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Magnetoresistance and localization in bosonic insulators

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Abstract – We study the strong localization of hard-core bosons. Using a locator expansion we find that in the insulator, unlike for typical fermion problems, nearly all low-energy scattering paths come with positive amplitudes and hence interfere constructively. As a consequence, the localization length of bosonic excitations shrinks when the constructive interference is suppressed by a magnetic field, entailing an exponentially large positive magnetoresistance, opposite to and significantly stronger than the analogous effect in fermions. Within the forward-scattering approximation, we find that the lowest-energy excitations are the most delocalized. A similar analysis applied to random field Ising models suggests that the ordering transition is due to a delocalization initiated at zero energy rather than due to the closure of a mobility gap in the paramagnet.

Most disordered insulators display some form of variable range hopping transport [1], reflecting the localization of carriers at low energies [2]. Naively, one might expect that in such insulators the quantum statistics of carriers is irrelevant, as particles essentially never exchange their places. However, when the hopping length becomes much larger than the distance between impurities, transport is in fact very sensitive to statistics, as probed, *e.g.*, by orbital magnetic fields. For *fermions* the latter suppresses the destructive interference among alternative virtual paths, leading to a strong negative magnetoresistance [3–8]. This is a very non-trivial manifestation of quantum interference in impurity bands. In this letter we address the *bosonic* counterpart of this effect, which, remarkably, has always the opposite sign. The quantum statistics also manifests itself in a non-trivial energy dependence of localization and in the way delocalization is approached.

The present study of disordered bosons is motivated by a variety of experimental situations involving bosonic insulators, such as in Josephson junction arrays, certain superconducting films, turned insulating by strong disorder, repulsive cold bosonic atoms in speckle potentials and artificial gauge fields [9,10], helium in porous media or random quantum magnets. [11]. In the presence of strong disorder, the respective insulators are expected to be Bose glasses [12], whose low-energy excitations are localized by disorder, but do not exhibit a spectral gap. Transport of such strongly disordered bosons is still scarcely studied, but poses a variety of interesting conceptual questions, which are not fully resolved yet [11,13,14].

A particularly interesting aspect of localization is the magnetoresistance in charged *bosonic* insulators. Recent experiments in strongly disordered, superconducting InO_x films [15,16] have shown that a magnetic field not only destroys rapidly the already weak superconductivity [17], but also induces a giant positive magnetoresistance in the ensuing insulating state. Similar effects in magnetoresistance have been reported in amorphous films of TiN [18], Bi [19], and in patterned films [20], cf. the review [21]. The giant positive magnetoresistance in the vicinity of the superconducting transition is intriguing. Mechanisms such as shrinking impurity wave functions or spin blocking of weakly interacting electrons, which may play a role in semiconductors [1], hardly apply to these systems [13]. Instead, experimental observations in transport [15–18,20,22–24] and tunneling microscopy [25,26], as well as theoretical model studies [27–29] suggest the importance of remnant electron pairing in the insulator, despite the absence of global phase coherence [30].

While it is natural to expect a magnetic field to increase the resistance of a bosonic insulator, in continuation of its destructive effect on superconductivity, there is no satisfactory microscopic explanation of the giant effects seen in experiments yet, despite attempts at phenomenological explanations [31] or model calculations for granular systems [32,33]. The latter do not account for the fact that in the experimental films [15] no well-defined granular structure exists, and the spectral gap for pairs is expected to be washed out by strong disorder [27]. This suggests that, most likely, it is Cooper pair (boson) localization due to disorder, which induces the insulating behavior, rather than the opening of a homogeneous gap in the insulator [13].

Here we study a microscopic model of an insulator of hard-core bosons, subject to strong disorder potentials. This captures, *e.g.*, electronic systems with a strong local electron pairing. By contrasting this model with similar fermionic models, we reveal the specific role of quantum statistics. As the simplest model containing all relevant ingredients we consider a lattice, whose sites can accommodate only one quantum particle due to strong onsite repulsion. For spinless fermions this is simply the non-interacting Anderson model for single-particle localization [2,3]. For hard-core bosons the model was introduced by Ma and Lee [34] who considered disordered superconductors in terms of preformed pairs (Anderson pseudospins). This is a faithful low-energy representation of single-band Hubbard models with a strong negative U attraction [35]. Similar models were recently studied in refs. [11,36], using approaches based on large lattice connectivity. Our calculation scheme below can easily be generalized to grains or islands hosting many particles, as long as the charge gap on typical grains is much bigger than the hopping amplitude between grains.

