Anomalous Diffusion and Griffiths Effects Near the Many-Body Localization Transition

Kartiek Agarwal,[1,*] Sarang Gopalakrishnan,[1] Michael Knap,[1,2] Markus Müller,[3,4] and Eugene Demler[1]
[1]Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA
[2]ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138, USA
[3]The Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, 34151 Trieste, Italy
[4]Department of Physics, University of Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland
(Received 20 August 2014; revised manuscript received 20 February 2015; published 23 April 2015)

We explore the high-temperature dynamics of the disordered, one-dimensional *XXZ* model near the many-body localization (MBL) transition, focusing on the delocalized (i.e., "metallic") phase. In the vicinity of the transition, we find that this phase has the following properties: (i) local magnetization fluctuations relax subdiffusively; (ii) the ac conductivity vanishes near zero frequency as a power law; and (iii) the distribution of resistivities becomes increasingly broad at low frequencies, approaching a power law in the zero-frequency limit. We argue that these effects can be understood in a unified way if the metallic phase near the MBL transition is a quantum Griffiths phase. We establish scaling relations between the associated exponents, assuming a scaling form of the spin-diffusion propagator. A phenomenological classical resistor-capacitor model captures all the essential features.

DOI: 10.1103/PhysRevLett.114.160401 PACS numbers: 75.10.Pq, 05.30.Rt, 05.60.Gg, 72.15.Rn

Noninteracting electrons in disordered media display a uniquely quantum phenomenon known as Anderson localization [1]; when all electronic states are Anderson localized, dc transport is absent. Evidence from perturbative [2–4], numerical [5–8], and rigorous mathematical approaches [9] suggests that the main features of Anderson localization (in particular, the absence of diffusion and dc transport) persist in the presence of interactions. The resulting phase, known as the many-body localized (MBL) phase [10], has a number of remarkable features: a system in the MBL phase is nonergodic—i.e., its many-body eigenstates violate the eigenstate thermalization hypothesis [8,11,12]—and supports extensively many local conserved quantities [13–18]. Consequences of MBL such as slow entanglement growth [13,19–21] and unconventional phase transitions [22–29] have been analyzed and their experimental implications discussed [30–35].

While there has been a great deal of recent work establishing the existence and properties of the MBL phase, little is known about the transition between the MBL and delocalized phases. It is expected that, for sufficiently weak disorder and strong interactions [3,4], eigenstates should remain ergodic and transport should be diffusive, as in clean nonintegrable systems [36]. However, it has been proposed that diffusivity and/or ergodicity may break down as the MBL transition is approached [37,38], even before transport vanishes. Thus, there might be an intermediate phase, or phases, between the conventional metallic phase and the MBL phase.

In this Letter, we provide numerical evidence that an intermediate, nondiffusive phase, indeed exists. To this end, we examine the dynamical properties of the random-field, spin-$1/2$ *XXZ* chain at intermediate disorder strengths (i.e., in the vicinity of the MBL transition), using exact diagonalization. In particular, we examine the infinite-temperature, low-frequency behavior of the optical conductivity $\sigma(\omega)$, and the long-time dependence of the return probability $C_{zz}(t)$. These probes are complementary: $\sigma(\omega)$ probes long-wavelength behavior, while $C_{zz}(t)$ is a local probe.

Our numerical results indicate that both of these quantities exhibit anomalous power laws that vary smoothly as a function of the disorder strength in this intermediate regime. Specifically, $\sigma(\omega) \sim \omega^{\alpha}$ and $C_{zz}(t) \sim t^{-\beta}$, with the scaling relation $\alpha + 2\beta = 1$ (Fig. 1). Furthermore, we

