Absence of many-body mobility edges

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Localization transitions as a function of temperature require a many-body mobility edge in energy, separating localized from ergodic states. We argue that this scenario is inconsistent because inclusions of the ergodic phase in the supposedly localized phase can serve as mobile bubbles that induce global delocalization. Such inclusions inevitably appear as rare fluctuations in any typical state. We conclude that the only possibility for many-body localization occurs in lattice models that are localized at all energies. Building on a close analogy with a two-particle problem, where interactions induce delocalization, we argue why hot bubbles are mobile and do not localize upon diluting their energy. Numerical tests of our scenario show that the previously reported mobility edges cannot be distinguished from finite-size effects.

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It is now almost mathematically proven that many-body localization, i.e., the absence of long range transport in a thermodynamic many-body system, occurs in certain quantum lattice models with sufficiently strong quenched disorder, at any energy density. [1] In these cases it comes along with a complete set of conserved quasi-local quantities [2–4]. However, it remains less clear whether the originally predicted localization transition at finite temperature [5, 6] exists as a genuine dynamical phase transition defining a sharp many-body mobility edge in energy density. Even though several numerical investigations in small 1d systems have reported such mobility edges [7–9], recent theoretical considerations [4, 10–13] have raised doubts about non-perturbative effects which might reduce the putative transition to a crossover. A related open issue concerns the many-body analogue of Mott's argument, which forbids the coexistence of localized and delocalized states at the same energy in single particle problems.

In this Letter, we address these issues, which are fundamental for a complete understanding of localization, equilibration and transport in closed many-body quantum systems. We argue that for systems with short range interactions many-body mobility edges cannot exist, thus ruling out sharp transitions from a conducting to a completely insulating phase as a function of temperature. These considerations also imply a strong many-body version of Mott's argument, which rules out the coexistence of localized and delocalized states, even at extensively different energies. To this end, we first introduce and discuss a simple two-particle model of assisted hopping, which illustrates several important features that this problem has in common with the rare events that induce delocalization in many-body systems and wash out finite mobility edges whenever there is a ergodic state at some finite temperature.

Assisted hopping model - Consider particles on a hypercubic lattice, hopping with amplitude t_1 and subject to a disorder potential ϵ_x, i.i.d. uniformly in $[-W, W]$. A particle on site x interacts with others by inducing assisted hoppings of strength t_2 along the diagonals of plaquettes that x belongs to (e.g. due to lattice distortions)

$$\begin{aligned} H \;=\; & -t_1 \sum_{\langle x,y\rangle} (c_x^\dagger c_y + \text{h.c.}) + \sum_x \epsilon_x n_x \\ & -t_2 \sum_x \sum_{s,s'=\pm} \sum_{1\le\alpha<\beta\le d} n_x (c^\dagger_{x+s\vec{e}_\alpha} c_{x+s'\vec{e}_\beta} + \text{h.c.}). \end{aligned} \tag{1}$$

We consider parameters $t_1 \ll W$, for which the single particle problem is localized in the whole spectrum. For $t_2 \gg W$, the two-particle problem has several interesting features: In dimensions $d > 2$, the assisted hopping term induces a delocalization of close pairs which will move together diffusively as a composite light particle and overcome Anderson localization. This effect is closely related to the interaction-induced increase of the localization length in sufficiently weakly localized systems [14, 15]. A single particle analogue of the phenomenon is the solvable case of two coupled Bethe lattices [16]. The delocalization in (1) seems natural, since all configurations of two particles at distance one are strongly resonant with each other. They thus form a percolating, delocalized resonant subgraph in configuration space, which supports delocalized wavefunctions with inverse participation ratios that vanish as the inverse volume. This type of effect is confirmed numerically in [17–19]. In a system of only two particles the eigenstates come in two kinds: The overwhelming number of states is strongly concentrated on a configuration with two distant, immobile particles. Only a vanishing fraction of order $\log(L)/L$

of all two-particle states are delocalized as dynamically bound, mobile pairs. In this example localized and delocalized states coexist at the same energy. This is possible because the matrix elements that couple the two kinds of states through a random perturbation of the Hamiltonian are typically exponentially small in the system size and thus negligible as compared to the relevant level spacings.

Let us now discuss how a finite density of particles modifies the situation. In the thermodynamic limit, there is a finite density of close pairs in typical configurations. These pairs diffuse through the sample. Initially well isolated and localized particles scatter inelastically off these pairs and thus move as well, leading to complete delocalization. Even in exponentially rare configurations where initially all particles are far from each other, particles eventually tunnel together and decay into the continuum of diffusive pair states. We thus do not expect any localized eigenstates to survive at finite density.