We consider a lattice of sites i with random energies ε_i, uniformly distributed in $[-W,W]$, and weakly coupled by a tunneling amplitude $t_{ij}=t$ between nearest neighbors,

$$H=\sum_i \varepsilon_i n_i-\sum_{\langle i,j\rangle} t_{ij}(b_j^\dagger b_i+b_i^\dagger b_j),\quad n_i=b_i^\dagger b_i. \qquad (1)$$

$b_i^\dagger, b_i$ are creation and annihilation operators of fermions or hard-core bosons, respectively. They satisfy $b_i^2=0$, and the commutation relations $[b_i,b_j]_B=0$, $[b_i^\dagger,b_j]_B=\delta_{ij}(-1)^{B(1-n_i)}$, where $[.,.]_B$ denotes the commutator for bosons ($B=1$) and the anticommutator for fermions ($B=0$), respectively[1]. In the presence of a magnetic field, the hopping acquires a phase $t_{ij}=te^{-i\phi_{ij}}$, the sum of ϕ_{ij} around a plaquette being proportional to the flux threading it.

The important role of quantum statistics on magnetoresistance was noted early on by Zhao *et al.* [4], where low-energy excitations were discussed. Here, we introduce an efficient new formalism, which allows us to give a rigorous derivation of their prediction, generalize it to finite-energy excitations and gain insight on bosonic delocalization. The formalism is easily extendible to treat subleading corrections [37], and can be applied to many other disordered systems as well, as we will exemplify on the Ising model in random transverse fields.

We focus on the strongly insulating regime $t\ll W$. In the limit $t=0$ elementary excitations correspond to the addition or removal of a particle on given sites. For small hopping $t/W\ll 1$, these adiabatically deform into dressed excitations, which are still well localized in space. The spatial properties of such many-body excitations are well captured by the retarded Green's function

$$G^R_{i,0}(t-t')=-i\Theta(t-t')\langle[b_i(t),b_0^\dagger(t')]_B\rangle, \qquad (2)$$

where $A(t)=e^{iHt}A(0)e^{-iHt}$. It describes the amplitude of finding an extra particle at site i, after a time t of adding a particle on site 0. To characterize the spatial decay of an excitation of given energy, one preferably works in frequency space, $G^R_{i,0}(\omega)=\int_{-\infty}^{\infty}G^R_{i,0}(t)e^{i\omega t}\mathrm{d}t$, and extracting the relevant pole. Having hopping conductivity in mind, we are interested in low-energy excitations, $\omega\ll W$.

We now analyze $G^R_{i,0}$ perturbatively in t_{ij}, which is justified deep in the insulator. Similarly as in early works of the Hubbard model [38], we study the equation of motion

$$i\frac{\mathrm{d}}{\mathrm{d}t}G^R_{i,0}(t-t') = \delta(t-t')\delta_{i,0}\langle[b_0(0),b_0^\dagger(0)]_B\rangle -i\Theta(t-t')\langle[i\dot b_i(t),b_0^\dagger(t')]_B\rangle, \qquad (3)$$

as a convenient starting point for a locator expansion in powers of the hopping t/W [2]. This technique can easily be generalized to analyze other random field systems, too. It is easy to show that

$$i\dot b_i(t)=[b_i(t),H]=\varepsilon_i b_i(t)-(-1)^{Bn_i(t)}\sum_{j\in\partial i}t_{ij}b_j(t), \qquad (4)$$

where the sum runs over all neighboring sites of i.