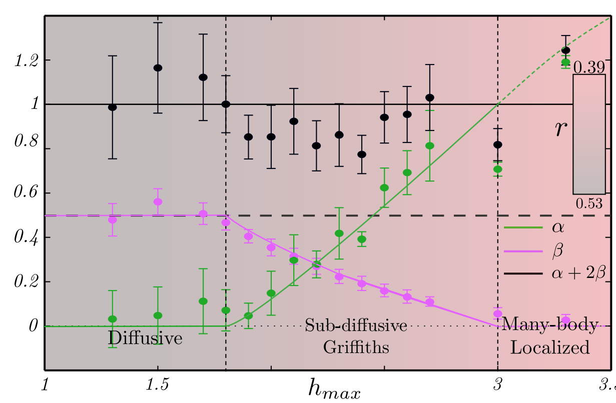

FIG. 1 (color online). The phase diagram of the random field, XXZ model for $J_z = 1$. $h_{\max}$ characterizes disorder strength. As the disorder is increased, the system transitions smoothly into a subdiffusive Griffiths-like phase with an anomalous diffusion exponent β and exponent α characterizing low-frequency optical conductivity, which satisfy the scaling relation $\alpha + 2\beta = 1$. The MBL transition is predicted to occur where $\sigma(\omega) \sim \omega$. Within precision, it coincides with the transition point determined from the level statistics parameter r (see main text).

compute the full distribution $D[\rho(\omega)]$ of resistivities ρ at a fixed sample size as a function of frequency; we find that the width of this distribution *diverges* in the low-frequency limit as $\Delta\rho(\omega) \sim 1/\omega^{\alpha'}$. Such behavior is characteristic of a quantum Griffiths phase [39], in which power-law correlations emerge due to the interplay between the exponential rareness of large insulating regions and their exponentially large resistance. We account for these scaling relations by postulating a scaling form of the spin-diffusion propagator and a phenomenological resistor-capacitor model with power-law-distributed resistors.

As our work was nearing completion, a related numerical study by Bar-Lev *et al.* [40] appeared. While our numerical results are consistent with those of Ref. [40], we are also able to provide an analytic understanding of the subdiffusive phase (see also Ref. [41]).

Numerical simulations.—We work with the XXZ model given by the Hamiltonian

$$H = \sum_i h_i S_i^z + \sum_{\langle ij\rangle} J(S_i^x S_j^x + S_i^y S_j^y) + J_z S_i^z S_j^z, \qquad (1)$$

where $\langle ij\rangle$ implies sites i and j are nearest neighbors. The local magnetic field values h_i are picked uniformly from the range $[-h_{\max}, h_{\max}]$; $h_{\max}$ characterizes the strength of the disorder. The exponents α, β presented in Fig. 1 were extrapolated from finite size results computed using system sizes $L = 12, 14, 16$ (see the Supplemental Material [42]), while results in Fig. 2 correspond to $L = 14$. We use $J = 1$ as the unit of energy, and choose the interaction strength to be close to the Heisenberg point, $J_z \lesssim 1$, as finite-size effects are more severe for $J_z/J \ll 1$. The XXZ chain is expected to exhibit an infinite-temperature transition to the MBL phase at a critical $h_{\max}$ [5]. In what follows, we restrict ourselves to infinite temperature and choose the subspace of total magnetization $\sum_i S_i^z = 0$.

The real part of the optical conductivity $\sigma(\omega)$ in linear response reads

$$T\sigma(\omega) = \frac{T}{L}\frac{1-e^{-\omega/T}}{\omega}\sum_{mn}\frac{e^{-\beta E_n}}{Z}\left|\langle m|\sum_i j_i|n\rangle\right|^2 \delta(\omega-\omega_{mn}),$$
$$\overset{T\to\infty}{\approx} \frac{1}{LZ}\sum_{mn}\left|\langle m|\sum_i j_i|n\rangle\right|^2 \delta(\omega-\omega_{mn}), \qquad (2)$$

where m, n are the many-body eigenstates of the system with energies E_m, E_n, which we evaluate using exact diagonalization, $\omega_{mn} = E_m - E_n$, and T is the temperature (we set $\hbar = k_B = 1$). The first line of Eq. (2) is the Lehmann representation of $T\sigma(\omega)$, given in terms of a sum over local current operators j_i, which are related to the spin operators using the continuity equations, $j_i - j_{i+1} = \partial_t S_i^z$. The second line of Eq. (2) is the limiting behavior of $T\sigma(\omega)$ as $T \to \infty$. In the remainder of the Letter, the factor T is implicitly understood when we refer to the conductivity $\sigma(\omega)$. In our numerics, we use a Lorentzian form for the δ function with a width $\eta \sim \Delta/10^2$, where $\Delta = h_{\max}\sqrt{L}/2^L$ is approximately the average level spacing $\sim 10^{-3}$–10^{-2} for the system size $L \sim 14$ and disorder strengths $h_{\max} \sim 1.5$–3.5 that we explore. The precise value of η is unimportant, so long as it is appreciably smaller than Δ (see the Supplemental Material [42]). The return probability, $C_{zz}^i(t)$, is defined as $C_{zz}^i(t) = 4\langle S_i^z(t) S_i^z(0)\rangle$, where i is any site on the chain. Since we are interested in describing the phase close to the MBL transition, we also require an additional, independent, method to identify the transition point.