Many-body systems: Delocalization from rare bubbles - The argument of Basko et al. [5] for a localization transition as a function of temperature, i.e., a many-body mobility edge, builds on the idea that conduction can set in only if the energy density exceeds a critical level essentially everywhere in the sample. We argue instead that delocalization occurs as soon as finite, but mobile excitations exist, even if they are very rare and atypical. They constitute the analogues of the diffusive pairs above. Examples of such excitations are large, albeit finite regions which are internally ergodic thanks to a higher than average energy density [27]. Hereby we assume that interactions are local, so that the internal ergodicity is only a function of the energy contained in that region.

Let us assume that at some temperature there is conduction and ergodicity [28]. In typical states and in any given place such ergodic regions occur with finite probability as spontaneous fluctuations of energy density, without being tied to a particular disorder realization. Thus, there exists a (possibly very low) finite density of ergodic spots. Below we will argue that these excitations are mobile and delocalize the whole system, akin to the diffusing pairs above. From this reasoning it follows that finite conduction at some temperature [29] implies finite conduction at any temperature in thermodynamic systems with local interactions.

To argue for the mobility of the hot bubble excitations we proceed in two steps: First we show that there exists a resonant, delocalized subset of bubble configurations. In a second step we argue that the delocalization remains robust when processes are taken into account that lead away from the resonant subgraph.

We consider a quantum lattice system with local interactions and a bounded energy density, possessing a putative many-body mobility edge at energy density ϵ_c, such that states below (above) ϵ_c are localized (ergodic). For simplicity, we assume the model to be one-dimensional. Now consider a rare hot bubble of a super-critical energy density above some $\epsilon_2 > \epsilon_c$, surrounded by "cold" regions of energy density below $\epsilon_1 < \epsilon_c$. If this energy fluctuation is large enough (much larger than a correlation length $\xi(\epsilon_2)$) and decoupled from the surrounding, it is internally ergodic by assumption.

We argue that this state can hybridize with a translate of the bubble by some length $\ell_0 > \max[\xi(\epsilon_1), \xi(\epsilon_2)]$ when the coupling between the hot region and its surrounding is switched on. It suffices to show that extending (or shortening) the hot region by a length ℓ_0 (by heating up or cooling down the relevant region) can occur as a resonant transition. For the latter it suffices to show that changing the energy in the boundary region by a finite amount is a resonant process. Let $H_1 = gO_h \otimes O_c$ be the interaction term coupling a hot (h) and a cold (c) region of size ℓ_0 across their common boundary. Let Ψ, Ψ' be eigenstates in the hot region and η, η' eigenstates in the cold region. For any hot eigenstate Ψ in a sufficiently large bubble we can find (many) Ψ' such that

$$\frac{|\langle \Psi\eta|H_1|\Psi'\eta'\rangle|}{|E(\eta) - E(\eta') + E(\Psi) - E(\Psi')|} \gg 1, \qquad (2)$$

because on the one hand, by the eigenstate thermalization hypothesis (ETH) [20], $|\langle \Psi\ |O_h|\Psi'\rangle| \sim d_h^{-1/2}$ where d_h is the dimension of an appropriate micro-canonical ensemble for the hot bubble at the energy density set by Ψ, while the matrix element $|\langle\eta|O_c|\eta'\rangle| = O(1)$ is finite and independent of d_h. On the other hand, we can pick Ψ' such that $|E(\eta) - E(\eta') + E(\Psi) - E(\Psi')| \leq W/d_h$, where W is the energy width of the ensemble. The ratio in (S6) thus scales as $\sim d_h^{1/2}$ and grows exponentially with the length of the bubble. It may thus become much larger than unity, indicating a resonant process. This is not surprising: it merely expresses that a sufficiently large ergodic bubble acts as a bath for small systems coupled to it. It follows that configurations with hot bubbles in different positions hybridize with each other.

Can bubbles freeze? - We now ask whether some processes that have not been taken into account in the previous analysis could impede the hybridization of bubbles. The first objection that can be raised is that hot bubbles should not survive dynamically, but should rather spread, dilute their energy and eventually localize, so that they could not evolve back to their original hot configuration. Though such a spreading is indeed entropically favored in real time dynamics, that argument is fallacious. At a fundamental level, Hamiltonian dynamics is micro-reversible. If a given transition is possible, then its reverse is as well. By invariance of the Gibbs ensemble, one can definitely rule out that initially present bubbles typically completely disappear with time for most initial configurations. Looking at it from a different perspective, there are eigenstates that have a significant overlap with bubble configurations and they assure that there is a finite, albeit small, probability per unit volume to observe bubbles at all times.