To leading order in t/W, we can restrict ourselves to forward-scattering paths, in analogy to the fermionic (single-particle) study by Nguyen *et al.* [3]. Hence, in eq. (4) we retain only the neighbors j, which are closest to 0, cf. fig. 1. To leading order in t/W we can decouple the sign factor in (4) and use $\langle(-1)^{n_i(t)}\ldots\rangle=\mathrm{sign}(\varepsilon_i)\langle\ldots\rangle+O((t/W)^2)$ in eq. (3) to obtain the recursion relation

$$G^R_{i,0}(\omega)\approx\sum_{j\in\partial i,\mathrm{dist}(j,0)<\mathrm{dist}(i,0)}\frac{t_{ij}\,[\mathrm{sign}(\varepsilon_i)]^B}{\varepsilon_i-\omega}G^R_{j,0}(\omega). \qquad (5)$$

This is easy to evaluate by a transfer matrix computation. Upon iteration of this forward-scattering approximation, we obtain $G^R_{i,0}$ as a sum over all shortest paths $\mathcal{P}$ (of length ℓ) between sites 0 and i, which is *exact* to leading order in t/W,

$$\frac{G^R_{i,0}(\omega)}{G^R_{0,0}(\omega)}=\sum_{\mathcal{P}=\{j_0=0,\ldots,j_\ell=i\}}\prod_{p=1}^{\ell}\frac{t_{j_{p-1},j_p}[\mathrm{sgn}(\varepsilon_{j_p})]^B}{\varepsilon_{j_p}-\omega}. \qquad (6)$$

[1] For hard-core bosons, after the standard mapping to spin $s=1/2$ particles, $n_i-1/2\to s_i^z$, $b_i\to s_i^-$, $b_i^\dagger\to s_i^+$, the commutation relations for $b,b^\dagger$ translate to the usual spin algebra: $[s_i^\alpha,s_j^\beta]=\delta_{ij}\epsilon_{\alpha\beta\gamma}s_i^\gamma$.

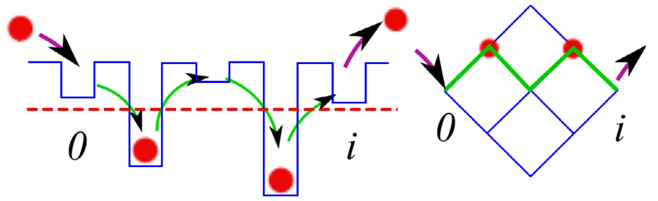

Fig. 1: (Colour on-line) In a coherent hopping process many particles move by one slot to the next negative-energy site (the process shown in the left panel corresponds to the highlighted paths on the right panel). The many-body nature of this process is responsible for the statistical sign difference between bosons and fermions. To leading order in the hopping, many alternative paths interfere in the Green's function G^R_{i0} between sites $0, i$ (6 paths in the right panel). The sign of fermion amplitudes depends on the number of occupied sites (indicated by filled circles) on each path whereas bosonic paths all have positive amplitudes at low energy. A magnetic field suppresses the maximally constructive interference of bosons.

By setting $\omega \to \varepsilon_0$ and extracting the residue of the corresponding pole in $G^R_{i,0}$, we find the "wave function" of the quasiparticle excitation, which is adiabatically connected to a boson insertion or removal at site 0 in the limit $t=0$. This is easily seen from a Lehman decomposition of the Green's function. Note that it would be highly non-trivial to derive the result (6) from a naive perturbation theory for $G^R_{i,0}$. This may be appreciated from fig. 1 which illustrates the forward scattering and its many-particle nature on a selected path. Remarkably, the exponentially many virtual trajectories between initial and final state sum up to the single product in eq. (6).

At low temperatures, transport of bosons is expected to proceed via variable range hopping, which is very sensitive to the localization length ξ of excitations. For non-interacting fermions it is defined as the (log-averaged) inverse spatial decay rate of single-particle wave function amplitudes. For hard-core bosons, ξ is naturally generalized to be the typical inverse decay rate of $G^R_{i,0}$ with distance

$$1/\xi(\varepsilon_0) = -\lim_{\vec{r}_i \to \infty} \frac{\overline{\ln |G^R_{i,0}(\omega)/G^R_{0,0}(\omega)|}_{\omega \to \varepsilon_0}}{|\vec{r}_i - \vec{r}_0|}, \qquad (7)$$

the overbar denoting disorder average. Setting $\omega \to \varepsilon_0$ selects the decay rate of the excitation centered at site 0.