Following Ref. [5], we consider the level statistics parameter $r_m = \delta_-^m/\delta_+^m$, where $\delta_\pm^m$ are the energy differences between eigenstate m and the two adjacent eigenstates with $\delta_-^m < \delta_+^m$. The average over all eigenstates m, $r = \langle r_m\rangle$, is known to assume different values, $r \sim 0.39$ and $r \sim 0.53$ in the cases of the MBL and the conducting phase, respectively. We crudely estimate the MBL transition as the point when r is halfway between these values, as determined for a $L = 16$ system (dashed line at $h_{\max} \approx 3$ in Fig. 1).

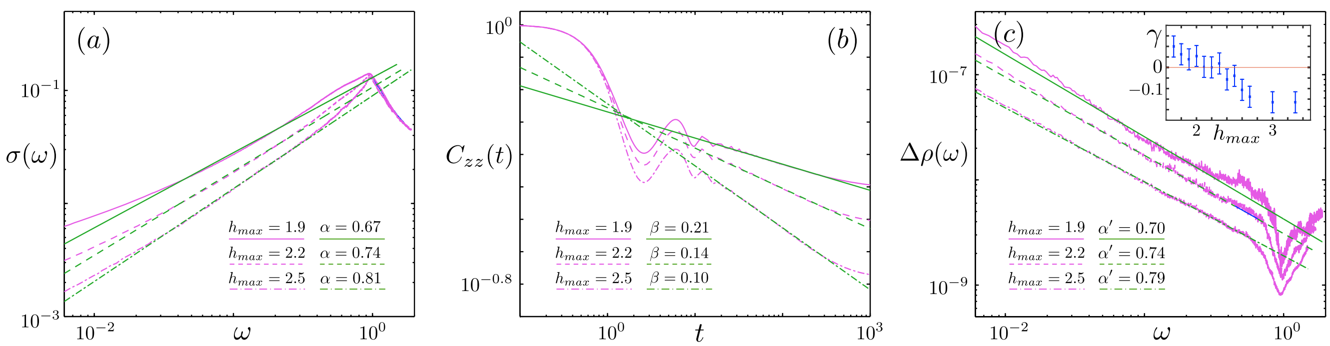

FIG. 2 (color online). Behavior of (a) optical conductivity $\sigma(\omega)$, (b) return probability $C_{zz}(t)$, and (c) width $\Delta\rho(\omega)$ of the distribution of resistivities as a function of frequency (magenta). All plots are for $J_z = 0.8$, and disorder strengths as indicated in the legend. The fits (green) are power laws of the form $C_{zz} \sim 1/t^\beta$, $\sigma(\omega) \sim \omega^\alpha$, and $\Delta\rho(\omega) \sim 1/\omega^{\alpha'}$. The inset in (c) shows the relative power $\gamma = \alpha' - \alpha$ governing the scaling of the ratio $\Delta\rho(\omega)/\langle\rho(\omega)\rangle$ of the width and the mean of the resistivity distribution.

Numerical results.—Our numerical results on the dynamic observables are summarized in Fig. 2. Both the optical conductivity and the return probability obey power-law behavior over multiple decades. For the exponents defined by $\sigma(\omega) \sim \omega^{\alpha}$ and $C_{zz}(t) \sim 1/t^{\beta}$, respectively, we numerically find the scaling relation $\alpha + 2\beta \approx 1$; the physical origin of this relation is discussed below. This scaling relation also holds in the diffusive regime, where one expects that $C_{zz}(t) \sim 1/\sqrt{t}$ and $\sigma(\omega) \sim \text{const}$ (thus, $\alpha = 0$, $\beta = 1/2$). Our results approach diffusive values at small disorder $h_{\max} \approx 1.5$, but we are unable to extract reliable power laws in this limit. As the system approaches the MBL transition, α continuously increases toward 1 and β decreases to 0; $\beta = 0$ implies the absence of relaxation, and marks the transition into the MBL phase. The trend suggests that at the MBL transition $\sigma(\omega) \sim \omega$; this differs sharply from the expectation for a noninteracting Anderson insulator [48], viz. $\sigma(\omega) \sim \omega^2 \log^2(\omega)$.

