Interaction effects in graphene

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collaborations with
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The Abdus Salam
ICTP Trieste

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Outline

- The many special facets of graphene
- Coulomb interactions are strong!
- Relativistic hydrodynamics and collision-dominated transport
- Boltzmann theory
- Strongly coupled relativistic fluids and AdS-CFT
- Graphene: an almost perfect quantum liquid: turbulence in electrons?

Ask questions all along the way, please!

Why should we care so much about graphene?

Is there more to do than repeating all calculations
for metals and semiconductors,
just with a different dispersion relation?

YES !

Graphene: a plethora of new phenomena

Graphene as a 2d crystal or membrane

- Real 2d monolayer!
- 2d ordering of adatoms, melting in 2d
- Buckling of membranes

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- Very clean 2d material, especially when suspended
- Semimetal (between semiconductor and metal)

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- Massless, chiral Dirac particles (Weyl equation)
- Klein tunneling, Zitterbewegung, Lensing (negative refractive index)
- non-trivial Berry phase when circling a cone → shift of Landau levels
- QHE at room temperature!

Useful thumb rule for estimates:

$$\boxed{\text{Fermi liquid } mv^2 \quad \rightarrow \quad \text{Graphene } k_B T, E}$$

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Landau levels:

$$\boxed{\omega_c^{(n)} = n\frac{eB}{mc} \quad \rightarrow \quad \omega_c^{(n)} = n\frac{v^2}{E}\frac{eB}{c} = n\frac{v^2}{\hbar\omega_c^{(n)}}\frac{eB}{c} \rightarrow \omega_c^{(n)} \sim v\sqrt{n\frac{eB}{\hbar c}}}$$

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Localization and disorder:

• Non-localization due to scattering within one Dirac cone, *despite* disorder ↔ Generalization of this phenomenon: surface states in topological insulators do not localize!

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Graphene = unrolled carbon nanotube

Are interactions weak all of a sudden?

1d versus 2d?

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This lecture:

Graphene with Coulomb interactions

- Interactions are surprisingly strong in neutral suspended graphene
- Nearly quantum critical behaviour, despite the simplicity of the material!
- Strongly coupled, highly relativistic Coulomb plasma:
→ similarities with the hot quark-gluon plasma of QCD!

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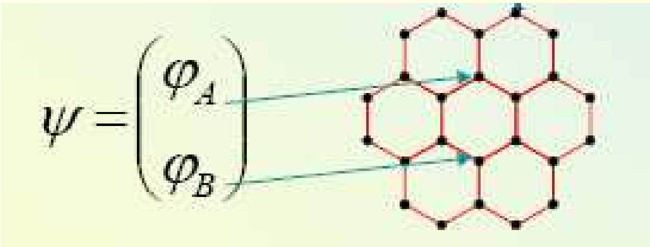
- **Experimental evidence**:

 - Fractional QHE predicted and observed (E. Andrei)
 - Coulomb broadening of cyclotron resonances

Dirac fermions in graphene

(Semenoff '84, Haldane '88)

Honeycomb lattice of C atoms

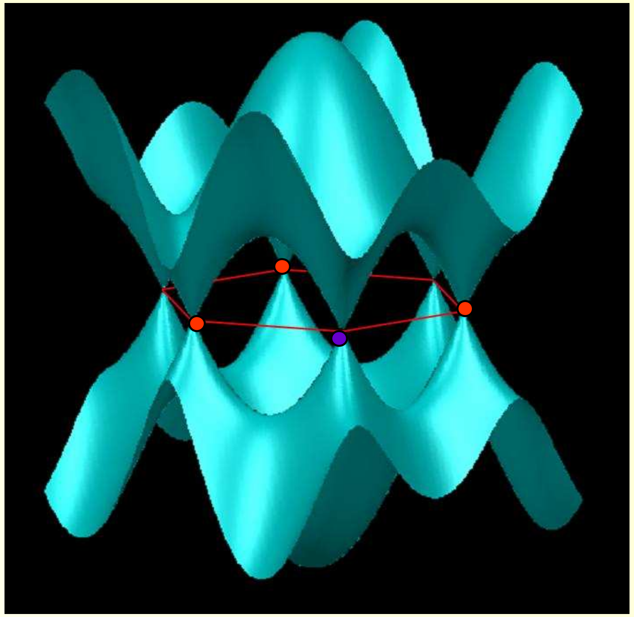


$$\psi = \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}$$

$$\hat{H} = v \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} = v\vec{\sigma}\cdot\vec{p}$$

$$\boxed{\mathbf{p} = \mathbf{k} - \mathbf{K} \quad \rightarrow \quad E_{\mathbf{p}} = \mathrm{v}_F |\mathbf{p}|}$$

Tight binding dispersion



2 massless Dirac cones in the Brillouin zone:
(Sublattice degree of freedom ↔ pseudospin)

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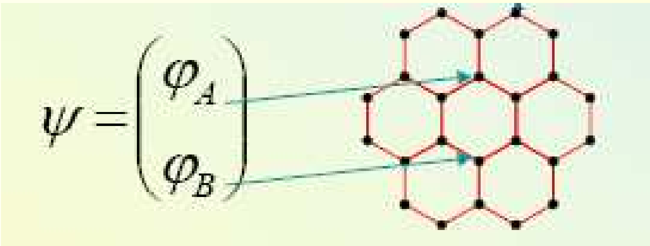

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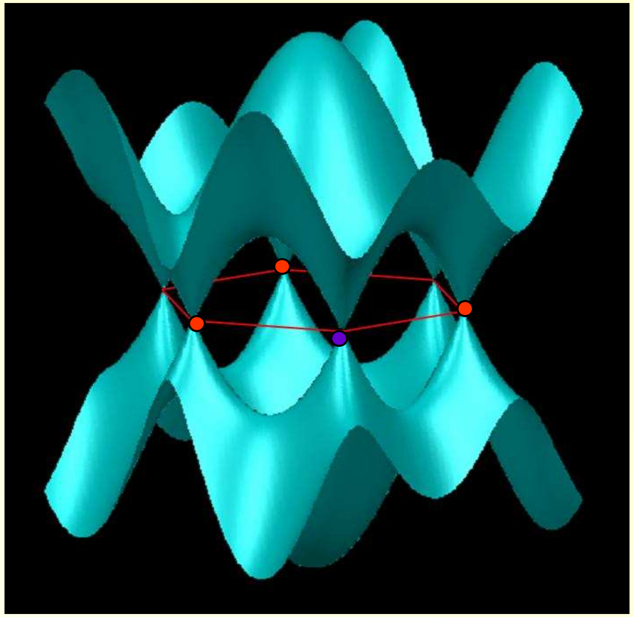

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Fermi velocity (speed of light")

$$\boxed{\mathrm{v}_F \approx 1.1 \cdot 10^6 \text{ m/s} \approx \frac{c}{300}}$$

Coulomb interactions: Fine structure constant

$$\boxed{\alpha \equiv \frac{e^2}{\varepsilon \hbar \mathrm{v}_F} = \frac{2.2}{\varepsilon}}$$

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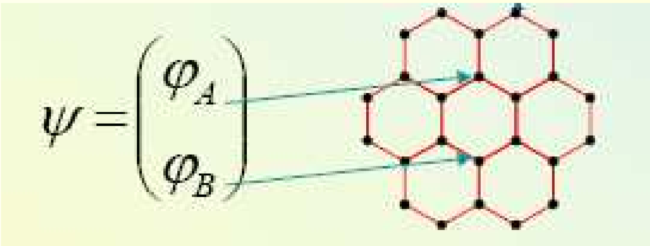

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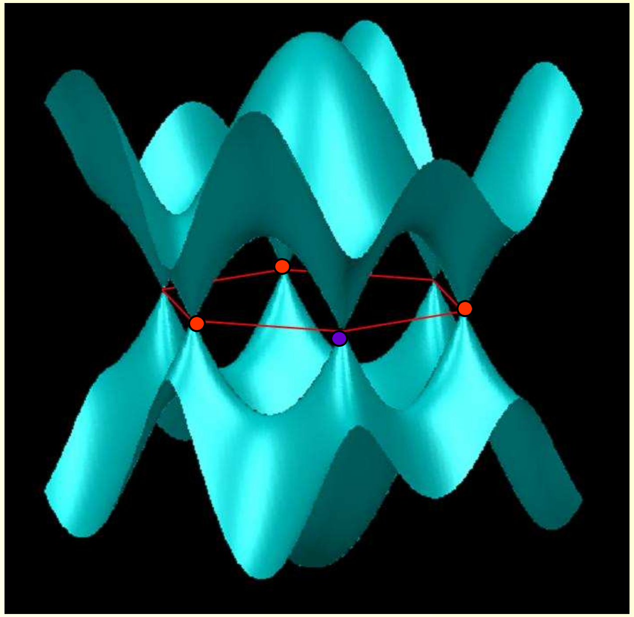

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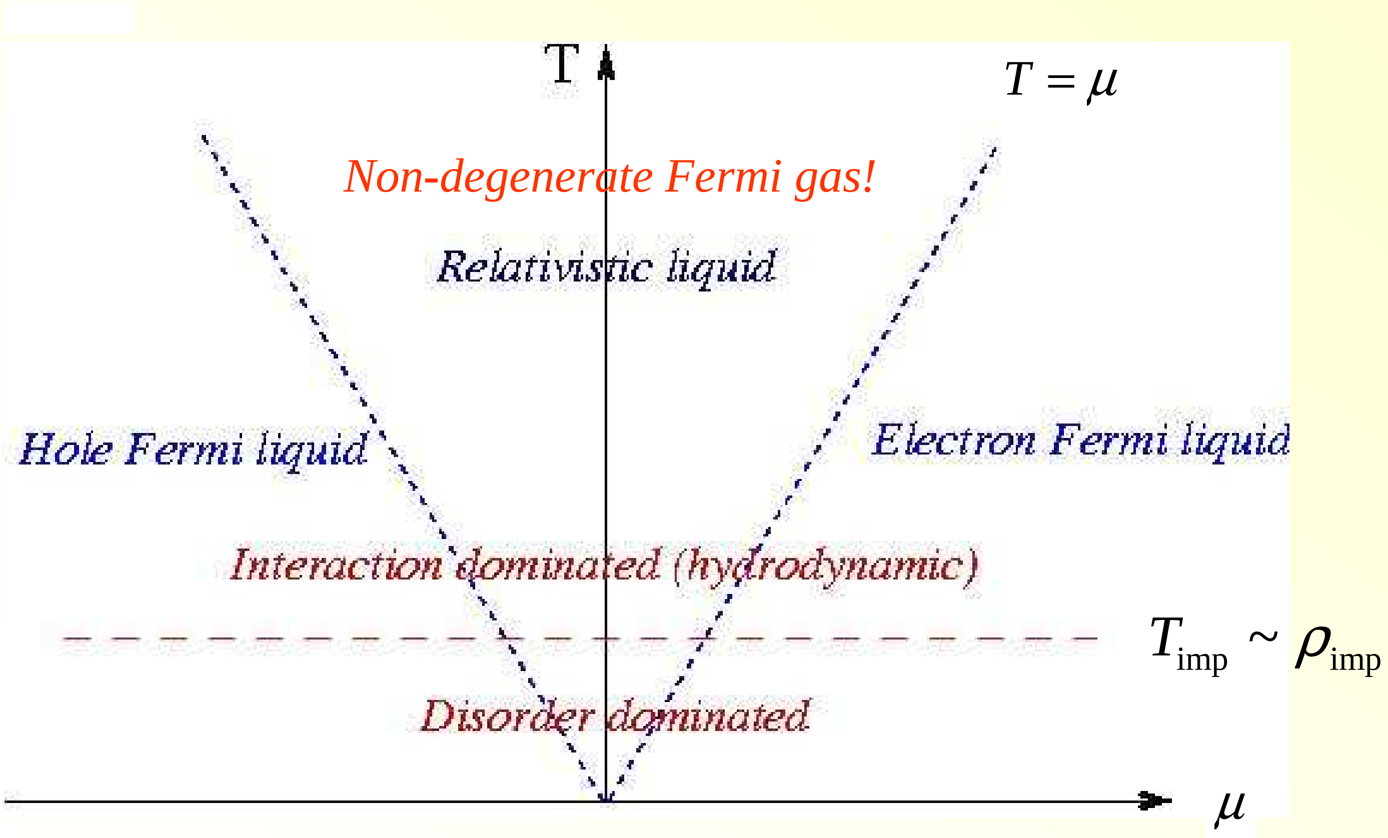
Relativistic fluid at the Dirac point

D. Sheehy, J. Schmalian, Phys. Rev. Lett. ***99****, 226803 (2007).*

- Relativistic plasma physics of interacting particles and holes!



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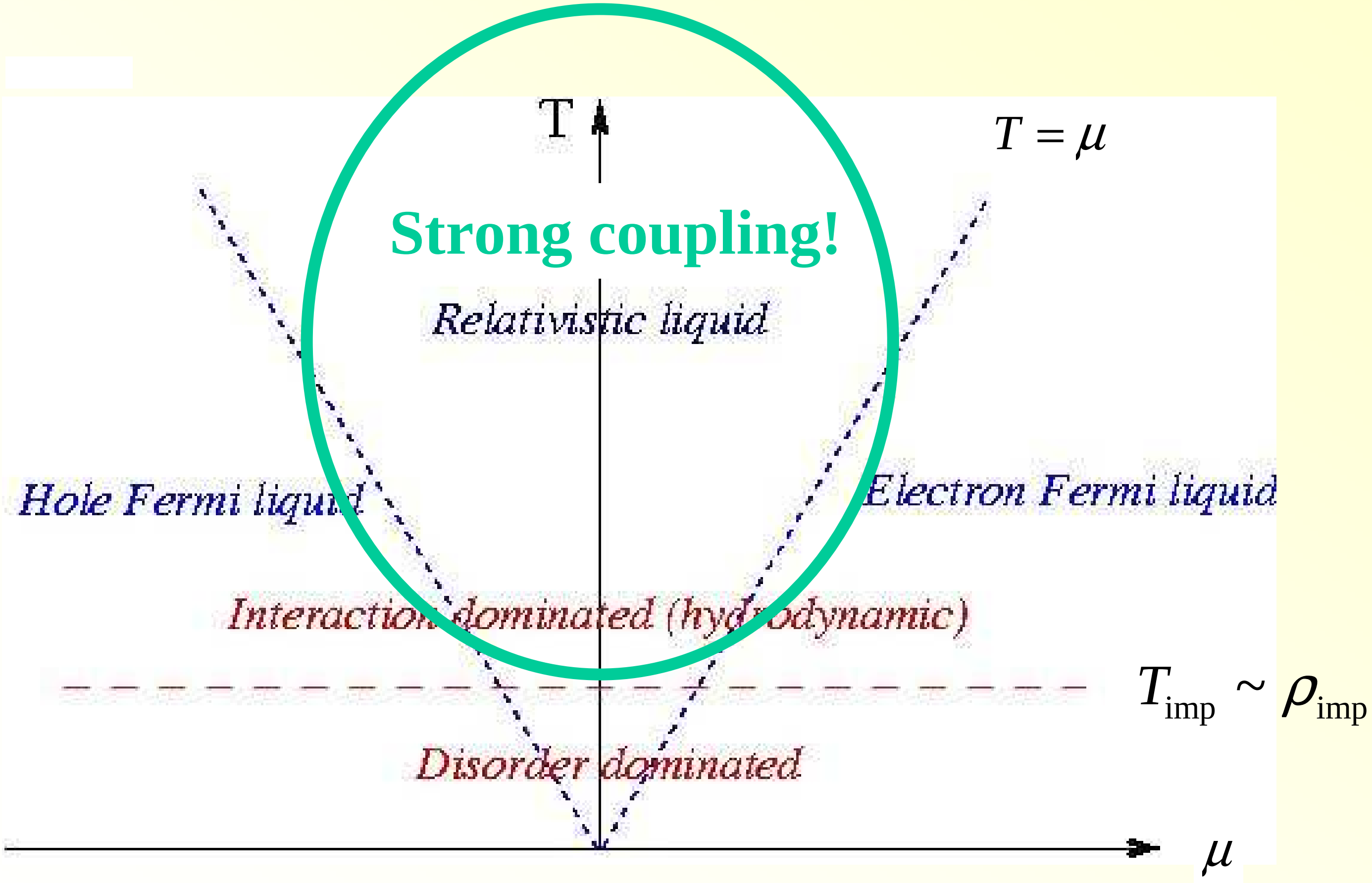