Note that setting $B=0$, eq. (6) reproduces the well-known result for non-interacting fermions [2,3], which can also be extended to repulsive interactions [6]. In contrast, hard-core bosons differ crucially in the sign of the amplitude contributed by the paths. The difference is easy to understand, cf. fig. 1. In order to observe a particle at site i after inserting a particle at 0, all the $n_{\mathcal{P}} \equiv \sum_{k=1}^{\ell} n_k \approx \sum_{k=1}^{\ell} [1 - \mathrm{sgn}(\varepsilon_k)]/2$ particles on the path $\mathcal{P}$ have to move to the next negative-energy site closer to site i. Upon retrieving a particle at site i, a ring exchange of $n_{\mathcal{P}}$ particles has been carried out in the ground state, which causes the sign difference $(-1)^{n_{\mathcal{P}}}$ between bosonic and fermionic amplitudes.

In the impurity band model of eq. (1), this feature distinguishes clearly between bosons and fermions. However, we should mention that sums over positive paths can also occur in fermionic problems [39]. Such situations arise, when all sites between 0 and i have energies above the considered ω. This occurs, *e.g.*, in lightly doped semiconductor solutions, where impurity states tunnel through the bottom of the disordered conduction band; or in impurity bands with chemical potential very close to the band edges.

Effects of quantum statistics and magnetoresistance. – Equation (6) shows that for low-energy bosonic excitations, $\omega \to 0$, in the absence of a magnetic field, all paths interfere constructively, unlike in typical fermionic situations. This may be seen as a precursor of the establishment of global phase coherence in a superfluid phase. A simple consequence of this difference is that in the same disorder potential hard-core bosons always have larger localization length than fermions of the same mass.

The difference in path signs has also a crucial effect on magnetoresistance, as was also noted (for $\omega = 0$) in refs. [4,36]. It manifests itself prominently in a strong opposite response to a magnetic field H depending on the statistics of carriers. It is well known that hopping fermions experience an (exponentially strong) negative magnetoresistance due to the suppression of destructive interference [3,5,7]. In contrast, the magnetoresistance of bosons is positive, since the phases in the hopping amplitudes reduce the constructive interference of paths that connect the low-energy sites relevant for transport.

As long as the relevant hopping distance $r = R_{\rm hop}$ is small, magnetoresistance is weak, since only a small fraction of a flux quantum threads the wave function on that scale. However, while fermions react with a non-analytic increase $\Delta\overline{\log G} \sim |H|$ (due to destruction of nearly perfect negative interference of competing paths [3]), bosons display a smaller, analytic response of opposite sign, $\Delta\overline{\log G} \sim -p_R \Phi_R^2 \sim H^2 R^3_{\rm hop}$. Here, $\Phi_R \sim H R^{1+\zeta}$ is the flux through a pair of typical tunneling paths of length R, while $p_R \sim R^{-\theta}$ is the probability for two paths to interfere significantly. $\zeta \geqslant 1/2$ is the wandering or roughness exponent of directed paths in a disordered environment, while the exponent $\theta = 2\zeta - 1$ follows from a standard scaling relation [40]. In the earlier literature on fermion scattering problems with positive amplitudes [39], the result $\Delta\overline{\log G} \sim H^2 R^3_{\rm hop}$ was obtained for the special case of weak disorder where the paths behave like random walks with $\zeta = 1/2$.

For larger $R_{\rm hop}$ or stronger H (see footnote [2]) the interference effects actually *shrink* the boson localization

[2]The crossover occurs when $Hr^{1+\zeta} = O(1)$, contrary to assumptions in the earlier literature [8], which predicted it at $Hr^{3/2} = O(1)$, viewing paths as random walks with $\zeta = 1/2$. This is, however, not appropriate in strong disorder [41].

length, in analogy to the opposite effect in fermions, *i.e.*, $\xi(H)-\xi(0)\propto(-1)^B H^\alpha$ [4,8]. This leads to *exponentially amplified, giant magnetoresistance* in the low-temperature hopping regime [41], where the hopping length is proportional to an inverse power of T. Hence, under a magnetic field, the typical hopping resistance $\sim\exp[R_{\rm hop}/\xi(B)]$ increases by a large factor, while the resistance of fermions typically decreases by a (significantly smaller) factor.