Next, we look at the distribution $D[\rho(\omega)]$ of resistivities ρ, at a fixed frequency ω. We find that the distribution of resistivities at a fixed sample size ($L = 14$) grows increasingly broad at low frequencies; the standard deviation of sample resistivities at a fixed frequency ω diverges as $\omega^{-\alpha'}$; see Fig. 2. At low frequencies, sufficiently high moments of the resistivity distribution become ill-defined and we expect that the distribution approaches a power law $P(\rho) \sim \rho^{-\tau}$ in this limit. While we are unable to reliably extract the exponent τ from the data, we will argue below that our numerical results can be explained using a resistor-capacitor network which predicts that $\tau = 2/(\alpha + 1)$ in the Griffiths regime.

Origin of scaling relation $\alpha + 2\beta = 1$.—The origin of the relation $\alpha + 2\beta = 1$ can be understood as follows. The relation between length and time in the subdiffusive phase can be written in the "diffusive" form $x^2 \sim D(t)t$, where $D(t)$ is a time-dependent diffusion constant. At long times, the scaling of return probability implies that $x^2 \sim t^{2\beta}$ in the subdiffusive phase. Thus, $D(t) \sim t^{2\beta-1}$, and from the Einstein relation $\sigma(\omega) \sim D(t = 2\pi/\omega) \sim \omega^{1-2\beta}$.

More generally, the relation $\alpha + 2\beta = 1$ follows if one assumes that the average spin density propagator takes the scaling form $G(x,t) \sim t^{-\beta}\phi(x/t^{\beta})$ [or, equivalently, $G(k,\omega) \sim \omega^{-1}\tilde{\phi}(k/\omega^{\beta})$], i.e., if one stipulates that lengths and times are related exclusively through the dynamical exponent $z = 1/\beta$. A dynamical exponent that smoothly varies with disorder strength is reminiscent of the zero-temperature Griffiths phase in random magnets (see Ref. [43] and references therein). We assume further that the static compressibility of the system evolves smoothly near the MBL transition as expected for a high-temperature system with short-range interactions. From these assumptions, it follows (see the Supplemental Material [42]) that the dynamic structure factor $S(k,\omega) \sim \omega^{-1}h(k/\omega^{\beta})$ (where h is another scaling function; $h(0)$ is finite). The dynamic structure factor is closely related to the momentum-dependent conductivity [49]: specifically, $\sigma(k,\omega) \sim \omega^2\partial_k^2 S(k,\omega) = \omega\partial_k^2 h(k/\omega^{\beta})$. Again, it follows that $\sigma(k = 0, \omega) \sim \omega^{1-2\beta}$.

Griffiths-phase interpretation.—The reasoning above related α to β, but did not account for the subdiffusive behavior itself. We now provide an interpretation of subdiffusion in terms of Griffiths effects. Near the MBL transition, one expects the system to consist of metallic segments separated by insulating barriers, i.e., local regions where the system parameters favor localization. Barriers through which the tunneling time is $\gtrsim t$ confine the magnetization at the time scale t. The scaling between length and time suggests that the average distance between such insulating barriers is $d(t) \sim t^{\beta}$. As long as $\beta < 1/2$, the time t spent to tunnel through the insulating regions is parametrically larger than the time scale to diffuse between barriers, $t \gg d(t)^2 \sim t^{2\beta}$. The long time dynamics is therefore limited by these rare insulating regions. When approaching the diffusive limit $\beta = 1/2$, the separation between barrier tunneling and diffusion time does not exist. Thus rare barriers cannot be defined and transport is simply governed by diffusion. From these considerations it follows that a local charge excess decays to $1/d(t)$ in time t which yields a return probability $C_{zz}(t) \sim 1/t^{\beta}$.