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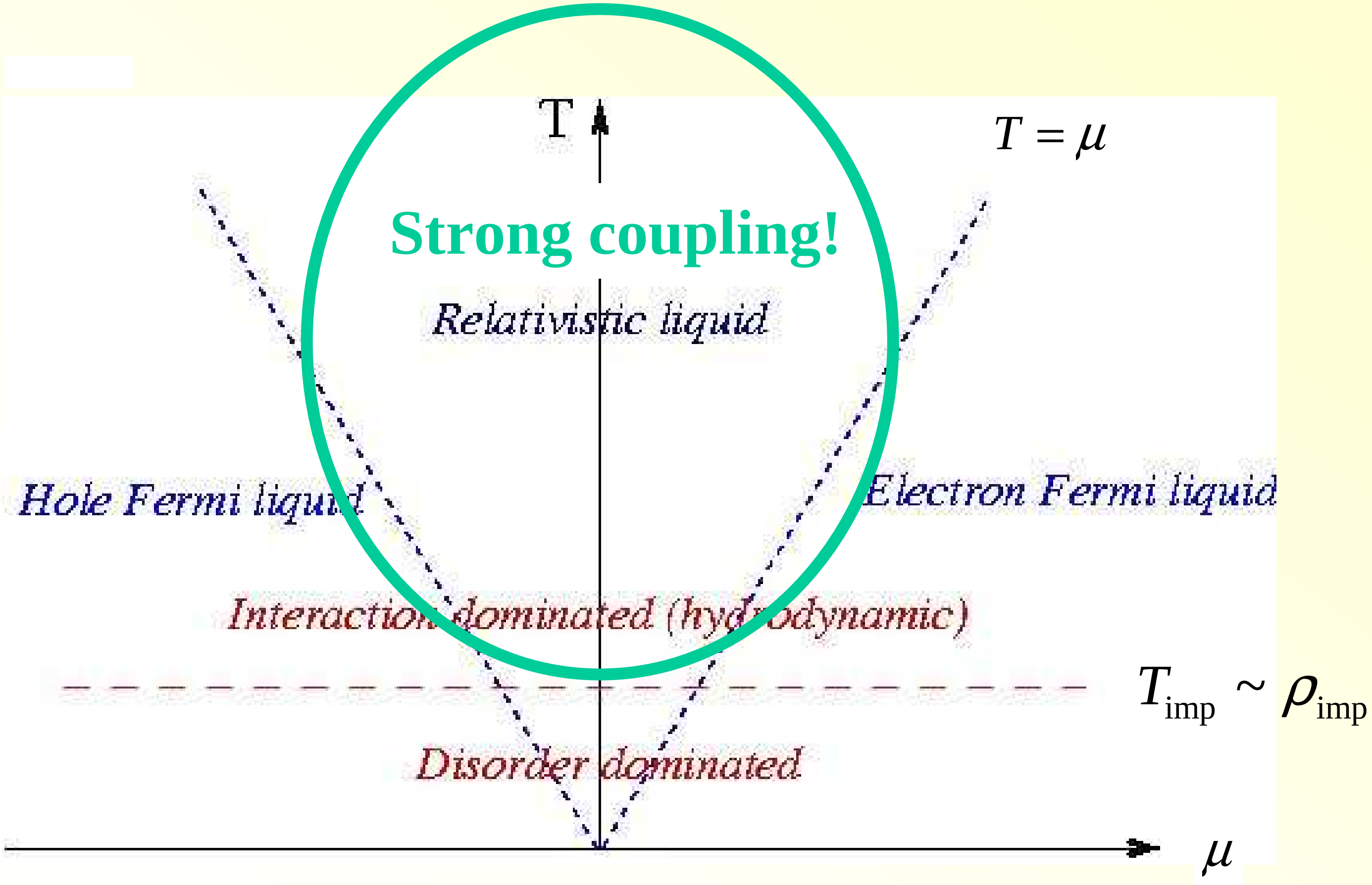

Very similar as for quantum criticality (e.g. SIT) and in their associated CFT's

Other relativistic fluids:

- Bismuth (3d Dirac fermions with very small mass)

- Systems close to quantum criticality (with z = 1)

Example: Superconductor-insulator transition (Bose-Hubbard model)

Maximal possible relaxation rate!

$$\boxed{\tau_{rel}^{-1} \approx \frac{\hbar}{k_B T}}$$

Damle, Sachdev (1996)
Bhaseen, Green, Sondhi (2007).
Hartnoll, Kovtun, MM, Sachdev (2007)

- Conformal field theories (critical points)

E.g.: strongly coupled Non-Abelian gauge theories (akin to QCD):
→ Exact treatment via AdS-CFT correspondence!

C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son (2007)
Hartnoll, Kovtun, MM, Sachdev (2007)

Are Coulomb interactions strong?

Fine structure constant (QED concept)

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r_s (Wigner crystal concept)

$$\boxed{r_s \equiv \frac{E_{Cb}(n)}{E_F(n)} = \frac{\sqrt{n}\, e^2/\varepsilon}{\hbar \text{v}_F \sqrt{\pi n}} = \frac{\alpha}{\sqrt{\pi}}}$$

Small!?

n-independent!

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Recall QED/QCD:

- The coupling strength α depends on the scale.
- Different theories have different scale behavior!

α is the high energy limit of the coupling.
But we care about α(T)!

Are Coulomb interactions strong?

Coulomb interactions:
Unexpectedly strong!
→ nearly quantum critical!

$$\boxed{V(\mathbf{q}) = \frac{2\pi e^2}{\varepsilon|\mathbf{q}|}}$$

$$H_1 = \frac{1}{2}\int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\Psi_a^\dagger(\mathbf{k}_2-\mathbf{q})\Psi_a(\mathbf{k}_2)V(\mathbf{q})\Psi_b^\dagger(\mathbf{k}_1+\mathbf{q})\Psi_b(\mathbf{k}_1)$$

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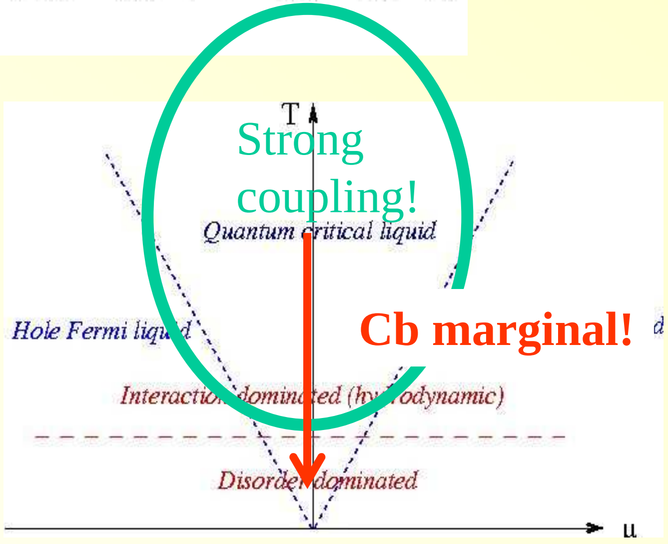
RG:
(μ = 0)



Gonzalez et al.,
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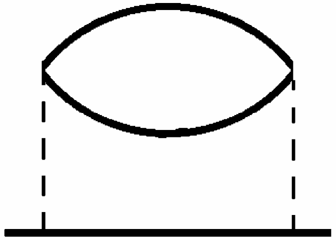
(μ = 0)

$$\frac{d\alpha}{d\ell} = -\frac{\alpha^2}{4} + \mathcal{O}(\alpha^3)$$

$$\boxed{\alpha \equiv \frac{e^2}{\varepsilon\hbar v_F} = O(1)}$$

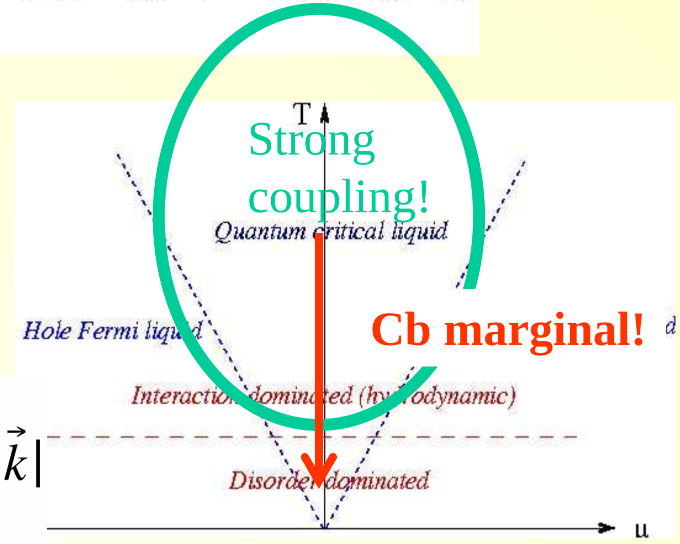
$$\alpha(T) = \frac{\alpha^0}{1+(\alpha^0/4)\ln(\Lambda/T)} \overset{T\to 0}{\sim} \frac{4}{\ln(\Lambda/T)}$$



$$\text{Im }\Sigma(\omega,\vec{k}) = \frac{1}{48}\left(\frac{e^2}{\epsilon_0\hbar v_{\rm F}}\right)^2\hbar v_{\rm F}|\vec{k}|$$

RG flow of v_F ↔ RG flow of α

Coulomb only marginally irrelevant for μ = 0!



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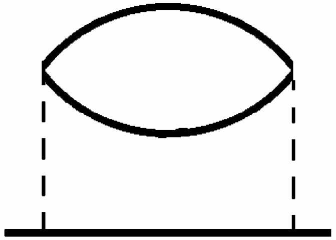

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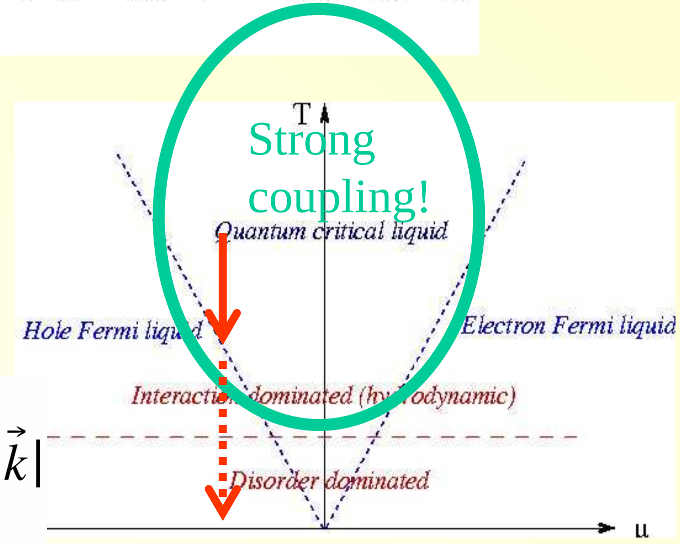

Coulomb **only marginally** irrelevant for $\mu = 0$!

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But: ($\mu > 0$)

For $T < \mu$: screening kicks in, short ranged Cb irrelevant

Are Coulomb interactions strong?

Several studies:

- 2-loop RG
- large N expansion (around N = 4 = 2*2 flavors)
- Numerical study on related square lattice problem

suggest proximity of a quantum critical point around $\alpha = O(1)$ between a Fermi liquid and a gapped insulator.

Experiments (Fractional QHE!) in suspended graphene also suggest strong Coulomb interactions.

Consequences for transport

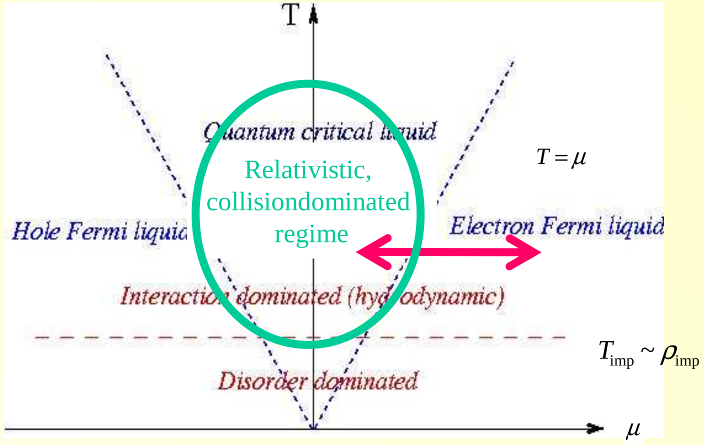
1. Collision-limited conductivity σ in clean undoped graphene

2. Emergent relativistic invariance at low frequencies!

3. Graphene is a perfect quantum liquid: very small viscosity η!

Questions

- Transport characteristics of the relativistic plasma in graphene and at quantum criticality?
- Connection between relativistic regime and standard Fermi liquid at large doping?
- Graphene as a nearly perfect fluid (like the quark-gluon plasma)?



Hydrodynamic approach to transport

Time scales

MM, L. Fritz, and S. Sachdev, PRB '08.

1. Inelastic scattering rate
(Electron-electron interactions)

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Relaxation rate ~ T,
like in quantum critical systems!
Fastest possible rate!

$\mu < T$: strongly coupled
relativistic liquid

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"Heisenberg uncertainty principle for well-defined quasiparticles"

$$E_{qp}(\sim k_B T) \geq \Delta E_{\text{int}} = \hbar \tau_{\text{ee}}^{-1} \sim \alpha^2 k_B T$$

As long as α(T) ~ 1, energy uncertainty is saturated, scattering is maximal
→ Nearly universal strong coupling features in transport,
Similarly as at the 2d superfluid-insulator transition

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2. Elastic scattering rate
(Scattering from charged impurities)
Subdominant at high T, low disorder

$$\boxed{\tau_{\text{imp}}^{-1} \sim \frac{\left(Ze^2/\varepsilon\right)^2 \rho_{\text{imp}}}{\hbar}\frac{1}{\max\left[T, \mu\right]}}$$

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3. Deflection rate due to magnetic field
(Cyclotron frequency of non-interacting
particles with thermal energy)

$$\boxed{\tau_{\mathrm{B}}^{-1} \sim \omega_c^{\mathrm{typ}} \sim \frac{eB\mathrm{v}_F^2}{\max[T, \mu]}}$$

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Hydrodynamic regime:
(collision-dominated)

$$\boxed{\tau_{\mathrm{ee}}^{-1} >> \tau_{\mathrm{imp}}^{-1}, \tau_{\mathrm{B}}^{-1}, \omega}$$

Hydrodynamics

Hydrodynamic collision-dominated regime

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Long times,
Large scales

$$\boxed{t >> \tau_{\mathrm{ee}}}$$

Hydrodynamics

Hydrodynamic collision-dominated regime $\boxed{\tau_{\mathrm{ee}}^{-1} >> \tau_{\mathrm{imp}}^{-1}, \tau_{\mathrm{B}}^{-1}, \omega}$

Long times,
Large scales $\boxed{t >> \tau_{\mathrm{ee}}}$

- Local equilibrium established: $\boxed{T_{\mathrm{loc}}(r) , \mu_{\mathrm{loc}}(r) ; \ \vec{u}_{\mathrm{loc}}(r)}$
- Study relaxation towards global equilibrium
- Slow modes: Diffusion of the density of conserved quantities:
  - Charge
  - Momentum
  - Energy

Relativistic Hydrodynamics

S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B ***76****, 144502 (2007).*

Reminder of special relativity:

Metric:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Indices:

$$\mu = 0 \ \ \text{time}$$

$$\mu = 1, 2 \ \ \text{2d-space}$$

Covariance: Physical laws are independent of the inertial frame and thus under Lorentz transformation.

Here: Lorentz group with “speed of light” $c \rightarrow \mathrm{v}_F$!