The opposite interference in bosons and fermions is very likely to be a key element for understanding the giant magnetoresistance peak in disordered films with remnant pairing. As long as the magnetic field does not destroy the localized pairs, it mainly reduces their localization length. Upon destruction of the pairs, *e.g.*, by the Zeeman effect, the predominant carriers are fermions, for which a negative magnetoresistance due to an increasing localization length is predicted [3,7]. Once the latter becomes large, the physics of loops (neglected in the forward-scattering approximation) is likely to play a role in the negative magnetoresistance, as well. In this regime, effects of Coulomb interactions [42], and the necessity of electrons to tunnel around or through remnant superconducting islands may enhance the negative magnetoresistance even further [31].

Note that the mechanism of positive magnetoresistance discussed above is based on transport via purely bosonic carriers (pairs of electrons). This differs from other theoretical scenarii [31] where the bottleneck of resistance is due to the transfer of single electrons between remnant superconducting islands, and the positive magnetoresistance is ascribed to the shrinking of those islands.

Purely bosonic transport is suggested by recent experiments on periodically patterned films of Bi or InO_x [20,24]. Indeed, the observed oscillations of magnetoresistance start with an upturn, as expected for bosons, in contrast to the downturn characteristic for fermions. More importantly yet, the oscillations come with a flux periodicity corresponding to carriers with charge $2e$, suggesting that transport is carried by "pairs" of electrons.

Energy dependence of ξ. – An interesting consequence of eq. (6) is the prediction that for bosons $\xi(\omega)$ has a non-trivial energy dependence around $\omega=0$. Indeed it reaches a *maximum* at $\omega=0$, as we confirmed numerically in fig. 2. The presence of other bosons thus enhances the delocalization tendency of an extra particle at low energy, in contrast to non-interacting fermions which are essentially insensitive to the position of the Fermi level. Note that at higher-energies bosonic excitations tend to behave like non-interacting particles, since paths through occupied sites become negligible.

So far the above discussion of $\xi(\omega)$ was based on the forward-scattering approximation, which yields the recursive relation (5) between Green's functions at the *same* ω, almost like in a non-interacting problem. This observation can be used to define an effective single-particle Hamiltonian with complex hopping amplitudes

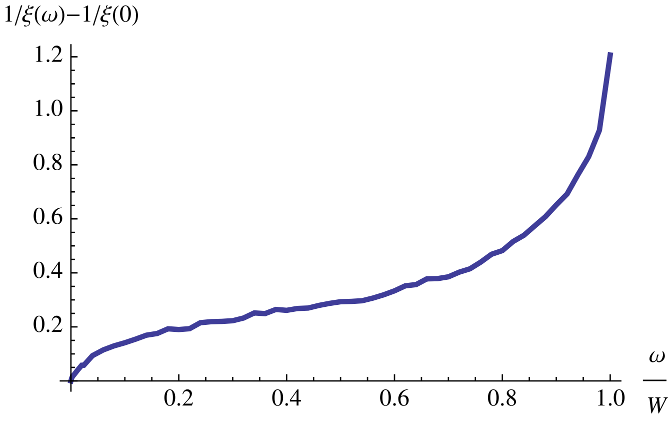

Fig. 2: (Colour on-line) Disorder averaged spatial decay rates ξ^{-1} of bosonic excitations along the diagonal of a square lattice. ξ is computed from eqs. (6), (7) as a function of energy ω and measured in units of inverse lattice spacing. Note that the excitations of lowest energy are the most delocalized.

$t_{ij}=t[\mathrm{sign}(\epsilon_i\epsilon_j)]^{1/2}$, which generates the same expression as in the interacting hard-core boson problem for *all*, not only the shortest, non-intersecting paths. The study of such effective non-Hermitian Hamiltonians is an interesting subject for future studies.