This picture of insulating barriers also yields the correct scaling of the optical conductivity. We imagine that we apply a field E that flips at a frequency $\omega \sim 1/t$. Between such flips, the charge equilibrates to the (approximately) linear potential gradient set up by the field between insulating barriers separated by a length $d(t)$. Thus, at a position x, the charge flips between $\pm Ex/T$ in time t. This requires average current densities of order $|j| \sim Ed(t)^2/tT$, a fraction of which is in phase with the applied field and thus gives rise to scaling of the real part of the conductivity, $\sigma(\omega) \sim \omega^{1-2\beta}$.

Note that this Griffiths picture is qualitatively *distinct* from the situation where only certain rare sites exhibit slow decay, while on most sites magnetization decays rapidly. In such a scenario, the subdiffusive behavior would only show up in the average and not the *typical* return probability. Instead, we find subdiffusive decay in both typical and average correlations (see the Supplemental Material [42]).

RC model.—We now introduce a classical RC network [50] that captures this Griffiths physics and reproduces all essential features of our numerical data. The model [Fig. 3(a)] consists of a chain of resistors with a distribution $P(R) \sim R^{-\tau}$, each connected to the ground by a capacitor with a constant capacitance C. A power-law distribution of resistances can arise naturally in the physical system, as follows. Suppose the resistance is dominated by randomly distributed but identical tunneling barriers, such that each site has a probability p of being a barrier. The probability of finding a string of N consecutive barriers is then p^N. Standard semiclassical arguments suggest that, if the

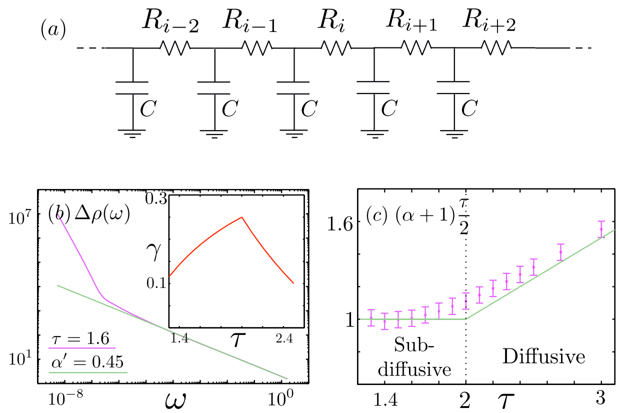

FIG. 3 (color online). (a) The RC model. (b) Width of the finite frequency resistivity distribution $\Delta\rho(\omega)$ (magenta) and its asymptotic form $1/\omega^{\alpha'}$ (green) plotted as a function of frequency. The inset shows $\gamma = \alpha' - \alpha \gtrsim 0$, where α is the exponent of the average resistivity $\bar{\rho}(\omega) \sim 1/\omega^{\alpha}$. (c) $(\alpha+1)\tau/2$ as a function of τ.

tunneling rate through any barrier of height W is e^{-W}, then the tunneling rate through a string of N barriers scales as e^{-NW}; consequently, if R is the (dimensionless) resistance of a single barrier, the resistance of a string of N barriers is R^N. Together, these observations imply that the distribution of resistances must satisfy the relation $P[R^N] \simeq \{P[R]\}^N$, and hence that $P[R] \sim R^{-\tau}$ for some τ whose value depends on microscopic details.

We now relate this RC model to our numerical results. To this end, we note that, for $1 < \tau < 2$, the average resistance of the chain in the dc limit $\bar{\rho}(\omega = 0) = \int_0^\infty P(R)RdR$ is divergent. At any nonzero frequency, however, the capacitors have a finite impedance $1/(\omega C)$, and a resistor with $R_i \gg 1/(\omega C)$ disconnects the circuit into separate blocks. Connecting to our arguments from the previous section, the average size of such blocks is given by $d(\omega) \sim \omega^{-\beta}$, where β is again the exponent of subdiffusive relaxation, as for the quantum model; probed on a time scale t, the circuit consists of disconnected regions of size $d(1/t)$; thus, the return probability scales as $C_{zz}(t) \simeq [d(1/t)]^{-1} \sim (1/t)^{\beta}$.