Though we show that entropic effects alone do not suffice to make bubbles disappear, it still remains to check that their mobility is not suppressed when all the diluted states of a bubble are taken into account, as a result of quantum mechanical effects. To this end, we first notice that, as the spreading of the bubble ends up in localized configurations, only finitely many degrees of freedom are affected, say ℓ_h around a fixed hot bubble position. Suppose then that the spreading mixes the original hot bubble states with states of a larger local Hilbert space of finite dimension $D_h \gg d_h$. [30] We also suppose that the resonant coupling between hot regions centered in different positions remains restricted to the original d_h configurations (because the spread-out bubbles may have lost their ability to translate directly). Assuming ergodicity within the larger space, the matrix elements get reduced by a factor D_h. However, the minimal denominators decrease by essentially the same factor to $\widetilde{W}/D_h$, except that now a slightly larger energy range $\widetilde{W}$ intervenes. Since the ratio $\widetilde{W}/W$ grows at best linearly with ℓ_h, which itself cannot grow faster than linearly with the length of the bubble because of local energy conservation, it cannot offset the exponential in (S6), and thus, the resonantly hybridizing subgraph persists. This contrasts with single particle problems where sufficiently strong coupling to a bath may induce localization due to a significant increase of the effective bandwidth, as discussed in [21]. These considerations are formulated more precisely in the Supplementary Material [19]. Similarly as in the assisted hopping model, the admixture of configurations that are not part of the resonant network cannot prevent hybridization, but it does increase the *timescale* for transitions between different positions of the bubble by a factor D_h/d_h.

So far we have reasoned that the construction of a resonant subgraph essentially implies delocalization. While we believe that in the present context this conclusion is correct, we nevertheless caution that this condition is not always sufficient, like in single-particle localization in weak disorder in low dimension, or in hopping problems without potential disorder on structurally disordered lattices close to classical percolation. However, in these cases localization is restored by specific mechanisms, which are not present in our many-body case: the proliferating amplitude of return to the origin in $d \leq 2$, and the generation of random self-energies from the structural disorder along barely percolating paths [24, 25]. Non-ergodic behavior is also known to occur in many-body systems due to orthogonality catastrophes, like in spin-boson systems at $T = 0$ and related spin problems at finite T. However, a closer examination [19] of the bubble situation suggests that such effects at best renormalize time scales, but do not fully suppress the motion of bubbles.

Bubbles are rare objects, as they correspond to large deviations from the average energy density. As discussed in Ref. [4], such effects are neglected in the analysis of Refs. [4–6], which focus on the set of the most numerous decay paths at a given order of perturbation theory. This set does not capture mobile high energy bubbles.

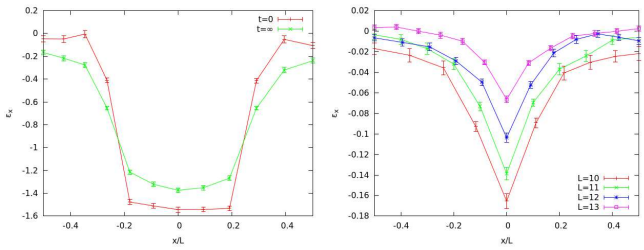

FIG. 1: *Left:* Disorder averaged energy per link ε_i at $t = 0$ (red) and averaged over time (green) for $L = 12$. Initially a cold region of length $L_c = L/2$ is prepared. The disorder strength is $\delta J = 3J$. *Right:* Same protocol, but for $\delta J = J$ and very short cold intervals ($L_c = 2$), at various L. The memory effects diminish with increasing L, but the hot region fails to thermalize the system well, even at the largest sizes. Results were averaged over 5000 disorder realizations.

Numerical studies - Our theoretical arguments contradict recent numerical data in favor of mobility edges [7–9]. The inconsistency is, however, only apparent. Indeed, we find that numerically accessible system sizes are not sufficiently large to host bubbles that are ergodic enough to be mobile. Therefore, delocalization by bubbles could not have been seen in numerics up to now. In other words, the numerical results do not contradict delocalization by rare bubbles, but rather confirm that available sizes are not large enough.