Approach to superfluidity and delocalization. – The predicted decrease of $\xi(\omega)$ with increasing ω may appear counterintuitive at first sight, since at higher energies more phase space is available, which generally favors delocalization. However, one should interpret the phenomenon of a decreasing $\xi(\omega)$ as a precursor of incipient long-range order, which will eventually establish at $\omega=0$, and favors propagation at low frequencies in local "precondensates". In the closely related random transverse field Ising model, cf. eq. (8) below, exact results for localization properties are available in 1d, due to the mapping to free fermions. Those exhibit indeed the same qualitative behavior of $\xi(\omega)$ [43]. These results contradict the predictions of refs. [11,13], which argued that the presence of a sea of hard-core bosons impedes the propagation of an extra boson injected at low energy[3]. However, while this reasoning would be correct for a *distinguishable* extra particle, it neglects exchange effects of identical bosons, which instead lead to *enhanced* propagation at low energies.

The locator expansion is helpful to understand qualitatively another aspect of bosonic (de)localization: How do bosons escape localization in $d=2$, while repulsive fermions are believed to always localize in the absence of special symmetries? Usually one argues that superfluids in 2d are stable to weak disorder, which proves their delocalization [44]. The approach of this work complements this view from the insulating side. At low energies all

[3]Reference [11] arrived at such a result by restricting the perturbation series of $G^R_{i0}(\omega)$ to virtual one-site excitations, instead of summing the exponentially many ways in which a minimal number of sequential moves lead to the many-particle rearrangment in fig. 1.

scattering paths interfere constructively, which is a precursor of the establishment of a global phase in the superfluid. This is very different from fermions where the various scattering paths have nearly random signs, such that quantum interferences essentially average out, *except* for paths returning back to the origin. For the latter, "time-reversed paths" (*i.e.*, sequences of scattering states encountered in perturbation theory, and their reverse) are guaranteed to have the same scattering amplitude in the absence of magnetic fields. Their positive interference thus systematically enhances the return to the origin and therefore localization. In contrast, boson propagation at low energy *always* involves positive interference of alternative paths, such that the return to the origin is not particularly enhanced as compared to other propagation channels.

Let us now attempt to obtain insight on the approach to superfluidity. We may use the locator expansion technique to revisit the problem of hard-core bosons on Cayley trees of large connectivity $K \gg 1$, as considered in ref. [11]. Such high-connectivity lattices are indeed interesting since they enable one to use the forward-scattering approximation even parametrically close to the superfluid transition. They thus yield insight into how bosonic excitations approach delocalization, and how this differs from the exactly solvable case of free fermions [45].

In finite dimensions, superfluidity sets in when $\xi(\omega=0)$ diverges. On the Cayley tree, the criterion generalizes to $\xi^{-1}_{\omega=0} = \lim_{R\to\infty} R^{-1}\overline{\ln[\sum_{i,\mathrm{dist}(i,0)=R} G^R_{i0}(\omega=0)]} = 0$. This can be evaluated by a mapping to a directed polymer [11,46], which is exact within the forward-scattering approximation. Due to the absence of loops, any two sites are connected by a unique shortest path. Hence, interference phenomena are subleading in the hopping. To leading order quantum statistics is therefore irrelevant, and one finds localization properties like for free fermions (as characterized by $|G^R_{i0}(\omega)|^2$ at large distances) and a superfluid transition at the same value at which non-interacting fermions delocalize, $(t/W)_c = O(1/K\ln(K))$ [45]. However, a study of subleading corrections shows that bosons actually delocalize already at a weaker hopping strength than fermions [37]. Like for the critical wave functions of the fermionic problem, one finds that the emerging Bose condensate is extremely sparse, as pointed out in ref. [11].