Relativistic Hydrodynamics

S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B ***76****, 144502 (2007).*

Energy-momentum tensor $T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}$ $\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$

Current 3-vector $J^\mu = \rho u^\mu + \nu^\mu$ $\begin{pmatrix} \rho \\ \rho u_x + \nu_x \\ \rho u_y + \nu_y \end{pmatrix}$

u^μ : 3-velocity: $u^\mu = (1,0,0) \longrightarrow$ No energy current

ν^μ : Dissipative current

$\tau^{\mu\nu}$: Viscous stress tensor (Reynold's tensor)

Relativistic Hydrodynamics

S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B ***76****, 144502 (2007).*

Energy-momentum tensor $T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}$

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

Current 3-vector $J^\mu = \rho u^\mu + \nu^\mu$

$$\begin{pmatrix} \rho \\ \rho u_x + \nu_x \\ \rho u_y + \nu_y \end{pmatrix}$$

u^μ : 3-velocity: $u^\mu = (1,0,0) \longrightarrow$ No energy current

ν^μ : Dissipative current

$\tau^{\mu\nu}$: Viscous stress tensor (Reynold's tensor)

+ Thermodynamic relations

$$\varepsilon + P = Ts + \mu\rho, \quad d\varepsilon = Tds + \mu d\rho,$$

Gibbs-Duheme

1st law of thermodynamics

Relativistic Hydrodynamics

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$$T^{\mu\nu} = (\varepsilon + P) u^{\mu} u^{\nu} + P g^{\mu\nu} + \tau^{\mu\nu}$$

Conservation laws (equations of motion):

$$\partial_{\mu} J^{\mu} = 0$$

Charge conservation

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Energy/momentum conservation

$$\partial_{\nu} T^{\mu\nu} = F^{\mu\nu} J_{\nu}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B \\ -E_y & -B & 0 \end{pmatrix}$$

$$\vec{E} = -i\vec{k}\frac{2\pi}{|k|}\rho_{\vec{k}}$$

Coulomb interaction

Relativistic Hydrodynamics

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Energy/momentum conservation

$$\partial_{\nu} T^{\mu\nu} = F^{\mu\nu} J_{\nu} + \frac{1}{\tau_{\text{imp}}} T^{0\nu}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B \\ -E_y & -B & 0 \end{pmatrix}$$

$$\vec{E} = -i\vec{k}\,\frac{2\pi}{|k|}\,\rho_{\vec{k}}$$

Coulomb interaction

Weak disorder → momentum relaxation

Relativistic Hydrodynamics

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Coulomb interaction

Dissipative current and viscous tensor?

Relativistic Hydrodynamics

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B **76**, 144502 (2007).*

Dissipative current and viscous tensor?

Heat current $Q^{\mu} = (\varepsilon + P)u^{\mu} - \mu J^{\mu}$

→ Entropy current $S^{\mu} = Q^{\mu} / T$

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Positivity of entropy production (Second law):

$$\partial_\mu S^\mu \equiv A_\alpha(\partial T, \partial\mu, F^{\mu\nu})\nu^\alpha + B_{\alpha\beta}(\partial T, \partial\mu, F^{\mu\nu})\tau^{\alpha\beta} \geq 0$$

$$\Rightarrow \nu^\mu = \text{const.} \times A^\mu(\partial T, \partial\mu, \partial u; F^{\mu\nu})$$

$$\tau^{\mu\nu} = \text{const.} \times B^{\mu\nu} + \text{const.} \times \delta^{\mu\nu} B_\alpha^\alpha$$

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→

$$\nu^{\mu} = \sigma_Q (g^{\mu\nu} + u^{\mu}u^{\nu})\left[(-\partial_{\nu}\mu + F_{\nu\lambda}u^{\lambda}) + \mu \frac{\partial_{\mu} T}{T} \right]$$

$$\tau^{\mu\nu} = -(g^{\mu\lambda} + u^{\mu}u^{\lambda})[\eta(\partial_{\lambda}u^{\nu} + \partial^{\nu}u_{\lambda}) + (\zeta - \eta)\delta_{\lambda}^{\nu}\partial_{\alpha}u^{\alpha}]$$

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Irrelevant for response at k → 0

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Irrelevant for response at k → 0

One single transport coefficient (instead of two)!

Meaning of σ_Q ?

- At zero doping (particle-hole symmetry):

$$\sigma_Q = \sigma_{xx}\left(\rho_{\text{imp}} = 0\right) < \infty \ !$$

→ Interaction-limited conductivity of the pure system!

How is it possible that $\sigma_{xx}\left(\rho_{\text{imp}} = 0\right)$ is finite ??

Collision-limited conductivity

Damle, Sachdev, (1996).
Fritz et al. (2008), Kashuba (2008)

Finite conductivity in a pure system at particle-hole symmetry ($\rho = 0$)!

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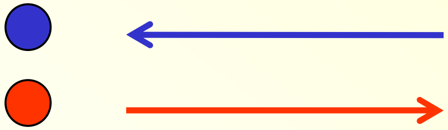
- Key: Charge current without momentum!



(particle)

(hole)

$$\vec{J} \neq 0, \quad \vec{P} = 0$$

Pair creation/annihilation
leads to current decay

- Finite collision-limited conductivity!

BUT:

- Infinite thermal conductivity!
  (whereas it is usually finite if J=0 is imposed!)

Collision-limited conductivity

Damle, Sachdev, (1996).
Fritz et al. (2008), Kashuba (2008)

Finite conductivity in a pure system at particle-hole symmetry (ρ = 0)!

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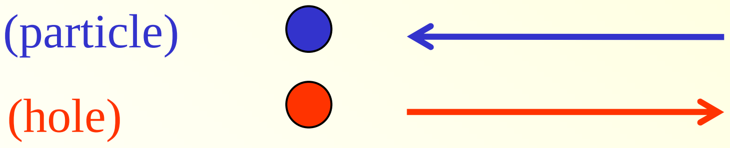

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- Marginal irrelevance of Coulomb: Maximal possible relaxation rate, set only by temperature

$$\boxed{\tau_{ee}^{-1} \approx \alpha^2 \frac{k_B T}{\hbar}}$$

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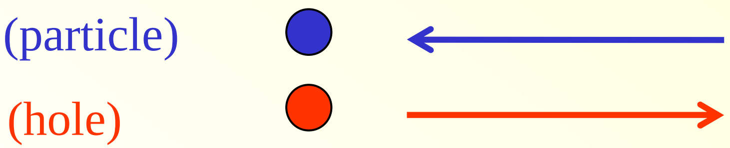
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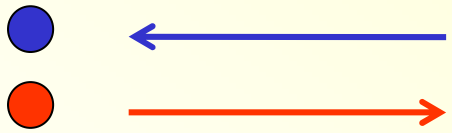

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→ Nearly universal conductivity

$$\boxed{\sigma_{Drude} = \frac{e}{m}\rho\tau \to \sigma(\mu=0) \sim \frac{e}{k_B T / \mathrm{v}^2}\left(e\frac{(k_B T)^2}{(\hbar \mathrm{v})^2}\right)\frac{\hbar}{\alpha^2 k_B T} \sim \frac{1}{\alpha^2}\frac{e^2}{h}}$$

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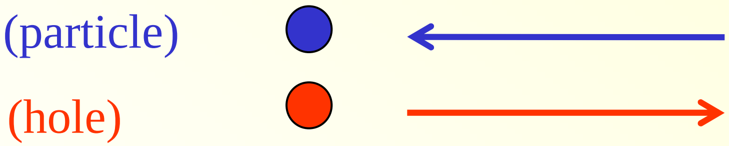

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Marginal irrelevance of Coulomb: $\boxed{\alpha \approx \frac{4}{\log(\Lambda / T)}}$

Back to Hydrodynamics

S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B ***76****, 144502 (2007).*

Elements discussed so far:

Conservation laws (equations of motion):

$$\partial_\mu J^\mu = 0$$

Charge conservation

$$\partial_\nu T^{\mu\nu} = F^{\mu\nu} J_\nu + \frac{1}{\tau_{\text{imp}}} T^{0\nu}$$

Energy/momentum conservation

Dissipative current (relating electrical and energy current)

$$\nu^\mu = \boxed{\sigma_Q}(g^{\mu\nu} + u^\mu u^\nu)\left[(-\partial_\nu \mu + F_{\nu\lambda} u^\lambda) + \mu \frac{\partial_\mu T}{T}\right]$$

Thermoelectric response

*S. Hartnoll, P. Kovton, MM, and S. Sachdev, Phys. Rev. B **76**, 144502 (2007).*

Charge and heat current:

$$J^{\mu} = \rho u^{\mu} - \nu^{\mu}$$

$$Q^{\mu} = (\varepsilon + P) u^{\mu} - \mu J^{\mu}$$

Thermo-electric response

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \hat{\sigma} & \hat{\alpha} \\ T\hat{\alpha} & \hat{\bar{\kappa}} \end{pmatrix} \begin{pmatrix} \vec{E} \\ -\vec{\nabla} T \end{pmatrix}$$

$$\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

etc.

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$$\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

etc.

i) Solve linearized conservation laws
ii) Read off the response functions from the dynamic
response to initial conditions! *(see Kadanoff & Martin, 1960)*

Results from Hydrodynamics

Response functions at B=0

Symmetry z → -z : $\sigma_{xy} = \alpha_{xy} = \kappa_{xy} = 0$

Longitudinal conductivity:

$$\sigma_{xx}(\omega, k; B=0) = \left(\sigma_Q + \frac{\rho^2}{P+\varepsilon}\frac{\tau}{1-i\omega\tau}\right)$$

Collision-limited conductivity at the quantum critical point ρ = 0

Drude-like conductivity, divergent for $\tau \to \infty, \omega \to 0, \rho \neq 0$

Momentum conservation (ρ≠0)!

Response functions at B=0

Symmetry z ⟶ -z : $\sigma_{xy} = \alpha_{xy} = \kappa_{xy} = 0$

Longitudinal conductivity:

$$\sigma_{xx}(\omega, k; B=0) = \left(\sigma_Q + \frac{\rho^2}{P+\varepsilon} \frac{\tau}{1 - i\omega\tau} \right)$$

Thermal conductivity:

$$\kappa_{xx}(\omega, k; B=0) = \sigma_Q \frac{\mu^2}{T} + \frac{s^2 T}{P+\varepsilon} \frac{\tau}{1 - i\omega\tau} + \mathcal{O}(k^2).$$

Relativistic Wiedemann-Franz-like relations between σ and κ in the quantum critical window!

Response functions at B=0

Symmetry z ⟶ -z : $\sigma_{xy} = \alpha_{xy} = \kappa_{xy} = 0$

Longitudinal conductivity:

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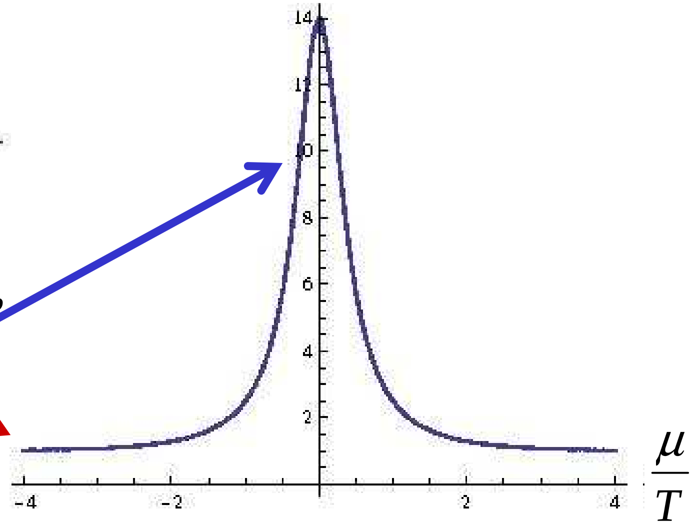
Thermopower:

$$\alpha_{xx}(\mu, \omega=0) = -\frac{\pi^2}{3e} k_B^2 T \frac{d\sigma(\mu, \omega=0)}{d\mu}$$

Only valid in the degenerate e-gas regime, but violated in the relativistic window.

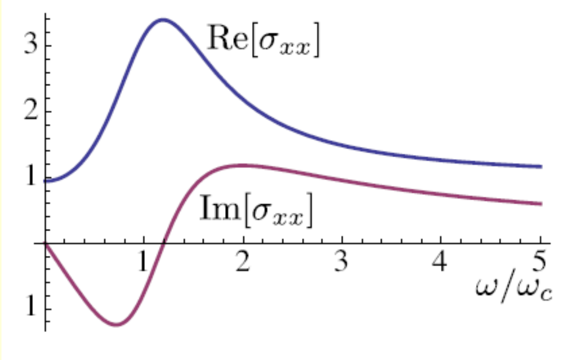
$$-\frac{3e}{\pi^2}\frac{1}{k_B^2 T}\frac{\alpha_{xx}}{d\sigma_{xx}/d\mu}$$



B > 0 : Cyclotron resonance

E.g.: Longitudinal conductivity

$$\sigma_{xx}(\omega) = \sigma_Q \frac{\omega\left(\omega + i\gamma + i\omega_c^2/\gamma\right)}{(\omega + i\gamma)^2 - \omega_c^2}$$

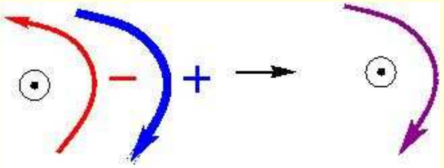


Pole in the response

$$\boxed{\omega = \pm\omega_c^{\text{QC}} - i\gamma}$$

Collective cyclotron frequency of the relativistic plasma

$$\boxed{\omega_c^{\text{QC}} = \frac{\rho\, B/c}{(\varepsilon + P)/\text{v}_\text{F}^2} \leftrightarrow \omega_c^{\text{FL}} = \frac{e\, B/c}{m}}$$



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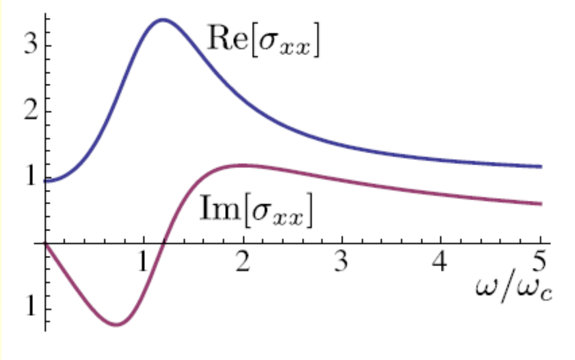

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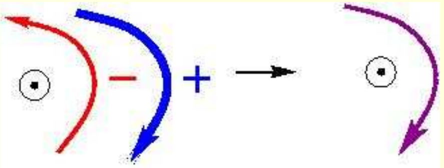

Intrinsic, interaction-induced broadening
(↔ Galilean invariant systems:
No broadening due to Kohn’s theorem)

$$\gamma = \sigma_{\mathrm{Q}} \frac{(B/c)^2}{(\varepsilon + P)/\mathrm{v}_{\mathrm{F}}^2}$$

Observable at room temperature in the GHz regime!

Can the resonance be observed?

$$\boxed{\omega = \pm\omega_c^{\mathrm{QC}} - i\gamma - i/\tau}$$

$$\boxed{\omega_c^{\mathrm{QC}} = \frac{\rho\, B/c}{(\varepsilon+P)/\mathrm{v}_{\mathrm{F}}^2}} \qquad \boxed{\gamma = \sigma_{\mathrm{Q}} \frac{(B/c)^2}{(\varepsilon+P)/\mathrm{v}_{\mathrm{F}}^2}}$$

Conditions to observe collective cyclotron resonance

Collison-dominated regime	$\hbar\omega_c << \alpha^2 k_B T$
Small broadening	$\gamma, \tau^{-1} < \omega_c^{QC}$
Quantum critical regime	$\rho \le \rho_{th} = \frac{(k_B T)^2}{(\hbar \mathrm{v}_F)^2}$
High T: no Landau quantization	$E_{LL} = \hbar \mathrm{v}_F \sqrt{\frac{2eB}{\hbar c}} << k_B T$

Parameters:

$T \approx 300K$

$B \approx 0.1T$

$\rho \approx 10^{11} cm^{-2}$

$\omega_c^{QC} \approx 10^{13} s^{-1}$

Does relativistic hydrodynamics apply?

- Do T and μ break relativistic invariance?
- Validity at large chemical potential?
- Larger magnetic field?