We study the disordered Ising chain with next-to-nearest neighbor interaction considered in Ref. [7],

$$H = -\sum_{i=1}^{L} \left[(J + \delta J_i)\, \sigma_i^z \sigma_{i+1}^z + J_2 \sigma_i^z \sigma_{i+2}^z + h_z \sigma_i^z + h_x \sigma_i^x \right],$$

where $\delta J_i \in \left[-\frac{\delta J}{2}, \frac{\delta J}{2}\right]$ are independent random variables, and periodic boundary conditions are taken. We choose parameters $J = 1$, $J_2 = 0.3$ and $h_x = 0.6$ as in Ref. [7], but add a finite $h_z = 0.1$ to remove the Ising symmetry and the associated degeneracies. The phase diagram in Ref. [7] predicts a mobility edge in the thermodynamic limit at disorder strength $\delta J = 3$. To test our ideas, we prepare the system at $\delta J = 3$ in a product state of the form $|\psi(0)\rangle_L = |\phi_c\rangle_{L_c} \otimes |\chi_h\rangle_{L-L_c}$, where $|\phi_c\rangle$ is the ground state of an interval of L_c sites, while $|\chi_h\rangle$ is an eigenstate of the complement close to the middle of the spectrum (a hot bubble). We choose $L - L_c$ as large as possible but such that the resulting global energy density is below the putative mobility edge. We then compute the time-evolving energy density on link $(i, i+1)$,

$$\varepsilon_i(t) \equiv -(J + \delta J_i) \langle \psi(t) | \sigma_i^z \sigma_{i+1}^z | \psi(t) \rangle . \qquad (3)$$

Our theory of mobile bubbles would predict that the $\varepsilon_i(t)$ profile becomes approximately flat as $t \to \infty$. Via ex-

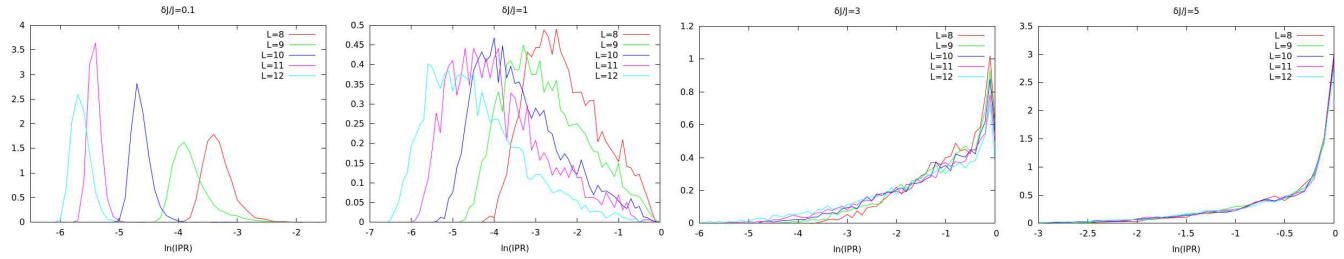

FIG. 2: Distribution of ln(IPR) associated with matrix elements of σ_1^z evaluated on eigenstates randomly picked from the middle of the spectrum, for $\delta J/J = 0.1, 1, 3, 5$. In the ergodic phase, the typical IPR is exponentially small in the size L. In the localized phase, the distribution is size independent. At $\delta J = J$ and the considered L, the distribution is very wide as compared to the typical IPR. $\delta J = 3J$ is nearly critical: the 'localized' peak at IPR $= O(1)$ slowly decreases with increasing L.

act diagonalization, we evaluated its time average, but almost no energy spreading from the initial state is observed cf. Fig. 1 (left). For tiny cold regions ($L_c = 2$) and bubbles of almost the system size the global energy density is supercritical. Yet, still only a very small fraction of the bubble energy spreads to the cold region at $L = 12$ (not shown), while in the thermodynamic limit, the energy profile would obviously thermalize and become flat. Therefore, this data shows unambiguously that at our system size the hot region is still unable to act as a bath.

To document this further, we calculated the inverse participation ratio (IPR) of an eigenstate $|\alpha\rangle$ of the full system, acted upon by a local unitary operator such as σ_1^z, in a basis of eigenstates $|\beta\rangle$:

$$\mathrm{IPR}_\alpha \equiv \sum_\beta |\langle\beta|\,\sigma_1^z\,|\alpha\rangle|^4 . \qquad (4)$$

At strong disorder, eigenstates are nearly eigenstates of σ_i^z as well, and thus $\mathrm{IPR}_\alpha \approx O(1)$, whereas deep in the delocalized phase, one expects eigenstate thermalization and behavior akin to random matrix theory, $|\langle\beta|\,\sigma_1^z\,|\alpha\rangle| \propto \exp[-sL/2]$, leading to a typical value $\mathrm{IPR}_\alpha \sim \exp[-sL]$, with a narrow distribution. The results shown in Fig. 2 confirm the absence of a truly ergodic phase at $L = 12$ and $\delta J = 3$, in accordance with results of [7]. In fact, the distribution of IPR's at these parameters looks more characteristic of localization. Nevertheless a slight, but clear tendency towards enhanced delocalization with increasing size is seen. This hints that in the thermodynamic limit the system will become ergodic, in agreement with the finite size extrapolations in [7].