In the insulator, the leading-order locator expansion shows that the typical propagator at finite energies, $G^R_{i0}(\omega>0)$, always decays faster than $G^R_{i0}(0)$, if a uniform distribution of random energies ($\rho(\epsilon\approx\mu)=\text{const}$) and chemical potential $\mu=0$ (half-filling) is assumed. However, the range of ω, for which this leading-order result is controlled, gradually decreases to zero upon approaching the phase transition. A complete description of criticality at finite energies would require the resummation of very high orders of perturbation theory. Nevertheless, this result is suggestive of the possibility that the superfluid emerges out of the insulator by a delocalization phenomenon at $\omega=0$, while slightly higher (intensive) excitations are still localized —a scenario which we indeed find below for Ising models. This contrasts with the scenario of a mobility gap in the insulator, that closes at the transition, as proposed in [11,13], and similar early ideas by Hertz *et al.* [47]. All of those neglected the above-discussed exchange effects of identical particles at low energies.

The present calculations do not provide any evidence for such a mobility edge at higher (intensive) energies in the presence of uniform disorder. However, an intensive mobility edge, and even a closing mobility gap, does arise rather trivially if the density of on-site energies increases sufficiently strongly across the chemical potential. In such cases the forward-scattering approximation on the Cayley tree indeed predicts an intensive mobility edge that closes at criticality, very much like at a standard Anderson transition of fermions.

Locator expansion for Ising spins. – Some of the qualitative features found for hard-core bosons also hold for the closely related random transverse field Ising model,

$$H_{\text{Ising}} = \sum_i \varepsilon_i s^z_i - 4\sum_{\langle i,j\rangle} t_{ij} s^x_i s^x_j, \tag{8}$$

even though its critical behavior turns out to be rather different [48]. This model differs from the disordered XY model of eq. (1) only by the replacement $2(s^x_i s^x_j + s^y_i s^y_j) \to 4 s^x_i s^x_j$ (after applying to eq. (1) the standard mapping between hard-core bosons and $s=1/2$ spins (see footnote [1]) In this case, analogous steps as in eqs. (3)–(6), applied to correlators $\langle [s^x_i(t), s^x_0(0)]\rangle$, yield a sum over shortest paths, with amplitudes given by products of factors $2|\varepsilon_i|/(\varepsilon_i^2-\omega^2)$, that replace the XY locator $\mathrm{sign}(\varepsilon_i)/(\varepsilon_i-\omega)$ in eq. (6). This differs from the factor $2/(|\varepsilon_i|-\omega)$ postulated in refs. [11], which incorrectly predicted an intensive mobility edge that closes at the transition point. Taking into account the effects of higher-order terms in the expansion in exchange coupling and the special role of rare events in Ising models, which are well known from 1d chains and strong randomness approaches [43,48–50], we find instead that the ordering transition is initiated by a delocalization at $\omega=0$, while slightly higher-lying intensive excitations are still localized in the paramagnet. On the other hand, we cannot reliably analyze high but finite energies, as this requires full control over very high orders of perturbation theory.

Reference [51] attempted to address the question of a mobility edge in the disordered phase. Numerically studying certain Ising models on small random graphs with $K=2$, the authors claimed to find a mobility edge at intensive energies [52]. However, it remained unclear whether this mobility edge was actually found on the paramagnetic side of transition, and if so, whether the mobility gap remains finite at the phase transition (as predicted here). Similar studies in finite-dimensional Ising and XY models would therefore be very desirable to determine whether also there delocalization is initiated at $\omega=0$ in Ising models, and whether the same holds for XY models.

Conclusion. – We have shown that strongly disordered bosons respond oppositely to a magnetic field than fermions in impurity bands, which makes magnetoresistance a measurement of choice to detect the statistics of the charge carriers in an insulator. We hope that the non-trivial dependence of the localization length $\xi(\omega, H)$ on energy, magnetic field and statistics will be studied in superconducting films or in cold bosonic atoms [9], where artificial "magnetic" gauge fields can be generated by various techniques [10]. The positivity of bosonic tunneling amplitudes at zero energy furnishes an intuitive understanding of why bosons escape localization in 2d. Applying the locator expansion to Ising systems as well, we predict that long-range–ordered phases may emerge from quantum disordered phases by delocalizing and condensing at $\omega = 0$, without the closure of a pre-existing intensive mobility edge.

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