic liquids?

Are there further similarities?
Quark-gluon plasma is described by QCD (nearly conformal, critical theory)

Low viscosity fluid!

\[ \frac{\eta}{s} \sim 1 \]
Graphene – a nearly perfect liquid!

Anomalously low viscosity (like quark-gluon plasma)

\[ \frac{\eta}{s} \sim E_{qp} \tau \geq 1 \]

\( \text{“Heisenberg”} \)

Measure of strong coupling:

\[ \frac{\eta}{s} > \frac{\hbar}{k_B} \frac{1}{4\pi} \]

What about graphene?
Graphene – a nearly perfect liquid!

(AdS-CFT) conjecture from MM, J. Schmalian, and L. Fritz, (PRL 2009)

Anomalously low viscosity (like quark-gluon plasma)

\[ \frac{\eta}{s} \sim E_{qp} \tau \geq 1 \]

“Heisenberg”

Measure of strong coupling:

\[ \text{shear viscosity} = \frac{\eta}{s} > \frac{\hbar}{k_B} \frac{1}{4\pi} \]

\[ \text{entropy density} = s > \frac{\hbar}{k_B} \]

(False) conjecture from AdS-CFT:

Doped Graphene & Usual metals: (Khalatnikov etc)

\[ \eta \propto n \cdot mv^2 \cdot \tau \rightarrow n \cdot E_F \cdot \frac{\hbar E_F}{(k_B T)^2} \]

\[ s \propto k_B n \frac{T}{E_F} \]

\[ \frac{\eta}{s} \sim \frac{\hbar}{k_B} \left( \frac{E_F}{T} \right)^3 \]
Graphene – a nearly perfect liquid!


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shear viscosity \quad entropy density

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Undoped Graphene:

\[ \eta \propto n \cdot mv^2 \cdot \tau \rightarrow n_{th} \cdot k_B T \cdot \frac{\hbar}{\alpha^2 k_B T} = \frac{\hbar}{\alpha^2} n_{th} \]

\[ s \propto k_B n_{th} \]

Boltzmann-Born Approximation:

\[ \frac{\eta}{s} = \frac{\hbar}{k_B} \cdot \frac{0.449 \pi}{9 \zeta(3) \alpha^2(T)} = \frac{\hbar}{k_B} \cdot \frac{0.13}{\alpha^2(T)} \]
Electronic consequences of low viscosity?

MM, J. Schmalian, L. Fritz, (PRL 2009)

Electronic turbulence in clean, strongly coupled graphene?
(or at quantum criticality!)
Reynolds number:

\[
Re = \frac{s/k_B}{\eta/\hbar} \times \frac{k_BT}{\hbar v/L} \times \frac{u_{\text{typ}}}{v}
\]
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Reynolds number:

$$Re = \frac{s/k_B}{\eta/\hbar} \times \frac{k_B T}{\hbar v/L} \times \frac{u_{typ}}{v}$$

Strongly driven mesoscopic systems: (Kim’s group [Columbia])

$\begin{align*}
L &= 1\mu m \\
u_{typ} &= 0.1v \\
T &= 100K
\end{align*}$

Complex fluid dynamics! (pre-turbulent flow)

New phenomenon in an electronic system!
Summary

• AdS-CFT helped establish hydrodynamic structure (crossover ballistic to collision-dominated can be described, too, tuning $\omega/T$)

• Interesting microscopic calculations of transport coefficients in strong coupling

• Guide for interesting strong coupling phenomenology in graphene:
  - Emergent relativistic hydrodynamics at low frequency
  - Nearly perfect quantum liquid with possible tendency to electronic turbulence!