To chart the lack of ergodicity at small sizes, we also look at $\delta J = 1$, where Ref. [7] suggests that most eigenstates are delocalized, even at $L = 12$. Nevertheless, here, too, we find strong deviations from fully ergodic behavior, using the same two protocols as above. Even in the extreme case of $L_c = 2$ in Fig. 1 (right), despite some energy transfer, the hot and cold regions are still clearly distinguishable. To quantify this effect, we consider the time average of the energy imbalance between hot and cold regions, $\Delta\varepsilon \equiv (L-3)^{-1}\sum_{i\notin\{c,c\pm1\}}(\varepsilon_i - \varepsilon_c)$, where c denotes the single link fully in the cold region. The imbalance decays exponentially with system size, $\Delta\varepsilon \sim \exp(-L/\xi)$ where ξ increases with disorder strength. For $\delta J/J$ in the range $[1, 1.5]$ we estimate $\xi \approx O(10)$ [19], which sets a characteristic scale required to observe genuine ergodic behavior. This suggests strongly that at reachable sizes the hot bubble is far from being ergodic. Also Fig. 2 illustrates that $\delta J = 1, L = 12$ is far from the thermodynamic limit: the distribution of $\ln(\mathrm{IPR}_\alpha)$ is much wider (as compared to the mean) than in a clearly ergodic sample.

Impossibility of MBL in the continuum - In the continuum the single-particle localization length becomes arbitrarily large at high energies. For dimensions $d > d_c$ it even diverges at a mobility edge, so that there is always some activated transport at finite T. For $d < d_c$ arbitrarily weak interactions render bubbles of high enough energy ergodic. Our arguments thus imply finite transport at any $T > 0$ in any continuum system (see also the discussion in Ref. [26]). However, for $d < d_c$ the conductivity of insulators is non-perturbative in the interaction, as the energy content of a mobile bubble diverges as the interaction strength vanishes.

Conclusion - We have argued that in the thermodynamic limit many-body-localized and ergodic states cannot coexist, not even at very different energies. This has important consequences on the nature of the MBL transition. On a lattice, it implies that a transition is possible at best upon tuning the interaction strength, but not the temperature. In the continuum, genuine MBL is replaced by a strong crossover in the conductivity instead, which is notoriously hard to distinguish from a genuine transition. Nevertheless, finite size systems will exhibit states, which for every practical purpose are localized - as they do not contain any of the rare bubbles that induce delocalization and weak transport.

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[27] We stress that the bubble excitations considered in this Letter are thermal, and are not tied to local, anomalous realizations of the disorder. Notice that here the strategy of Ref. [1] to show the existence of an MBL phase would fail. Indeed, it requires that the location of all possible resonant spots can be determined *independently* of the state of the system.

[28] We believe that, in the thermodynamic limit, the presence of finite conduction implies ergodicity, too. This follows by disregarding the exotic scenario of non-ergodic, but delocalized many-body systems, where transport is confined to a fractal support in real space. Indeed, in large enough systems, delocalized modes supported on a finite fraction of space will serve as a bath, which is expected to thermalize any finite set of degrees of freedom in finite times.

[29] An analogous argument applies to disorder-free, but strongly interacting systems, where the putative transition is tuned by density and the obviously delocalized phase is at low density, see [10]. There the analogues of the hot bubbles discussed above are bubbles of low density.

[30] Note that, in contrast, the entanglement of the bubble region with its surroundings grows logarithmically without limits, reflecting that an increasing number of degrees of freedom affects the relative phases of configurations in the bubble region. However, this does not alter the fact that the relevant number of hybridizing configurations remains finite.

SUPPLEMENTARY MATERIAL: ABSENCE OF MANY-BODY MOBILITY EDGES

This note contains supplementary material for the main paper 'Absence of many-body mobility edges', consisting of:
1) A formal argument for delocalization by bubbles
2) Numerical analysis of a two-particle model with assisted hopping
3) Numerical analysis of the (partial) thermalization of a 2-site cold region by a 10-site hot bath.