Boltzmann Approach

MM, L. Fritz, and S. Sachdev, PRB 2008

→ Recover and refine the hydrodynamic description

→ Describe relativistic-to-Fermi-liquid crossover

Boltzmann approach

L. Fritz, J. Schmalian, MM, and S. Sachdev, PRB 2008

Boltzmann equation in Born approximation

$$\left(\partial_t + e\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right] \cdot \frac{\partial}{\partial \mathbf{k}}\right) f_\pm(\mathbf{k}, t) = \alpha^2 I_{\text{coll}}^{Cb}\left[\mathbf{k}, t \mid \{f_\pm(\mathbf{k}', t)\}\right] + \Delta\, I_{\text{coll}}^{dis}\left[\mathbf{k}, t \mid \{f_\pm(\mathbf{k}', t)\}\right]$$

Boltzmann approach

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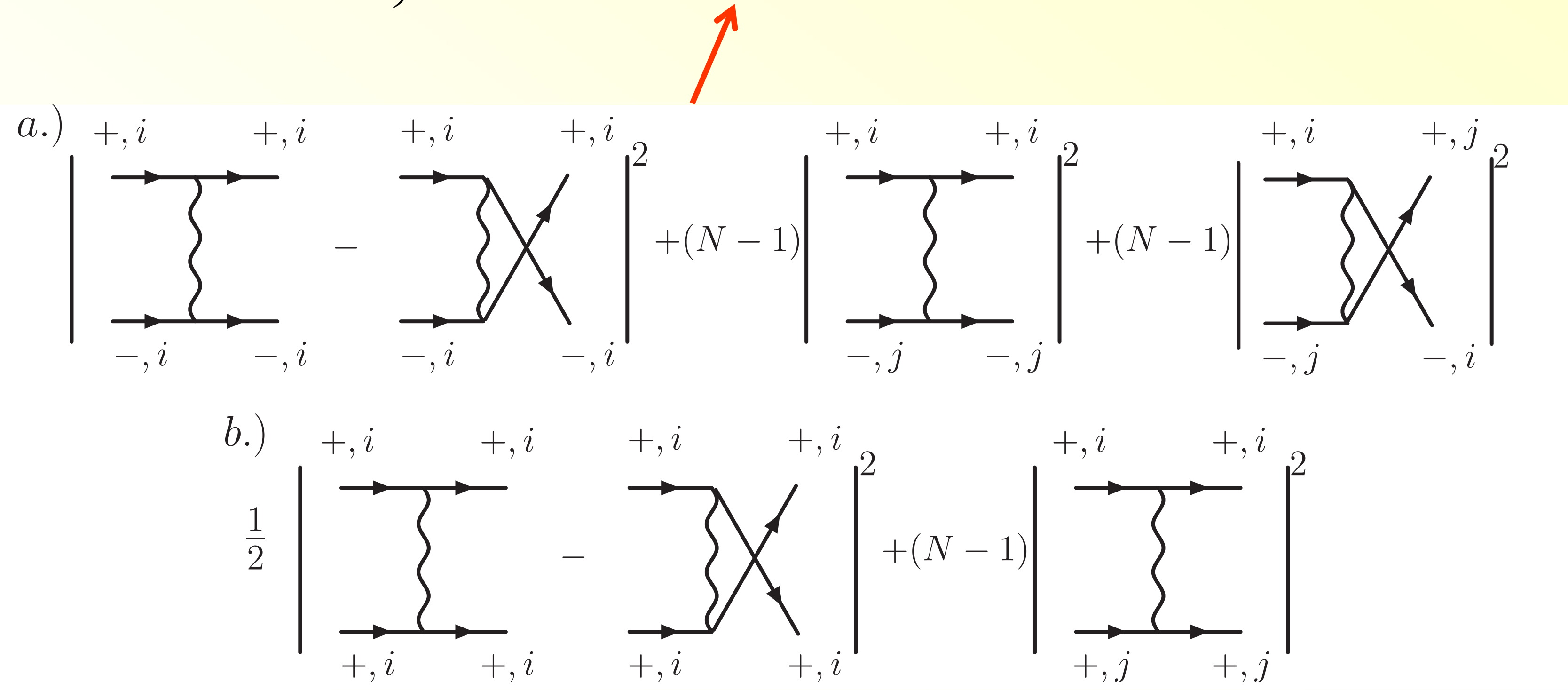

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L. Fritz, J. Schmalian, MM, and S. Sachdev, PRB 2008

Boltzmann equation in Born approximation

$$\left(\partial_t + e\left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]\cdot\frac{\partial}{\partial\mathbf{k}}\right) f_\pm(\mathbf{k},t) = \alpha^2 I_{\mathrm{coll}}^{Cb}\left[\mathbf{k},t\,|\,\{f_\pm(\mathbf{k}',t)\}\right] + \Delta\, I_{\mathrm{coll}}^{dis}\left[\mathbf{k},t\,|\,\{f_\pm(\mathbf{k}',t)\}\right]$$

1. Linearization: $f_\pm(\mathbf{k},t) = f_\pm^{eq}(\mathbf{k},t) + \delta f_\pm(\mathbf{k},t)$

Boltzmann approach

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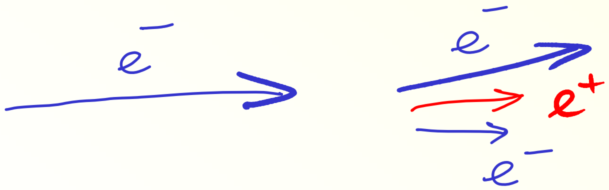
2. Forward scattering diverges logarithmically in 2d! (Cutoff at $\theta \approx \alpha \ll 1$)



Wilkins et al. (1971)

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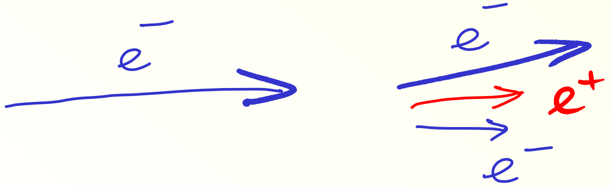
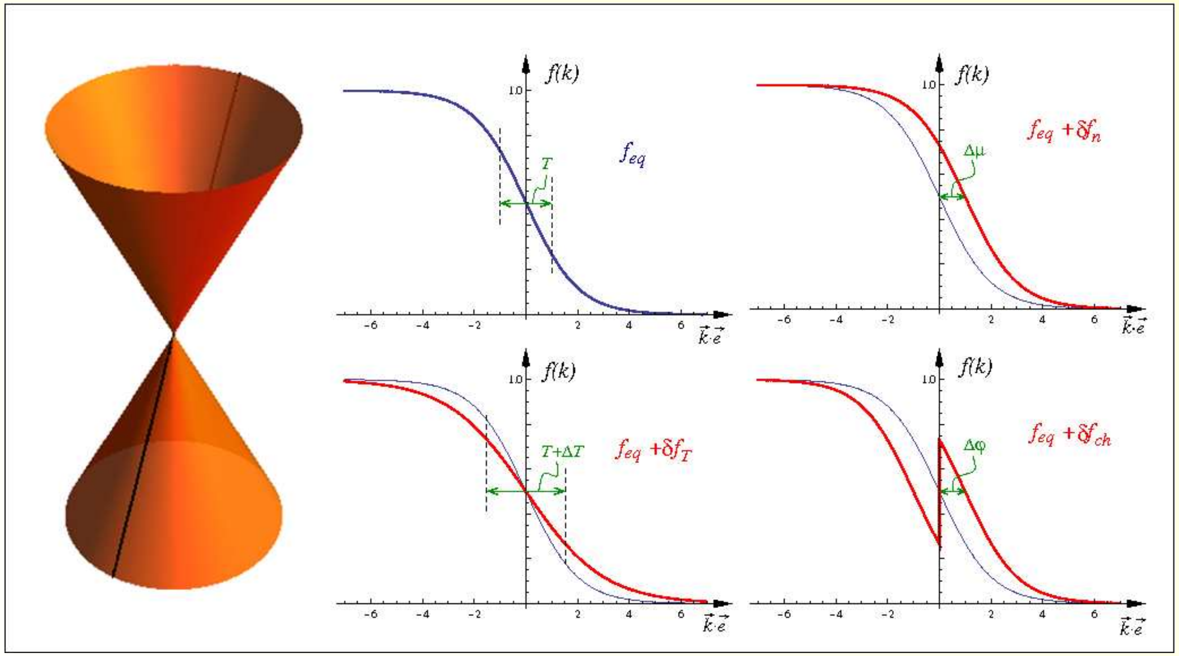


Wilkins et al. (1971)

→ Equilibration among particles with same group velocity



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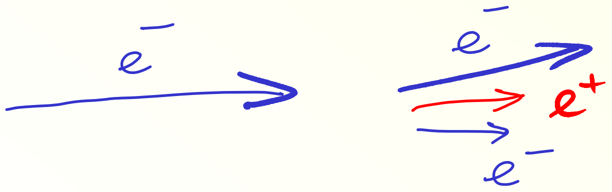
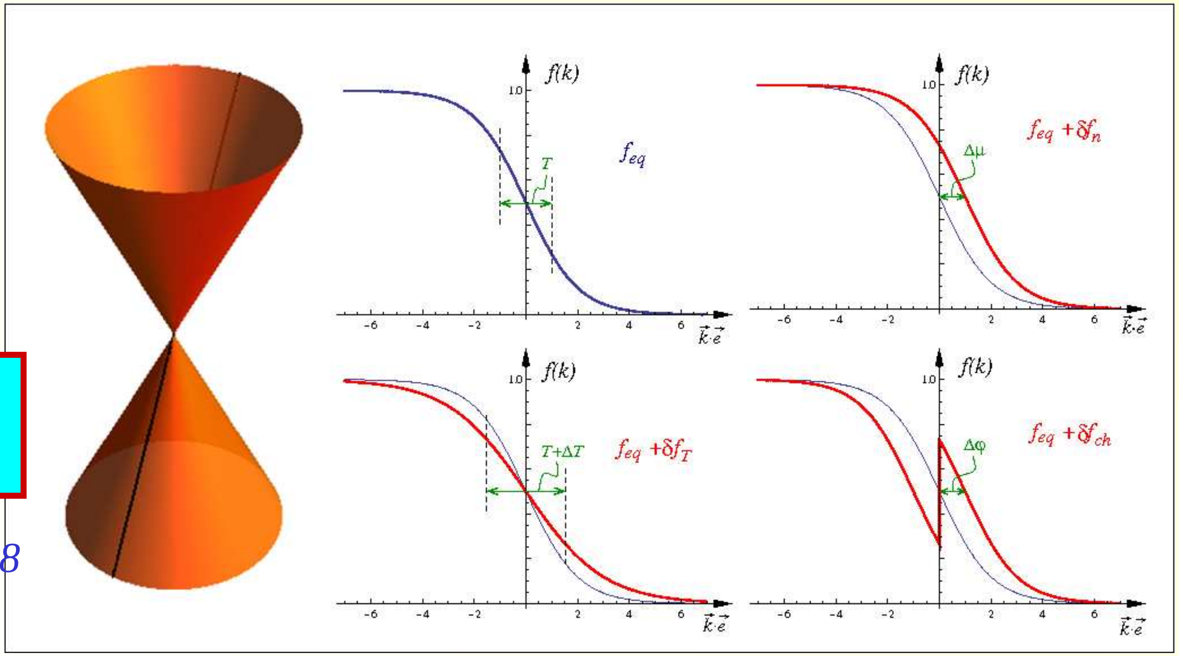


Wilkins et al. (1971)

Reduced to simple optimization problem for c_μ, c_T, c_φ !

MM, L. Fritz, and S. Sachdev, PRB 2008



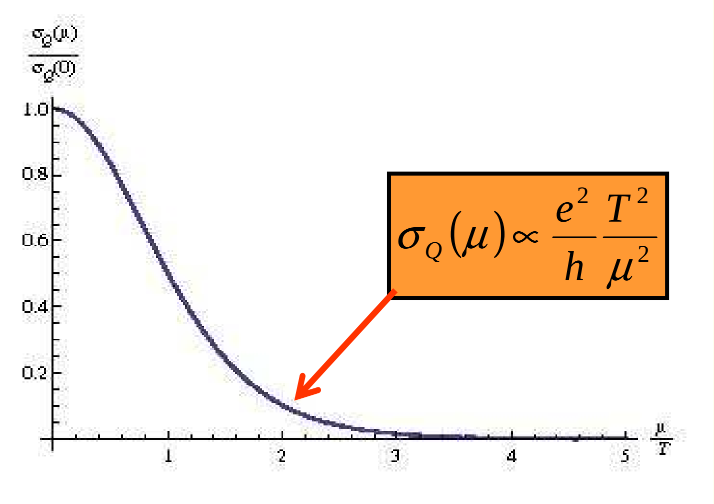
Boltzmann approach

MM, L. Fritz, and S. Sachdev, PRB 2008
Kashuba, PRB 2008

Collision-dominated conductivity

$$\sigma_Q(\mu=0) \approx \frac{0.76}{\alpha^2(T)} \frac{e^2}{h}$$

Gradual disappearance of relativistic physics as one crosses over to degenerate Fermi gas



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Boltzmann approach

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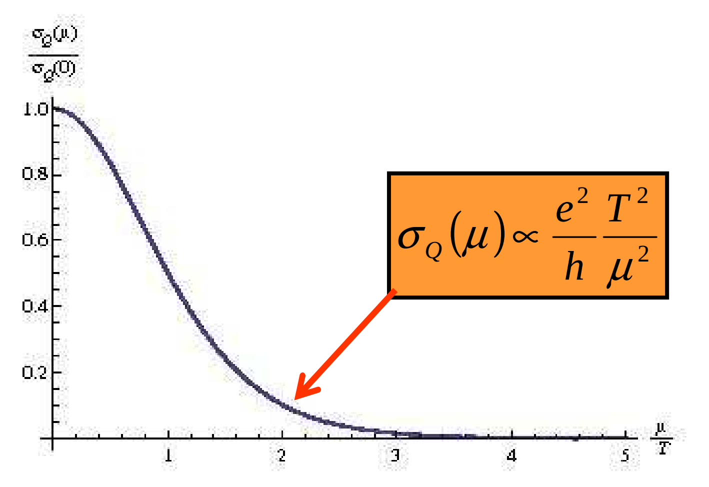

$$\boxed{\sigma_Q(\mu)\propto\frac{e^2}{h}\frac{T^2}{\mu^2}}$$

→ Recover Kohn’s theorem for width of cyclotron resonance:

$$\gamma\propto\sigma_Q(\mu)\overset{\mu>>T}{\longrightarrow}0$$

Sharp resonance in the degenerate limit $\mu >> T$!

Recovering magnetohydrodynamics

MM, L. Fritz, and S. Sachdev, PRB 2008

Momentum conservation →
Exact zero mode of the Coulomb collision integral!

$$\delta f_{\pm}^{(0)}(\mathbf{k}) = \pm\, c_T \mathbf{k} \cdot \mathbf{E} \;\; f_{\pm}^{eq}(\mathbf{k})\left[1 - f_{\pm}^{eq}(\mathbf{k})\right]$$

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Recover hydrodynamics by studying the dynamics of this **slowest** mode!

$$\longrightarrow \boxed{\sigma_{xx}(\omega, B) = \sigma_{xx}^{\mathrm{MHD}}(\omega, B) + \mathcal{O}(b/\alpha^2, \omega/\alpha^2)}$$

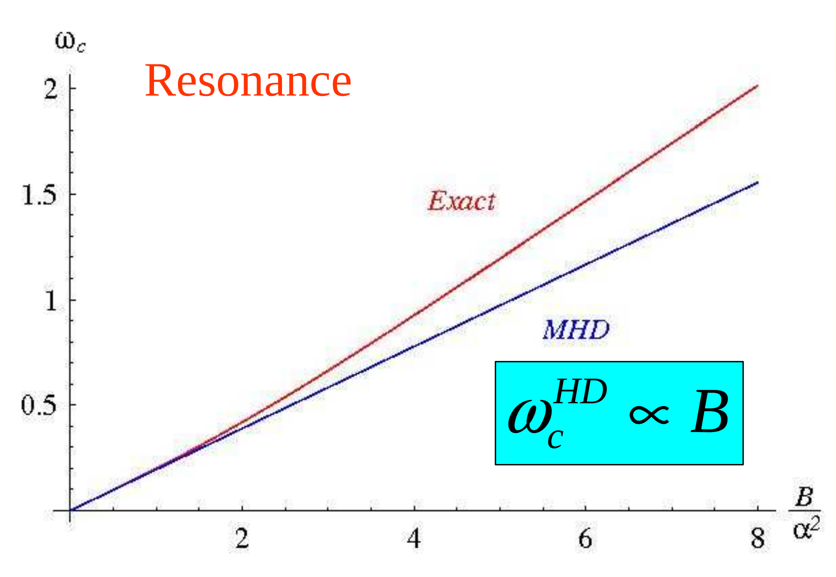
Corrections small if τ_{ee}^{-1} is large.