We can relate β to τ using the following argument. For $1 < \tau < 2$, the resistance of a block of size d is dominated by its largest *expected* resistor R_d, which is given by the criterion $d\int_{R_d}^\infty P(R)dR = 1$ yielding $R_d \simeq d^{1/(\tau-1)}$. The conductivity of such a block is given by $\sigma_d \simeq d/R_d$, and through the Einstein relation is related to the time of diffusion t_d across such a block, $t_d \sim d^2/\sigma_d \sim d^{-\tau/(1-\tau)}$. At any frequency ω, the size of independent blocks $d(\omega)$ is such that the diffusion time satisfies $t_d \simeq 1/\omega$. This yields $d(\omega) \sim \omega^{-(1-1/\tau)}$, i.e., $\beta = 1 - 1/\tau$. For $\tau > 2$, the distribution $P(R)$ yields a well-defined (length-independent) average resistance, and the Einstein relation yields $d(\omega) \sim \omega^{-1/2}$, or $\beta = 1/2$. These results concur with previous more rigorous analyses [50,51].

The conductivity exponent α is determined by reinstating frequency dependence in the result $\sigma_d \simeq d/R_d$; we find $\alpha = 2/\tau - 1$ and $\alpha = 0$ for $1 < \tau < 2$ and $\tau \geq 2$, respectively. Combining this with the previous result for β, we readily find that the scaling relation $\alpha + 2\beta = 1$ is satisfied for all values of τ. Next, we calculate the scaling of the width of the resistivity distribution. We again note that segments of the penetration depth $d(\omega)$ behave independently of one another and there are $L/d(\omega)$ such segments in a system of length L. The width of the resistivity distribution can then be shown (see the Supplemental Material [42]) to satisfy $\Delta\rho(\omega) \approx \Delta R(\omega)/\sqrt{L}$, where $\Delta R(\omega)$ is the width of the distribution of single resistors: $[\Delta R(\omega)]^2 \equiv \int_0^{R_d} P(R)(R - \bar{R})^2 dR \sim \omega^{1-3/\tau}$. This yields $\alpha' = 3/2\tau - 1/2$, which is slightly greater than α in the range of the Griffiths phase $1 < \tau < 2$ [this is reflected in the numerics for the XXZ model for $h_{\max} \lesssim 2.5$; see inset in Fig. 2(c)], and slightly beyond into the diffusive phase up to $\tau = 3$.

We have verified these scaling arguments by numerically solving the RC model (Fig. 3) in both the subdiffusive and diffusive regime and find good agreement with the analytical predictions. In particular, we calculate the mean and width of the finite frequency resistivity distribution and find that their asymptotic form is a power law; see Fig. 3(b). To test the relation $\alpha = 2/\tau - 1$, we plot $(\alpha+1)\tau/2$ as a function of τ [Fig. 3(c)] and confirm it to be constant in the subdiffusive regime $1 < \tau < 2$, while it increases linear in the diffusive regime $\tau > 2$, where $\alpha = 0$.

Conclusions.—In this work. we have numerically established the following facts about the delocalized phase near the MBL transition in the disordered XXZ chain. (1) The conductivity vanishes at low frequencies with the power law $\sigma(\omega) \sim \omega^{\alpha}$. (2) Spin transport is subdiffusive, and the return probability at long times decays as $C_{zz}^i(t) \sim 1/t^{\beta}$, with the scaling relation $\alpha + 2\beta = 1$. As the localized phase is approached, $\beta \to 0$, and $\alpha \to 1$ while as the diffusive phase is approached $\beta \to 1/2$ and $\alpha \to 0$. (3) The distribution of resistivities of a fixed-sized sample grows increasingly broad at low frequencies, and the width of this distribution diverges as a power law with exponent $\alpha' > \alpha$ at low frequencies. The distribution of resistivities becomes scale free and presumably power law in the dc limit. These general observations allow us to identify the phase as a Griffiths phase. We also derived the central scaling relation $\alpha + 2\beta = 1$, postulating a scaling form of the spin-diffusion propagator. We showed that a phenomenological, classical RC model allows us to capture the various features of the Griffiths phase in a simple manner. Our predictions can be directly tested in experiments with ultracold atoms in disordered potentials [52–59], polar molecules [60], nitrogen-vacancy centers in diamond [61], and thin films [62,63]. Two intriguing aspects of the Griffiths phase that remain to be addressed in future work are (i) whether it is ergodic, and (ii) whether any such phase exists in more than one dimension, where single local bottlenecks cannot block global transport. It would also be interesting to understand dynamics in the subdiffusive phase in the context of memory-matrix formalisms, in particular, the Mori-Lee approach where ergodicity (or lack of it) can be addressed directly [64,65].