FORMAL PRESENTATION OF THE BUBBLE ARGUMENT

We consider a quantum lattice system with local interactions, having a putative many-body mobility edge. For concreteness, we assume that the model is one-dimensional and that states below energy density ϵ_c are (putatively) localized, whereas those above ϵ_c are ergodic. We choose the energy density of the bottom of the spectrum as a reference and set it zero. We also impose a maximal energy density $\epsilon_m > \epsilon_c$, reflecting the fact that the Hilbert space is locally finite. For our argument it is important to have states at our disposal that are clearly ergodic or localized in a given finite volume and therefore we introduce, somehow arbitrarily, two other energy densities $\epsilon_{c1} < \epsilon_c < \epsilon_{c2}$. Let $\ell(\epsilon)$ be the localization length, diverging as $\epsilon \nearrow \epsilon_c$. We express lengths in units of lattice spacings, and energy densities ϵ as energy per site. We will now coarse-grain the model and group ℓ_0 adjacent sites into 'units'. The length ℓ_0 is chosen such that it is a) larger than the localization length $\ell(\epsilon_{c1})$, and b) large enough so that the interaction energy between two neighboring units is small compared to $\ell_0\epsilon_{c1}$, (maximal energy in a localized unit). The second constraint in particular ensures that the interaction of a unit with energy density below ϵ_{c1} with the surroundings does not trivially suffice to render the unit ergodic. To satisfy it, we used locality of the interaction. The coarse-graining provides a useful starting point, from which to proceed with perturbation theory. Within each unit, we compute the eigenstates which come in three kinds, cold (below ϵ_{c_1}), hot (above ϵ_{c2}) or intermediate (between ϵ_{c1} and ϵ_{c2}). If we consider the Hamiltonian without the interaction between units, then obviously the eigenstates are products of unit eigenstates. Let us focus on eigenstates at very low energy density $\epsilon \ll \epsilon_{c1}$. Then, typically, non-cold units appear only with a density ν that tends to 0 as $\epsilon/\epsilon_{c1} \to 0$. Chains of labeled units now serve as 'mesostates', i.e. *cccicchhhiiccc* would be a mesostate, with $c/i/h$ standing for cold/intermediate/hot. Hence, we can now write our model as

$$H = \sum_x \left[H_0(x) + H_1(x, x+1)\right], \tag{S1}$$

where x label the units, $H_0(x)$ acts on the Hilbert space $\mathcal{H}(x)$ at unit x only, and H_1 describes the coupling between neighboring units. Mesostates are eigenstates of the term H_0. We now consider switching on the coupling terms and evaluate their effect on the unperturbed eigenstates of H_0. This procedure is similar to the one followed in [1]. First we add the interaction terms between cold units (see below for what is meant precisely). Since we assumed that $\ell(\epsilon_{c1}) < \ell_0$, this will not have much effect on the localized eigenstates, which thus remain close to products. Note that by doing this, from the point of view of a typical state at energy density ϵ, we have already added most of the interaction terms. What remains is a small fraction $\sim 2\nu$ (which is controlled by the overall energy density) of all interaction terms. We now add the interaction terms between hot units. By assumption, sufficiently long stretches of such units $\ldots hhhh \ldots$ (which we call 'bubbles') are ergodic and we will assume that the resulting hot eigenstates in those bubbles satisfy the eigenstate thermalization hypothesis (ETH). The situation at this moment is hence that we have partitioned the Hilbert space into a big direct sum, and the Hamiltonian is block diagonal, with the blocks labelled by mesostates. Let P_x^r be the projector that restricts the value of $H_0(x)$ so that unit x is of type $r = h, i, c$. With this notation the interaction terms that have already been added are

$$P_x^{r'_x} P_{x+1}^{r'_{x+1}} H_1(x, x+1) P_x^{r_x} P_{x+1}^{r_{x+1}}, \tag{S2}$$

for $(r_x, r_{x+1}) = (r'_x, r'_{x+1}) = (c, c)$, and for $(r_x, r_{x+1}) = (r'_x, r'_{x+1}) = (h, h)$. Some terms are obviously very small (because the interaction is local in energy) and seem irrelevant, namely those corresponding to, $(r_x, r_{x+1}) = (c, c), (r'_x, r'_{x+1}) = (h, h)$ and with primes and no primes reversed. The main terms that we will be focusing on are those that allow bubbles to spread and move. Those are terms with

$$r_x = r'_x = h, \qquad \text{and arbitrary } r_{x+1}, r'_{x+1},$$

and they feature prominently in the next section.