Cyclotron resonance revisited

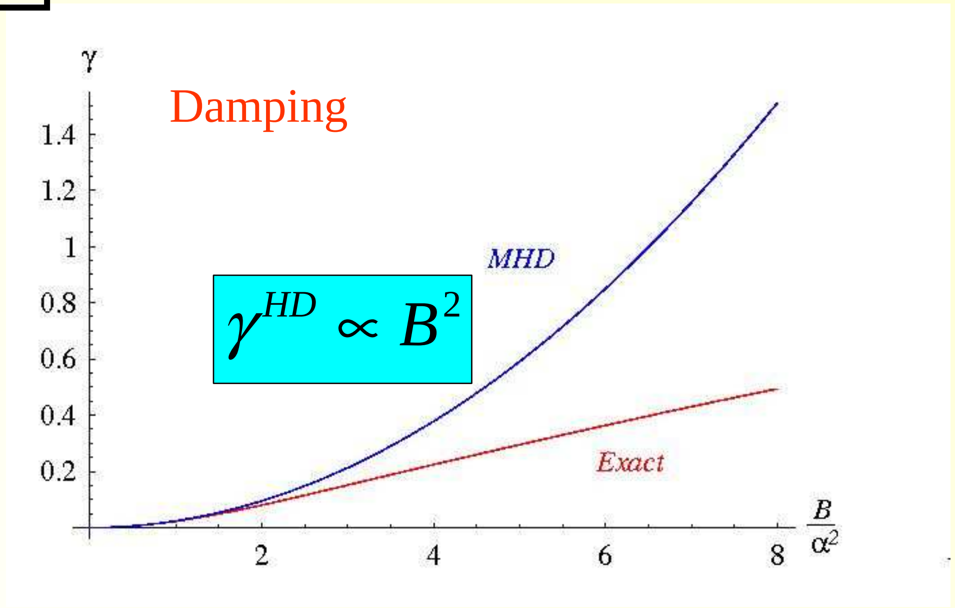
Cyclotron resonance at large fields: beyond hydrodynamics:

$$\boxed{\tau_{\mathrm{B}}^{-1} > \tau_{\mathrm{ee}}^{-1} >> \tau_{\mathrm{imp}}^{-1}, \omega}$$

$$\boxed{\mu = T}$$



$$\boxed{\omega_c^{HD} \propto B}$$



$$\boxed{\gamma^{HD} \propto B^2}$$

What to do beyond the weak coupling (Boltzmann) approximation [$\alpha(T) \rightarrow 0$] ??

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Answer until recently: Not much at all!

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Newest progress from string theory:

1) **Look at "similar" theories which are very strongly coupled, but can be solved exactly**

2) **Try to extract the "generally valid, universal" part of the result and use it as a guide**

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Newest progress from string theory:

1) **Look at "similar" theories which are very strongly coupled, but can be solved exactly**

2) **Try to extract the "generally valid, universal" part of the result and use it as a guide**

Take it with at least two grains of salt and just enjoy it!

Compare graphene to:
Strongly coupled relativistic liquids

S. Hartnoll, P. Kovtun, MM, S. Sachdev (2007)

Obtain exact results via string theoretical AdS–CFT correspondence

→ Response functions in particular strongly coupled relativistic fluids
(for maximally supersymmetric Yang Mills theories with $N \to \infty$ colors):

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Obtain exact results via string theoretical AdS–CFT correspondence

→ Response functions in particular strongly coupled relativistic fluids
(for maximally supersymmetric Yang Mills theories with $N \to \infty$ colors):

- Confirm the structure of the hydrodynamic response functions such as σ(ω).
- Calculate the transport coefficients for a strongly coupled theory!

SUSY - SU(N): $\sigma(\mu=0)=\sqrt{\frac{2}{9}}N^{3/2}\frac{e^2}{h}\ ;\quad \frac{\eta_{shear}}{s}(\mu=0)=\frac{1}{4\pi}\frac{\hbar}{k_B}$

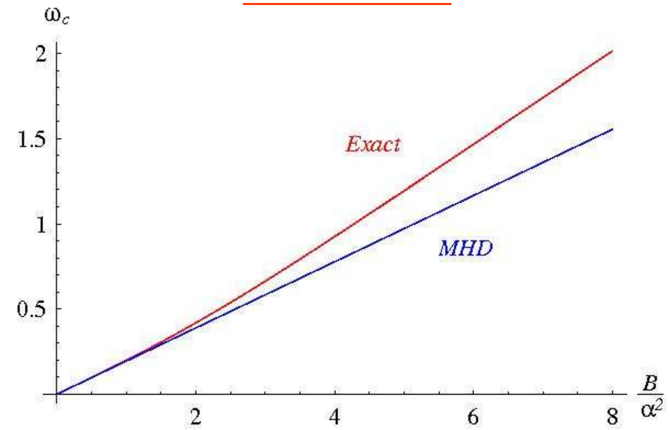
Strongly coupled liquids

Same trends as in exact (AdS-CFT) results for strongly coupled relativistic fluids!

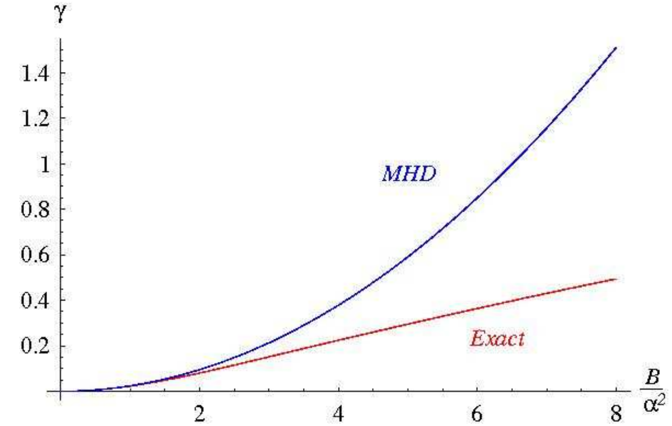
S. Hartnoll, C. Herzog (2007)

Graphene

Resonance



Damping



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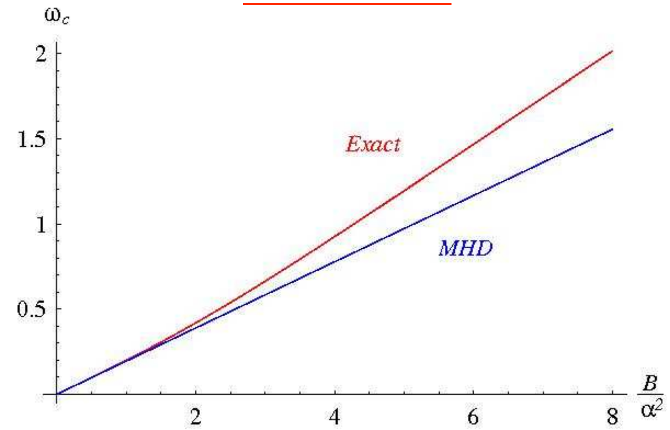

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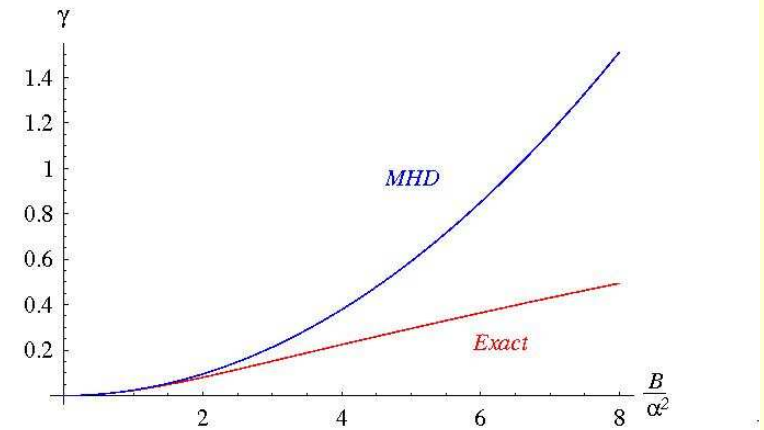
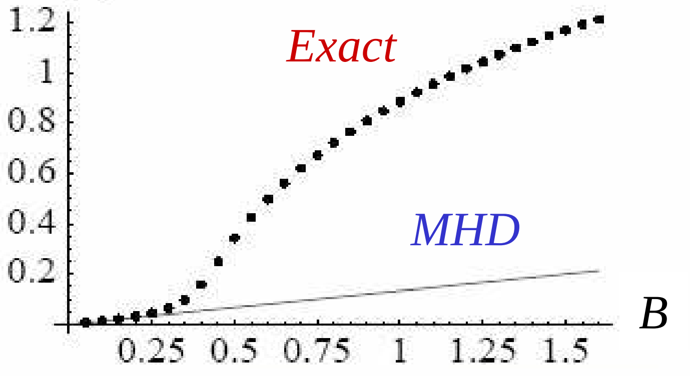
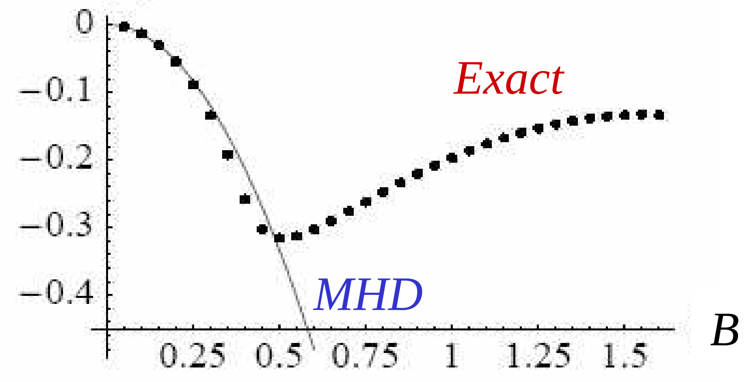


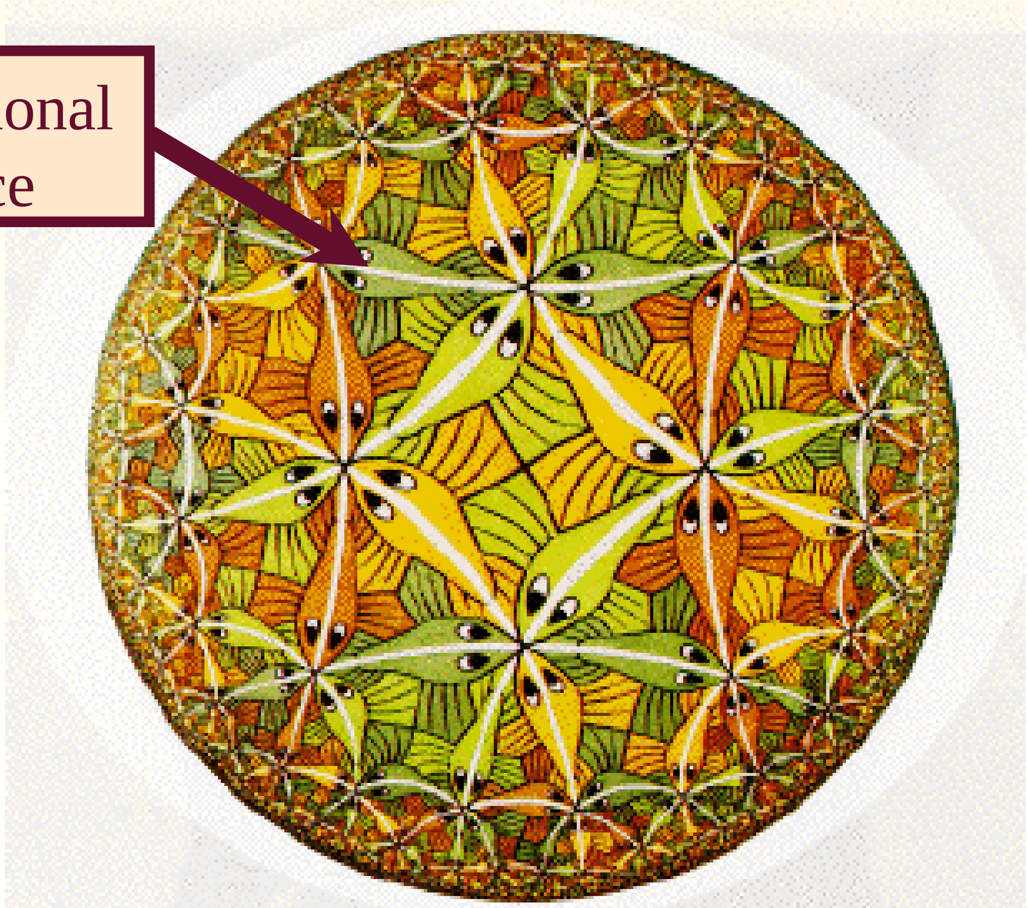
$\mathcal{N}$= 4 SUSY SU(N) gauge theory [akin to QCD: flows to a CFT at low energy]





AdS/CFT correspondence

The quantum theory of a black hole in a 3+1-dimensional negatively curved AdS universe is holographically represented by a CFT (the theory of a quantum critical point) in 2+1 dimensions



3+1 dimensional
AdS space

Maldacena, Gubser, Klebanov, Polyakov, Witten

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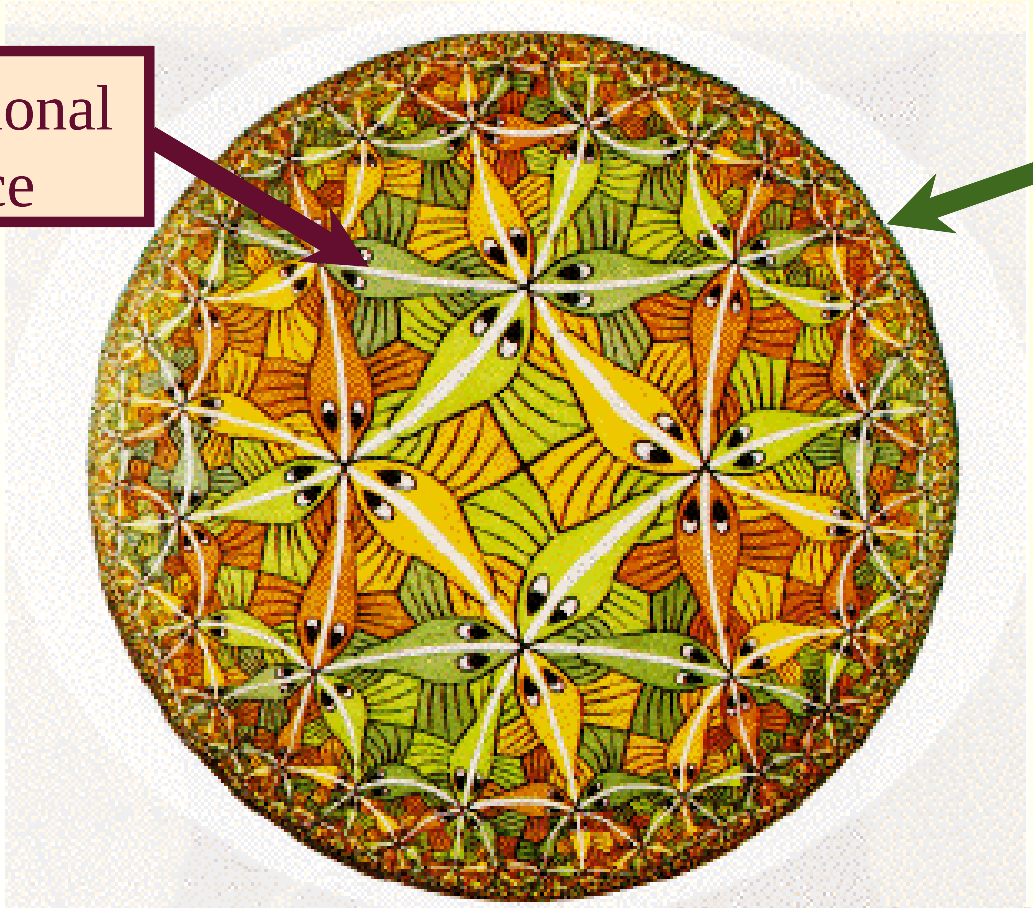