The authors acknowledge discussions with D. Abanin, E. Altman, A. Amir, I. Bloch, I. Gornyi, D. Huse, L. Ioffe, I. Lerner, M. Lukin, I. Martin, A. Mirlin, R. Modak, A. Pal, A. Polkovnikov, and N. Yao. The authors acknowledge support from Harvard Quantum Optics Center, Harvard-MIT CUA, ARO-MURI Quism program, ARO-MURI on Atomtronics, as well as the Austrian Science Fund (FWF) Project No. J 3361-N20. M. M. acknowledges the hospitality of the University of Basel.

*agarwal@physics.harvard.edu

[1] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[2] L. Fleishman and P. W. Anderson, Phys. Rev. B **21**, 2366 (1980).
[3] D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. (Amsterdam) **321**, 1126 (2006).
[4] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. Lett. **95**, 206603 (2005).
[5] V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007).
[6] C. Monthus and T. Garel, Phys. Rev. B **81**, 134202 (2010).
[7] T. C. Berkelbach and D. R. Reichman, Phys. Rev. B **81**, 224429 (2010).
[8] A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
[9] J. Z. Imbrie, arXiv:1403.7837 (2014).
[10] R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).
[11] A. De Luca and A. Scardicchio, Europhys. Lett. **101**, 37003 (2013).
[12] C. R. Laumann, A. Pal, and A. Scardicchio, Phys. Rev. Lett. **113**, 200405 (2014).
[13] R. Vosk and E. Altman, Phys. Rev. Lett. **110**, 067204 (2013).
[14] D. A. Huse and V. Oganesyan, Phys. Rev. B **90**, 174202 (2014).
[15] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **111**, 127201 (2013).
[16] B. Swingle, arXiv:1307.0507 (2013).
[17] V. Ros, M. Müller, and A. Scardicchio, Nucl. Phys. **B891**, 420 (2015).
[18] A. Chandran, I. H. Kim, G. Vidal, and D. A. Abanin, Phys. Rev. B **91**, 085425 (2015).
[19] M. Žnidarič, T. Prosen, and P. Prelovšek, Phys. Rev. B **77**, 064426 (2008).
[20] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. **109**, 017202 (2012).
[21] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013).
[22] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, Phys. Rev. B **88**, 014206 (2013).
[23] D. Pekker, G. Refael, E. Altman, E. Demler, and V. Oganesyan, Phys. Rev. X **4**, 011052 (2014).
[24] A. Chandran, V. Khemani, C.R. Laumann, and S.L. Sondhi, Phys. Rev. B **89**, 144201 (2014).
[25] B. Bauer and C. Nayak, J. Stat. Mech. (2013) P09005.
[26] Y. Bahri, R. Vosk, E. Altman, and A. Vishwanath, arXiv:1307.4092.
[27] R. Vosk and E. Altman, Phys. Rev. Lett. **112**, 217204 (2014).
[28] J. A. Kjäll, J. H. Bardarson, and F. Pollmann, Phys. Rev. Lett. **113**, 107204 (2014).
[29] R. Nandkishore and A. C. Potter, Phys. Rev. B **90**, 195115 (2014).
[30] N. Y. Yao, C. R. Laumann, S. Gopalakrishnan, M. Knap, M. Müller, E. A. Demler, and M. D. Lukin, Phys. Rev. Lett. **113**, 243002 (2014).
[31] R. Nandkishore, S. Gopalakrishnan, and D. A. Huse, Phys. Rev. B **90**, 064203 (2014).
[32] M. Serbyn, M. Knap, S. Gopalakrishnan, Z. Papić, N. Y. Yao, C. R. Laumann, D. A. Abanin, M. D. Lukin, and E. A. Demler, Phys. Rev. Lett. **113**, 147204 (2014).
[33] S. Gopalakrishnan and R. Nandkishore, Phys. Rev. B **90**, 224203 (2014).
[34] S. Johri, R. Nandkishore, and R. N. Bhatt, Phys. Rev. Lett. **114**, 117401 (2015).
[35] R. Vasseur, S. A. Parameswaran, and J. E. Moore, arXiv:1407.4476.
[36] S. Mukerjee, V. Oganesyan, and D. Huse, Phys. Rev. B **73**, 035113 (2006).
[37] A. De Luca, B. L. Altshuler, V. E. Kravtsov, and A. Scardicchio, Phys. Rev. Lett. **113**, 046806 (2014).
[38] T. Grover, arXiv:1405.1471.
[39] T. Vojta, J. Low Temp. Phys. **161**, 299 (2010).
[40] Y. Bar-Lev, G. Cohen, and D. Reichman, Phys. Rev. Lett. **114**, 100601 (2015).
[41] R. Vosk, D. A. Huse, and E. Altman, arXiv:1412.3117.
[42] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.114.160401, which includes Refs. [43,44–47], for a description of the scaling form of the anomalous-diffusion propagator, additional details on numerical methods used, and calculation of the width of the resistivity distribution in the RC model.
[43] O. Motrunich, K. Damle, and D. A. Huse, Phys. Rev. B **63**, 134424 (2001).
[44] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics*, Vol. 1 (Cambridge University Press, Cambridge, England, 2000).
[45] A. Miller and E. Abrahams, Phys. Rev. **120**, 745 (1960).
[46] V. Ambegaokar, B. I. Halperin, and J. S. Langer, Phys. Rev. B **4**, 2612 (1971).
[47] A. Amir, Europhys. Lett. **107**, 47011 (2014).
[48] N. Mott, Philos. Mag. **17**, 1259 (1968).
[49] Y. Imry, Y. Gefen, and D. J. Bergman, Phys. Rev. B **26**, 3436 (1982).
[50] J. P. Hulin, J. P. Bouchaud, and A. Georges, J. Phys. A **23**, 1085 (1990).
[51] J. Machta, J. Phys. A **18**, L531 (1985).
[52] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clement, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, Nature (London) **453**, 891 (2008).
[53] G. Roati, C. DErrico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature (London) **453**, 895 (2008).
[54] M. Pasienski, D. McKay, M. White, and B. DeMarco, Nat. Phys. **6**, 677 (2010).