Mobility of bubbles

Let us now consider states of the following form (bubble in a cold environment)

$$cccccccc\underbrace{hh\ldots hh}_{n \text{ units}}cccccccc, \tag{S3}$$

where n is sufficiently large so that the eigenstates in the bubble satisfy ETH. We now argue that this state

can hybridize with translates of the bubble when we add some of the missing coupling terms: In particular, we want to admit the mesostates (with x, y labelling units)

$$\ldots \underset{x}{c}\underset{y}{c}chhhhccc\ldots \leftrightarrow \ldots ccc\underset{x}{c}\underset{y}{c}hhhcc\ldots$$

in which the bubble has been translated by one unit. More precisely, we mean that most microstates (i.e., eigenstates of the Hamiltonian considered up to now) corresponding to the left mesostate can hybridize with a lot of microstates corresponding to the right mesostate. This in turn strongly suggests that we should expect all eigenstates to delocalize completely over these two mesostates. To obtain this, we have included the relevant coupling terms (S2) corresponding to two bonds $(x, x+1)$ and $(y-1, y)$. This hybridization process can be broken down into elementary steps, that is, transitions in first order of perturbation theory. First, by energy exchange with the hot region, the cold (c) unit at y is heated until it becomes i and finally hot (h). Second, the h unit at x is cooled down until it becomes c, via intermediate stages of i. Microscopically, let us consider a state Φ corresponding to the mesostate *ccchhhhccc* and such that $H_0(y)$ is not far below ϵ_{c1}. We will argue that Φ hybridizes with a lot of states Φ' corresponding to the mesostate *ccchhhhicc* where $r'(y) = i$. If instead $H_0(y)$ is far below ϵ_{c1}, then it hybridizes with a lot of states Ψ' which still corresponds to $r'(y) = c$ (*ccchhhhccc*), but now with $H_0(y)$ a bit closer to ϵ_{c1}. Here we care about not adding too much energy in one transition, even though this is basically redundant (see below). However, we want our argument to go through even if the simplifying assumption is made that the interactions are strictly local in energy. Finally, we need to increase the energy stepwise from i to h at unit y. The argument for all these transitions is essentially the same and for the sake of simplicity, we stick to $r(y) = r'(y) = c$. The next subsection below clearly shows the flexibility of the argument.

Obviously, it suffices to take eigenstates in Φ, Φ' in the region $[x, y]$ because of the essential product structure (exact at the left edge, approximate at the right edge around y, because we have already included the coupling between cold regions). They are of the form

$$\Phi = \Psi \otimes \eta, \qquad \Phi' = \Psi' \otimes \eta', \tag{S4}$$

where η, η' are the unperturbed eigenstates at unit y, while Ψ, Ψ' are hot bubble states in the region $[x, y-1]$ consisting of $n = y - x$ units. Consider Ψ' such that its energy (evaluated with H_0) is within a range $W \sim \epsilon_m$ of the energy of Ψ. The space spanned by such states has dimension $d_h \approx \exp[s\ell_0 n]$ which grows exponentially in n, s being the corresponding entropy density. Write $H_1(y-1, y) = gO_h \otimes O_c$, the first factor acting on $y-1$, the second on y. From ETH it then follows that

$$|\langle \Psi | O_h | \Psi' \rangle| \sim 1/\sqrt{d_h}. \tag{S5}$$

In other words, the (non-eigenstate) vector $O_h\Psi$ is essentially a random amplitude superposition of eigenstates Ψ'. Take now $\Delta E := E(\eta) - E(\eta')$ sufficiently small, i.e. not exceeding W, then $|\langle \eta | O_c | \eta' \rangle| \sim 1$. In fact, assuring the non-vanishing of $|\langle \eta | O_c | \eta' \rangle|$ is the main reason to choose W sufficiently small. We can then find many Ψ' (in fact, $\sim \sqrt{d_h}$ of them) such that

$$\frac{|\langle \Psi\eta | H_1 | \Psi'\eta' \rangle|}{|\Delta E + E(\Psi) - E(\Psi')|} \gg 1, \tag{S6}$$

because the energy spacings are of order W/d_h and $\langle \Psi\eta | H_1 | \Psi'\eta' \rangle \sim g/\sqrt{d_h}$. Hence the ratio in (S6) is huge since d_h grows exponentially in n. The outcome of this calculation should not come as a surprise: it merely expresses that an ergodic bubble can act as a bath for a small system (here unit y) that is coupled to it. Upon repeating the same calculation a few times, one easily convinces oneself that states with the bubble in different positions hybridize with each other.

Spatial range of direct hybridizations

In the above derivation, we focused on transitions that result in the translation of a bubble by one unit. This may appear to be a small translation if the bubble is very large, $n \gg 1$. Here we show, however, that direct hybridizations can take place at distances which are a finite fraction of the bubble length.