3+1 dimensional AdS space

A 2+1 dimensional system at its quantum critical point

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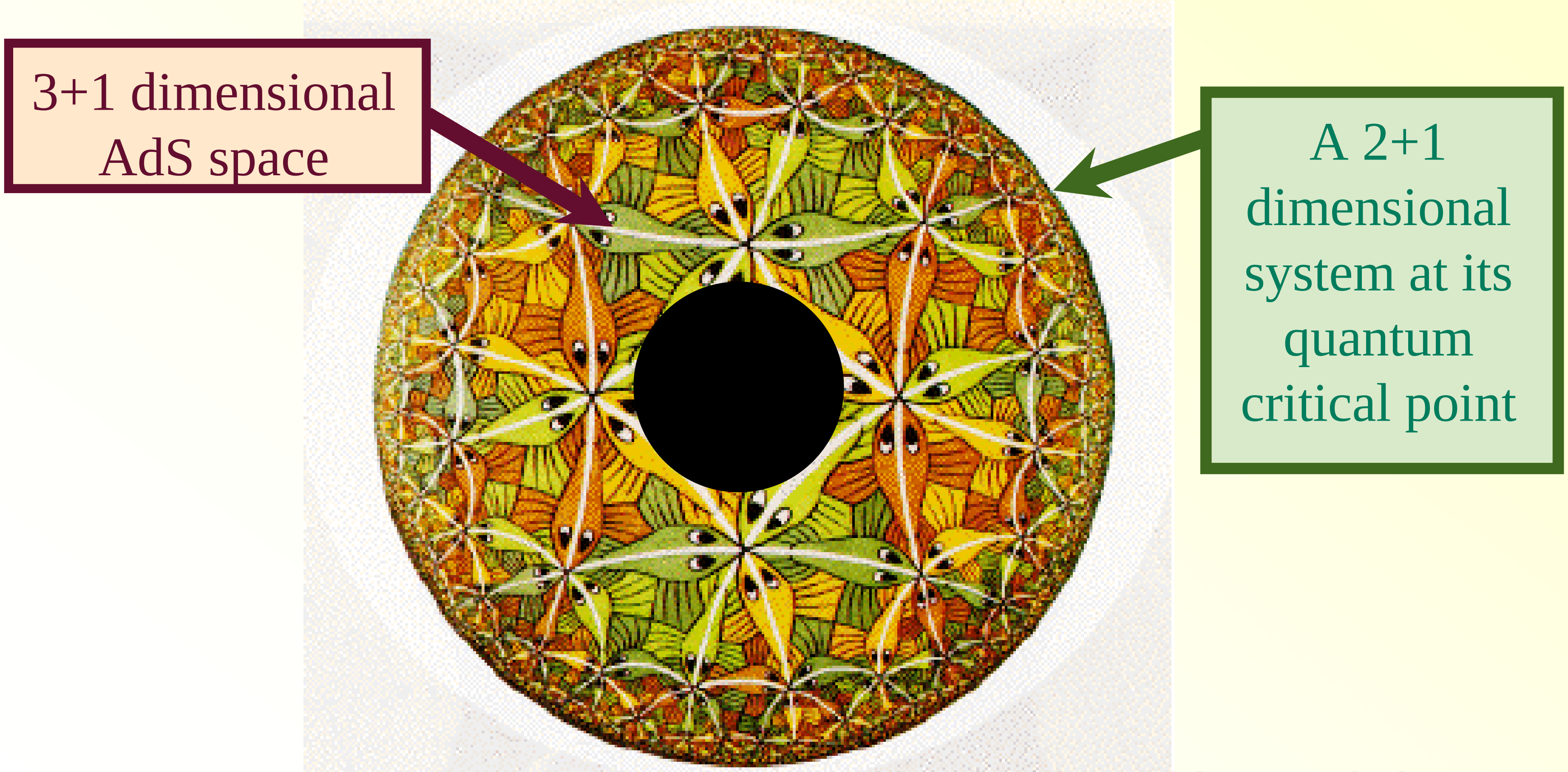

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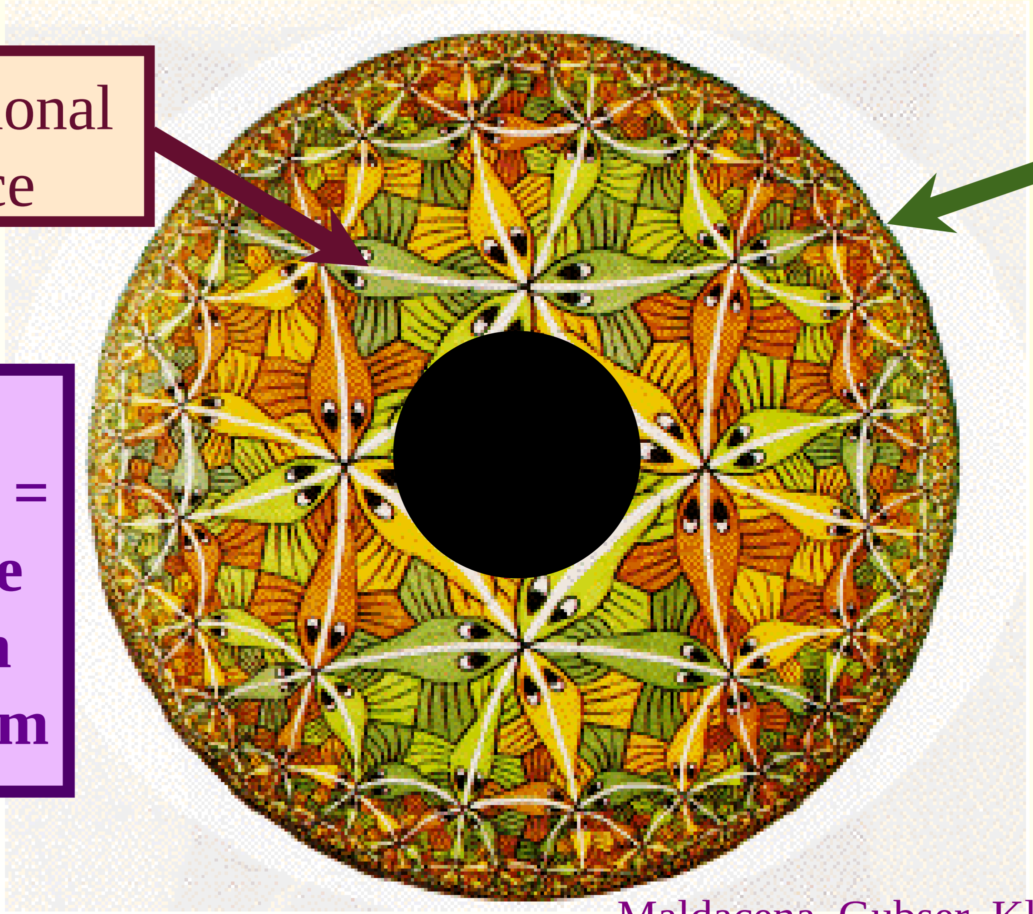

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Black hole temperature = temperature of quantum critical system

Maldacena, Gubser, Klebanov, Polyakov, Witten

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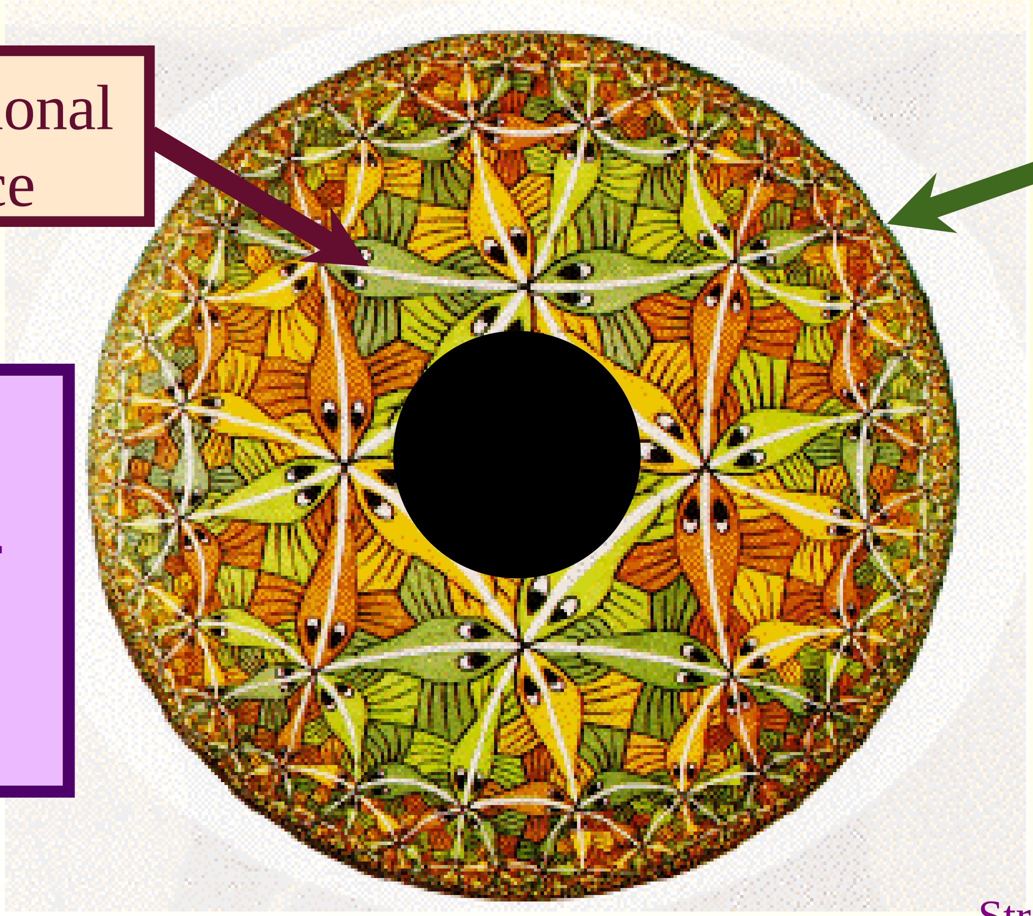

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Black hole entropy = Entropy of quantum criticality

Strominger, Vafa

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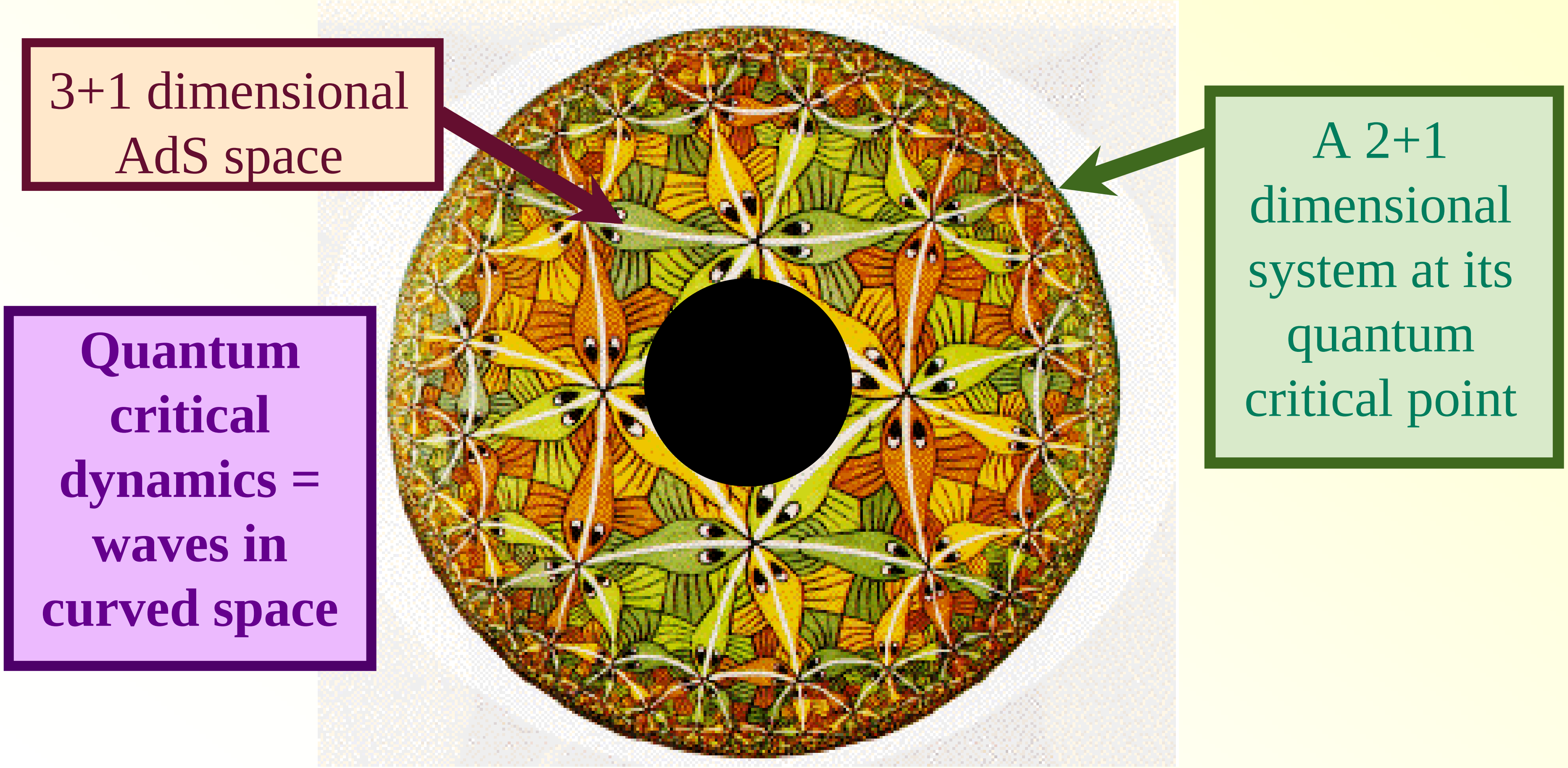

3+1 dimensional
AdS space

A 2+1
dimensional
system at its
quantum
critical point

**Quantum
critical
dynamics =
waves in
curved space**

Maldacena, Gubser, Klebanov, Polyakov, Witten

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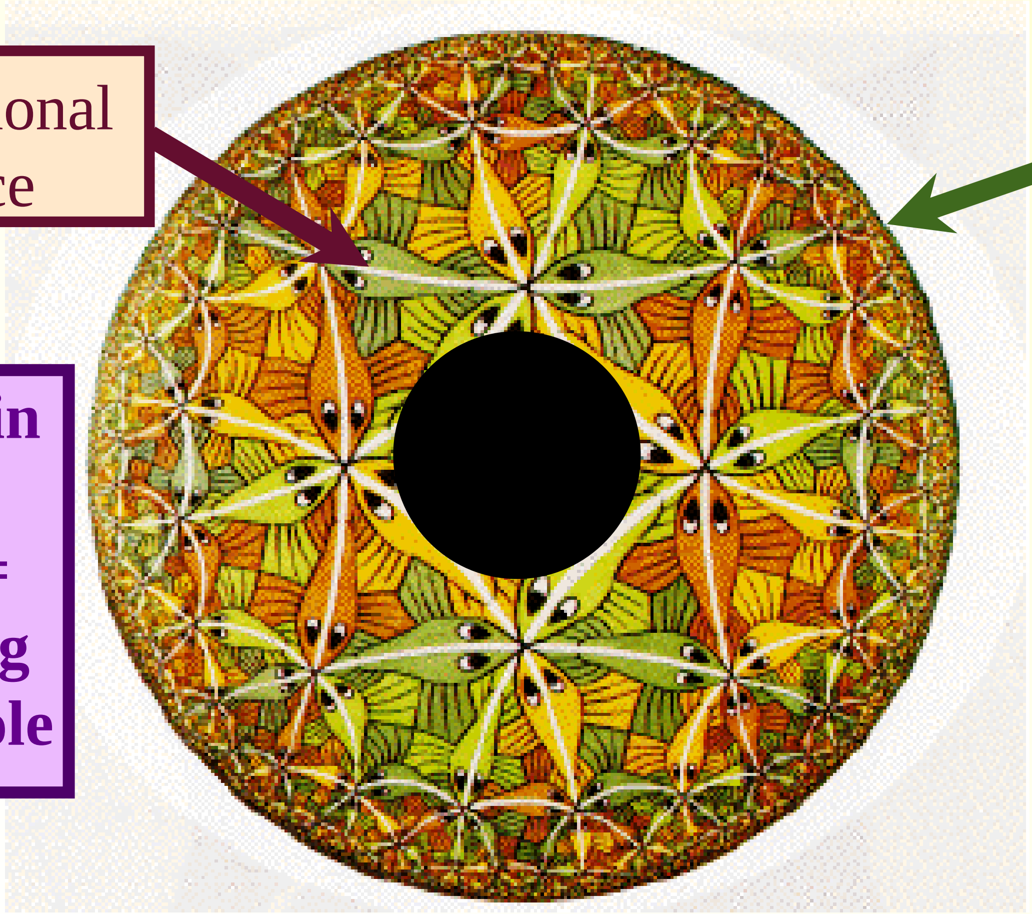