[55] S. S. Kondov, W. R. McGehee, and B. DeMarco, Phys. Rev. Lett. **114**, 083002 (2015).
[56] C. D'Errico, E. Lucioni, L. Tanzi, L. Gori, G. Roux, I. P. McCulloch, T. Giamarchi, M. Inguscio, and G. Modugno, Phys. Rev. Lett. **113**, 095301 (2014).
[57] W. S. Bakr, J. I. Gillen, A. Peng, S. Fölling, and M. Greiner, Nature (London) **462**, 74 (2009).
[58] J. F. Sherson, C. Weitenberg, M. Endres, M. Cheneau, I. Bloch, and S. Kuhr, Nature (London) **467**, 68 (2010).
[59] F. Andraschko, T. Enss, and J. Sirker, Phys. Rev. Lett. **113**, 217201 (2014).
[60] B. Yan, S. A. Moses, B. Gadway, J. P. Covey, K. R. A. Hazzard, A. M. Rey, D. S. Jin, and J. Ye, Nature (London) **501**, 521 (2013).
[61] M. W. Doherty, N. B. Manson, P. Delaney, F. Jelezko, J. Wrachtrup, and L. C. L. Hollenberg, Phys. Rep. **528**, 1 (2013).
[62] Z. Ovadyahu, Phys. Rev. Lett. **108**, 156602 (2012).
[63] Z. Ovadyahu, Phys. Rev. B **91**, 035113 (2015).
[64] M. H. Lee, Phys. Rev. Lett. **98**, 190601 (2007).
[65] M. B. Yu, Physica (Amsterdam) **398A**, 252 (2014).