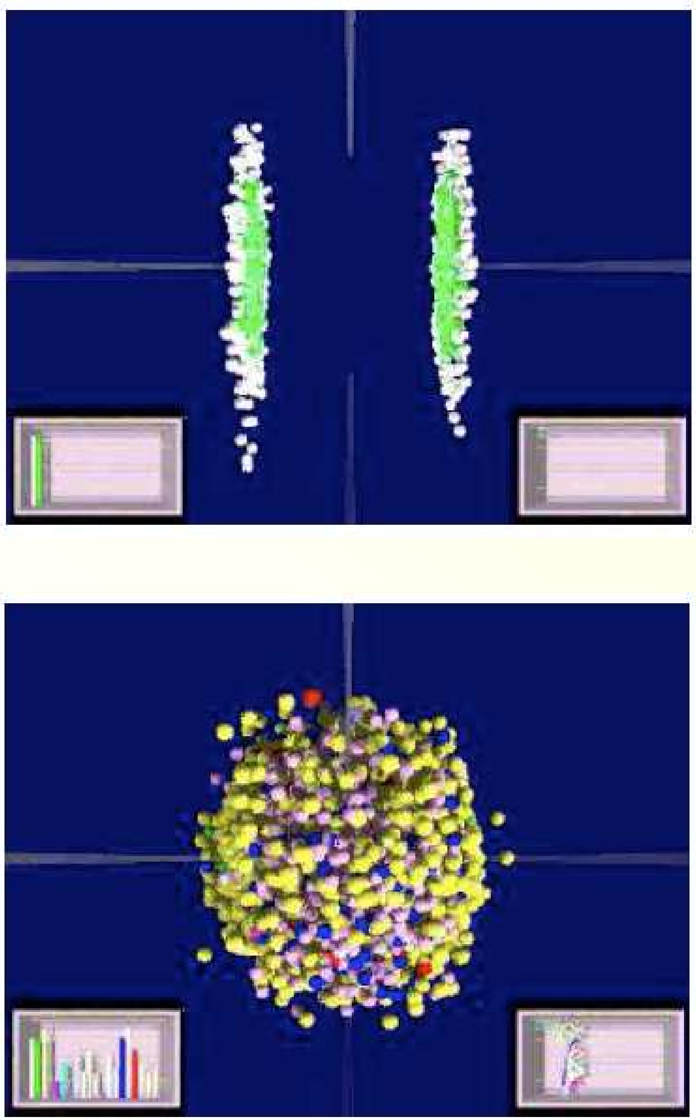
3+1 dimensional AdS space

A 2+1 dimensional system at its quantum critical point

Dissipation in quantum criticality = waves falling into black hole

Kovtun, Policastro, Son

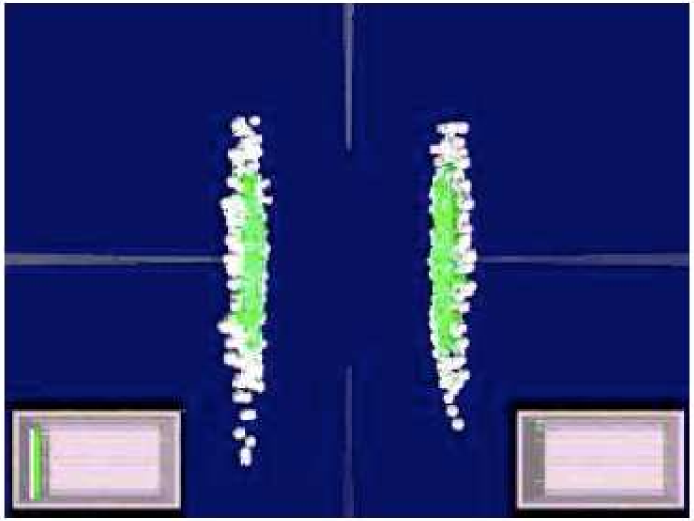
Au+Au collisions at RHIC



Quark-gluon plasma can be described by QCD (nearly conformal, critical theory)

Extremely low viscosity fluid!

Au+Au collisions at RHIC



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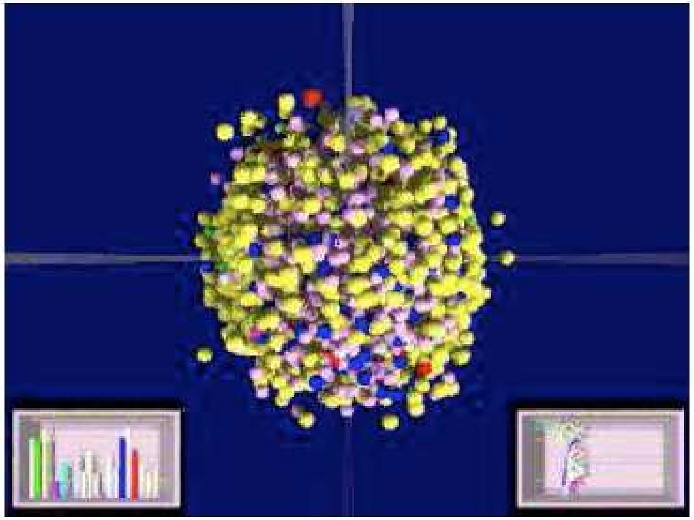

SUSY - SU(N):

$$\frac{\eta_{shear}}{s}(\mu = 0) = \frac{1}{4\pi}\frac{\hbar}{k_B}$$

This IS an extremely low value!
Is there a lowest possible value,
or a “most perfect” liquid?

Further analogy with AdS-CFT

MM and J. Schmalian, (2008)

Is quantum critical graphene a nearly perfect fluid?

↔

Anomalously low viscosity? – Yes!

Conjecture from black hole physics:

$$\boxed{\frac{\text{shear viscosity}}{\text{entropy density}} = \frac{\eta}{s} > \frac{\hbar}{k_B}\frac{1}{4\pi}}$$

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Undoped Graphene:

$$\boxed{\eta \propto n \cdot m\mathrm{v}^2 \cdot \tau \;\rightarrow\; n_{\text{th}} \cdot k_B T \cdot \frac{\hbar}{\alpha^2 k_B T} = \frac{\hbar}{\alpha^2} n_{\text{th}}}$$

$$\boxed{s \propto k_B n_{\text{th}}}$$

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$$s \propto k_B n_{\text{th}}$$

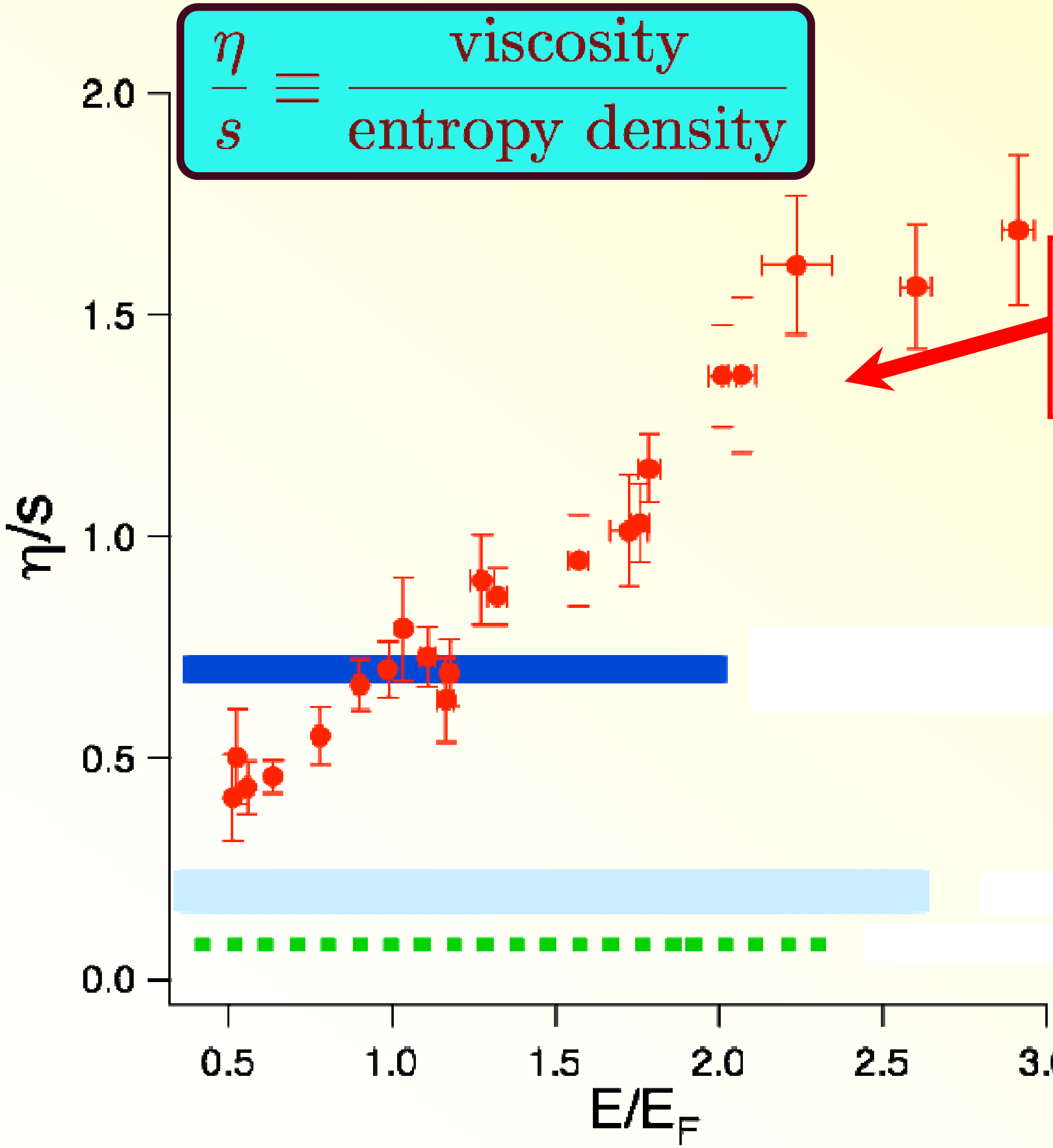
Doped Graphene:

$$\eta \propto n \cdot m\text{v}^2 \cdot \tau \;\rightarrow\; n \cdot E_F \cdot \frac{\hbar E_F}{(k_B T)^2}$$

$$s \propto k_B n \frac{T}{E_F}$$

$$\frac{\eta}{s} \sim \frac{\hbar}{k_B}\left(\frac{E_F}{T}\right)^3$$

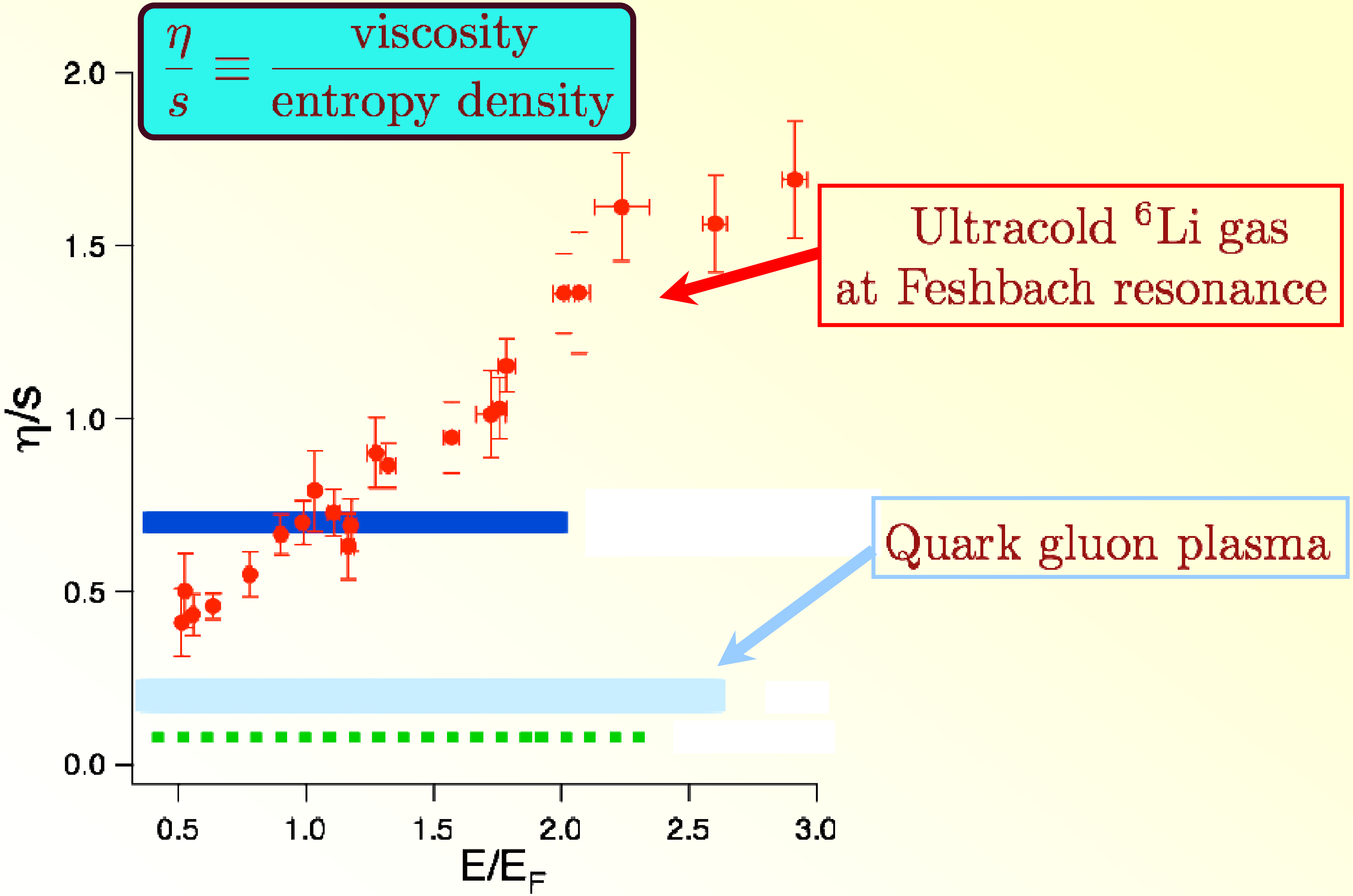
$$\frac{\eta}{s} \equiv \frac{\text{viscosity}}{\text{entropy density}}$$



Ultracold ^{6}Li gas at Feshbach resonance

T. Schäfer, Phys. Rev. A ***76****, 063618 (2007).*

A. Turlapov, J. Kinast, B. Clancy, Le Luo, J. Joseph, J. E. Thomas, J. Low Temp. Phys. ***150****, 567 (2008)*



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Quark gluon plasma

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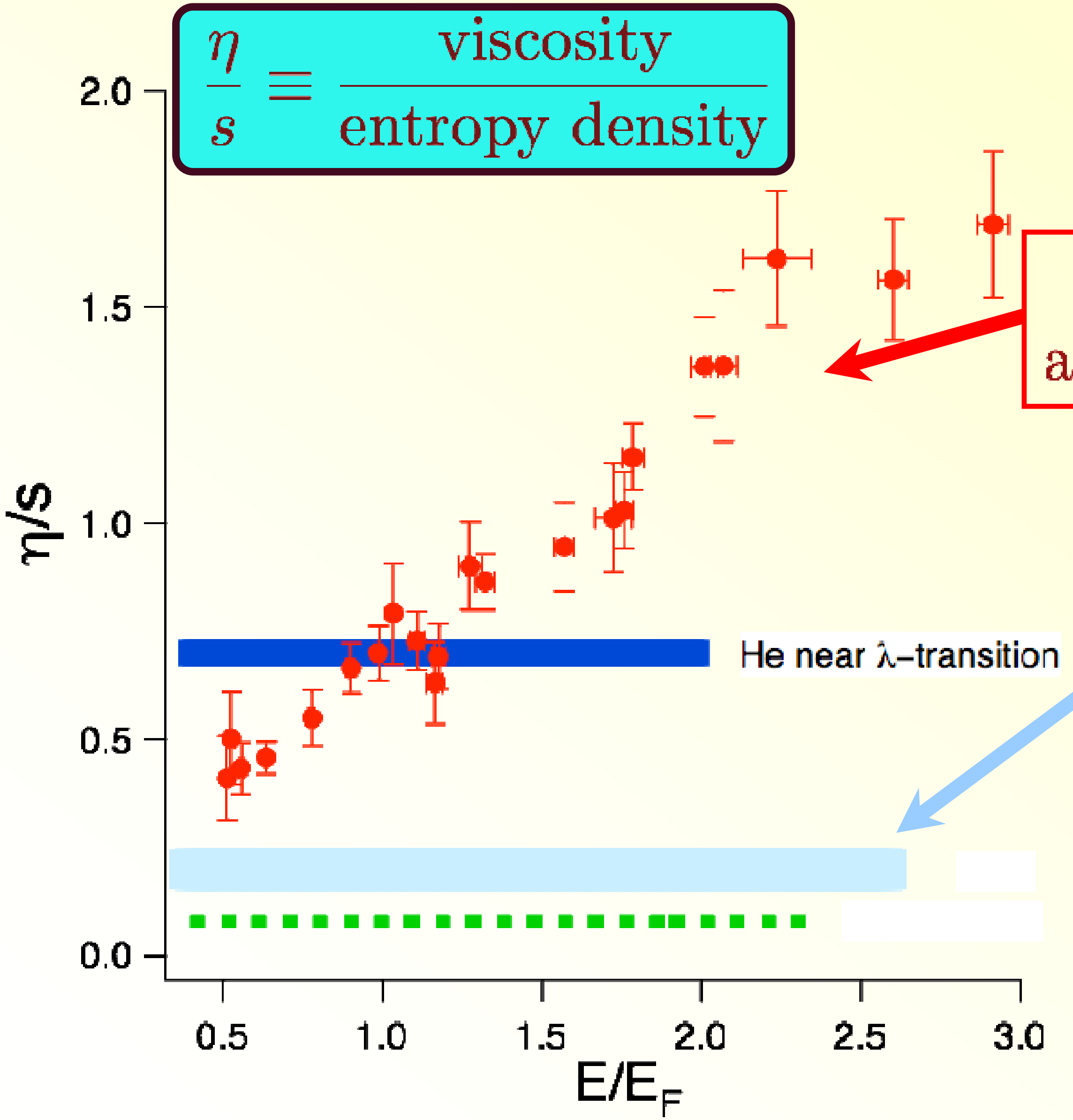

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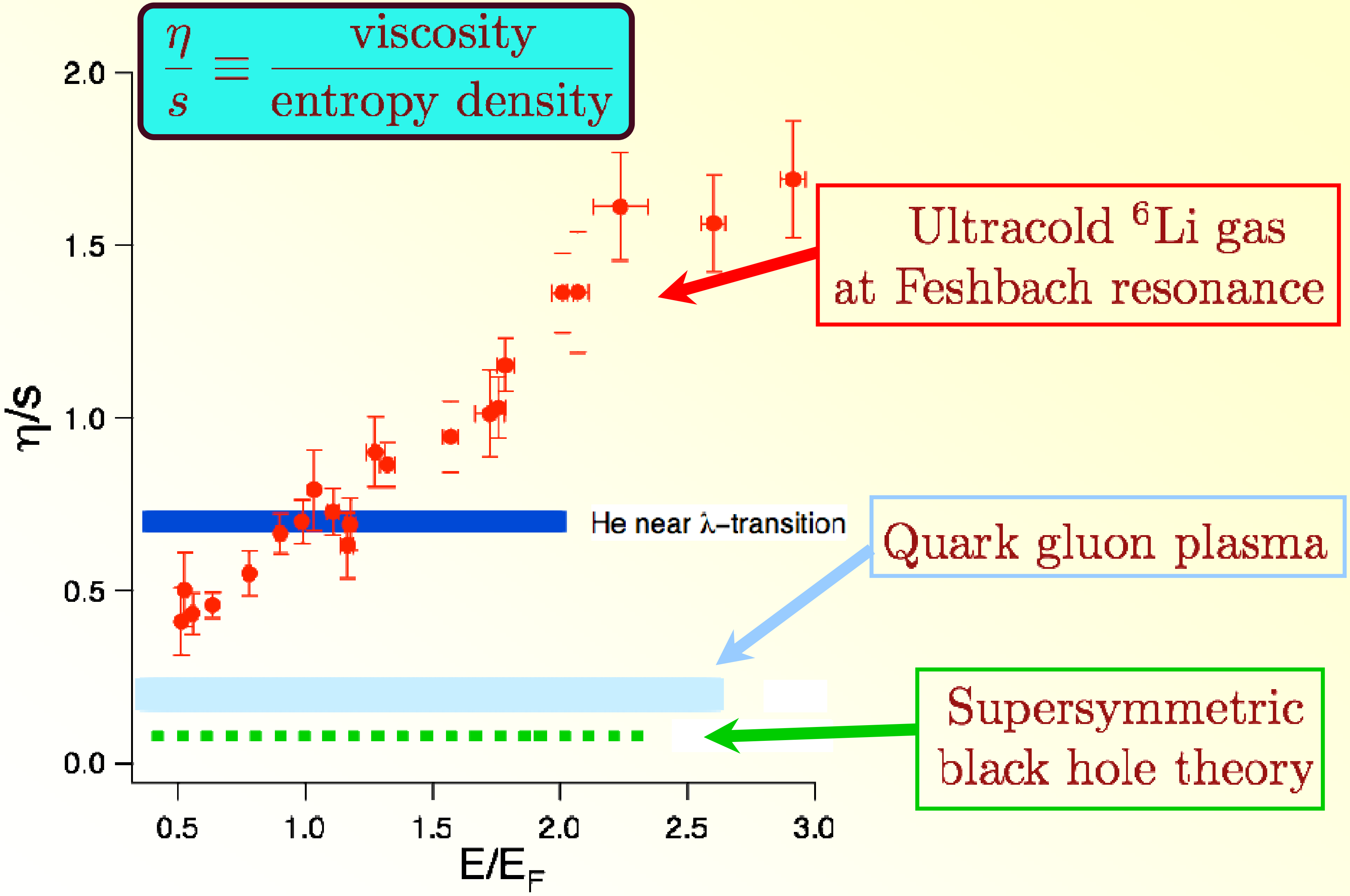

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Supersymmetric black hole theory

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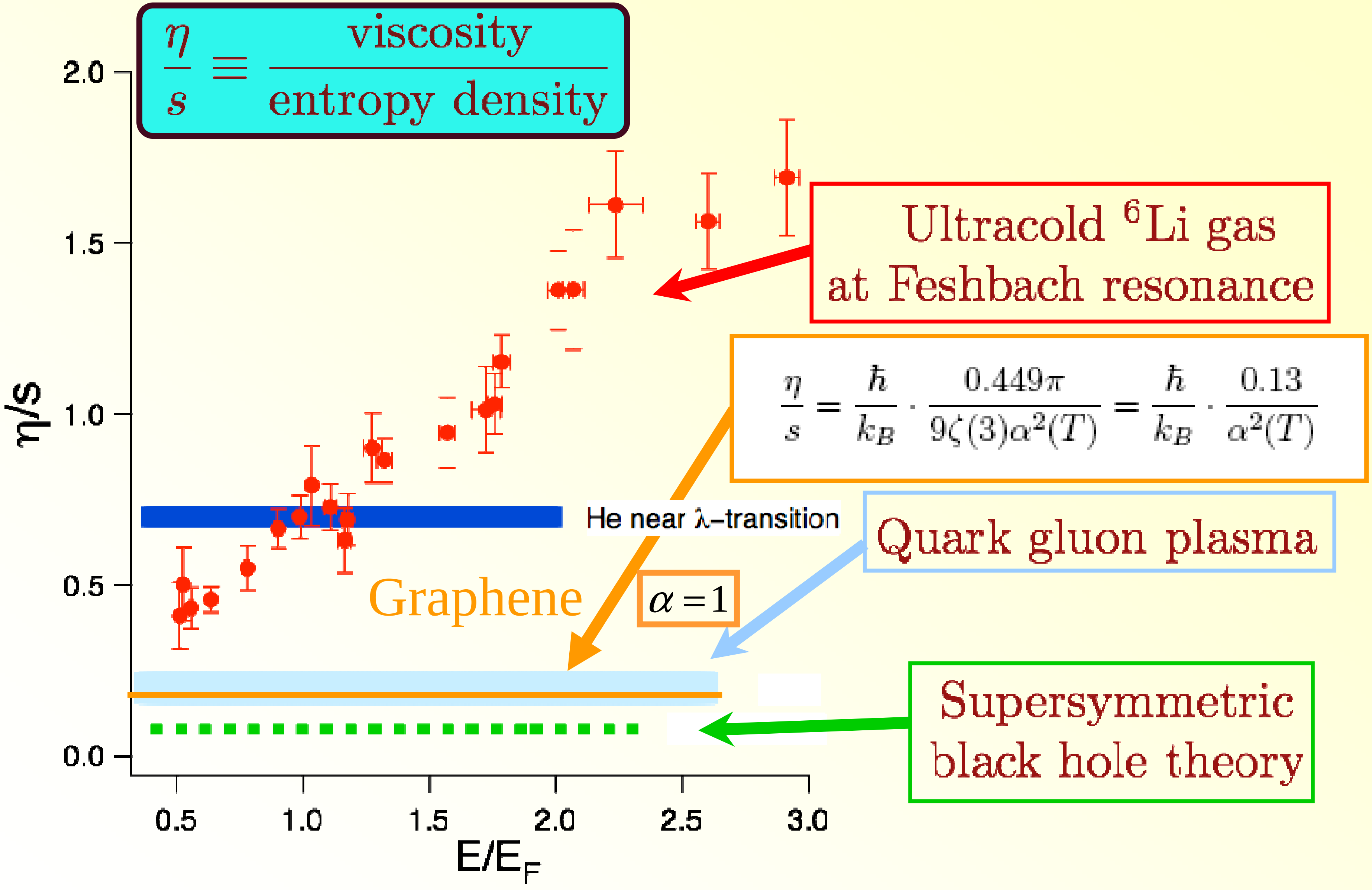

Ultracold ^{6}Li gas at Feshbach resonance

$$\frac{\eta}{s} = \frac{\hbar}{k_B}\cdot\frac{0.449\pi}{9\zeta(3)\alpha^2(T)} = \frac{\hbar}{k_B}\cdot\frac{0.13}{\alpha^2(T)}$$

Quark gluon plasma

Graphene

$\alpha = 1$

Supersymmetric black hole theory

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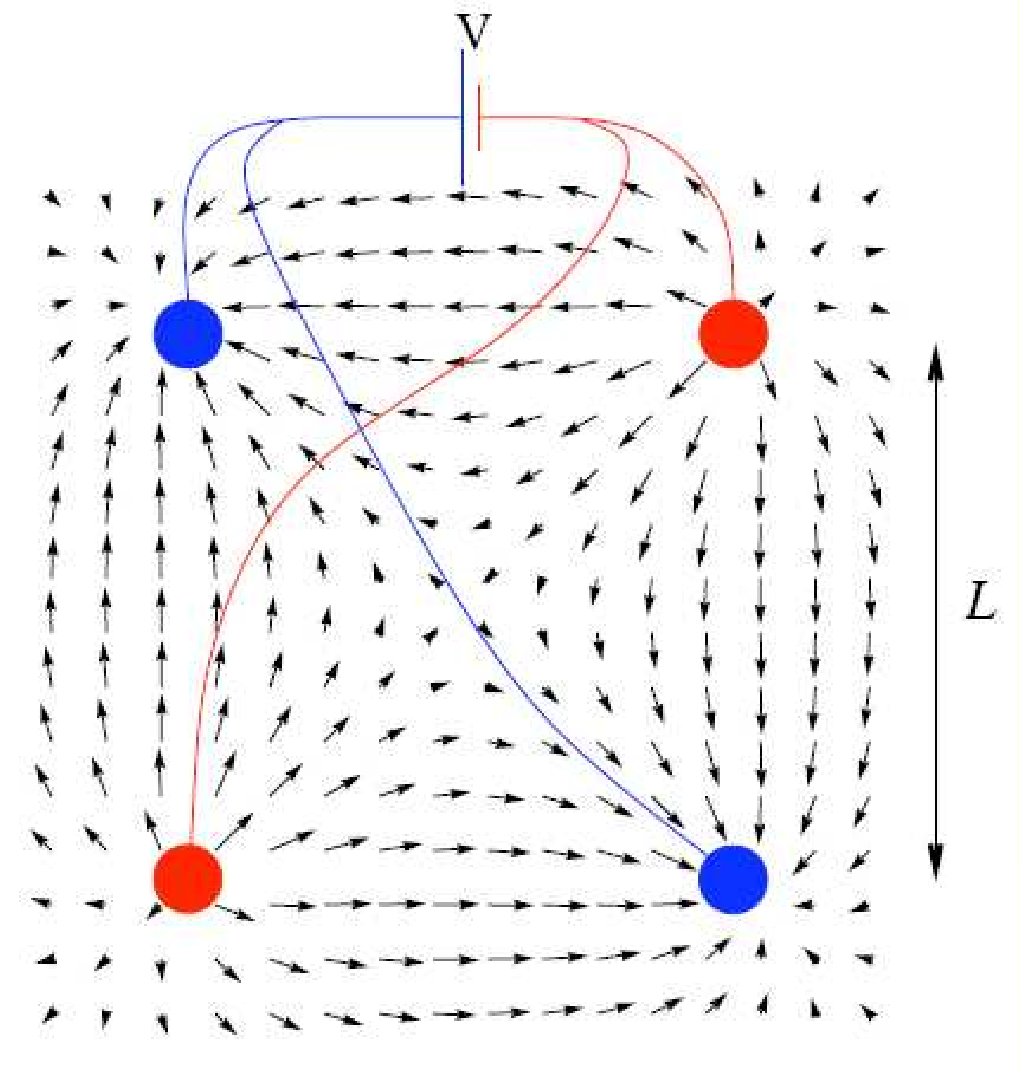
Electronic consequences of low viscosity?

MM, J. Schmalian, L. Fritz, (condmat:0903.4178)



Expected viscous effects on conductance in non-uniform current flow:

Decrease of conductance with length scale L

Electronic consequences of low viscosity?

MM, J. Schmalian, L. Fritz, (condmat:0903.4178)

Electronic turbulence in clean graphene?

Reynolds number:

$$\boxed{\mathrm{Re} = \frac{s/k_B}{\eta/\hbar} \times \frac{k_B T}{\hbar v/L} \times \frac{u_{\mathrm{typ}}}{v}}$$

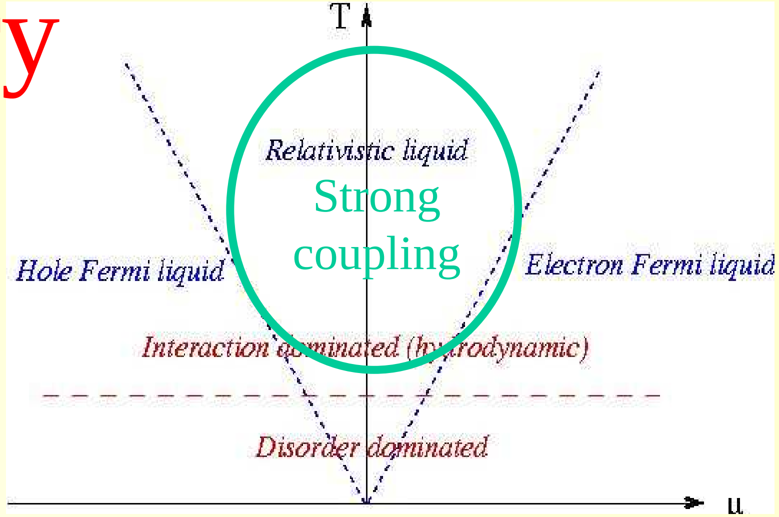
Strongly driven mesoscopic systems: (Kim's group)

$$\boxed{\begin{aligned} L &= 1\mu m \\ u_{typ} &= 0.1\mathrm{v} \\ T &= 100K \end{aligned}} \longrightarrow \mathrm{Re} \sim 10-100$$

Complex fluid dynamics!
(pre-turbulent flow)

New phenomenon in an
electronic system!

Summary



- Undoped graphene is strongly coupled in a large temperature window!
- Nearly universal, strong coupling features in transport
- Emergent relativistic hydrodynamics at low frequency
- Graphene: Nearly perfect quantum liquid!
  → Possibility of complex (turbulent?) current flow at high bias