As already pointed out, in the above derivation, we were careful to pick states η, η' whose energy difference was small enough so that $|\langle \eta, O_c \eta' \rangle| \sim 1$. This is, however, not crucial, and if $r'(y) = i, h$, then it cannot be assured anyhow. The matrix element $|\langle \eta | O_c | \eta' \rangle|$ will typically decay exponentially in the energy difference $E(\eta) - E(\eta')$. Hence, it can be as small as $\mathrm{e}^{-l_0\epsilon_m}$, but obviously this number decreases with ℓ_0 and not with n, so it is irrelevant. To determine at what distance direct hybridizations are possible, we proceed as follows. Instead of making the transition $\eta \to \eta'$ at unit y, we now make a transition $\underline{\eta} \to \underline{\eta'}$ in a stretch of ℓ units starting at y. By the structure of localized states, we know that

$$|\langle \underline{\eta} | O_c | \underline{\eta'} \rangle| \sim (g/\epsilon_m)^{\ell\ell_0}. \tag{S7}$$

The transition is possible as long as this small number is larger than $\sqrt{1/d_h}$, so that we find

$$\ell \sim \frac{s}{2\log(\epsilon_m/g)} n. \tag{S8}$$

This shows us that the bubble hybridizes with translates by a finite fraction of its size. However, this fraction becomes parametrically small as the coupling becomes weak $g/\epsilon_m \to 0$.

Can bubbles freeze due to further couplings?

However, so far we have not yet added all omitted coupling terms. Indeed, the bubble cannot only move through the cold background, it can also spread its energy, and entropically, this is of course much more likely. In particular, one sees that starting from a bubble configuration, the most likely thing to happen is that the bubble spreads until its energy density is intermediate or just below the putative mobility edge, at which point we cannot expect it to spread further as the involved states are now localized. The question arises whether these further couplings may induce a localization of bubbles, despite the above construction of an apparently resonant, delocalized network of bubble configurations. This issue is addressed below.

Hybridization

In the notation of the previous section, the relevant type of transitions are as follows

$$\begin{array}{ccc} ccchhhhccc & \leftrightarrow & cccchhhhcc \\ \updownarrow & & \updownarrow \\ \ldots \leftrightarrow \ldots cciiiiiiiicc & & ccciiiiiiic \ldots \leftrightarrow \ldots, \end{array} \tag{S9}$$

whereby the states on the lower line represent a multitude of mesostates. Let us pretend that they themselves do not communicate with each other. This simplifying assumption favors maximally the possibility that the coupling to such states could localize the bubble and thus invalidate our preliminary conclusion above. Simplifying a bit, we consider the following situation. We consider the two subspaces, each of dimension d_h, that correspond to the mesostates on the upper line, the eigenstates of which are hybridized by the perturbation H_1. Let us refer to them as left and right subspaces. We now couple each of them to a space of dimension $d'_h \gg d_h$ and we ask whether the perturbation H_1 is still able to induce hybridization between left and right subspaces. Concretely, the subspace $\mathbb{C}^{d_h}$ is now embedded in the space $\mathbb{C}^{d_h} \oplus \mathbb{C}^{D'_h}$ of dimension $D_h \equiv D'_h + d_h$, and O_h becomes $O_h \oplus 0$. We focus on the transitions between the ergodic states Ψ, Ψ' (notation as above), and just consider the operator O_h which acts on the hot bubble. Let us assume that after diagonalizing within the larger spaces of dimension D_h, the eigenstates $\widetilde{\Psi}, \widetilde{\Psi}'$ are completely ergodic and well captured by random matrix theory. (In practice, this defines the relevant space to be added to the bubble subspace, and its dimension D_h.) We now have to discuss how the ratio

$$\frac{|\langle \widetilde{\Psi}|O_h|\widetilde{\Psi}'\rangle|}{|\Delta E + E(\widetilde{\Psi}) - E(\widetilde{\Psi}')|} \tag{S10}$$

differs from the original ratio

$$\frac{|\langle \Psi|O_h|\Psi'\rangle|}{|\Delta E + E(\Psi) - E(\Psi')|} \sim \frac{\sqrt{d_h}}{W} \tag{S11}$$

with given $|\Delta E| \leq W$. We find a suppression of the numerator because now

$$|\langle \widetilde{\Psi}|O_h|\widetilde{\Psi}'\rangle| \sim \frac{\sqrt{d_h}}{D_h}. \tag{S12}$$

Indeed, the simplest way to derive this is by remarking that

$$\sum_{\widetilde{\Psi},\widetilde{\Psi}'} |\langle \widetilde{\Psi}|O_h|\widetilde{\Psi}'\rangle|^2 = \mathrm{Tr}(O_h^\dagger O_h) \sim d_h. \tag{S13}$$

as O_h acts only in the original subspace (with dimension d_h) and it is zero on the attached space with dimension D_h. On the other hand, the energy spacing $|\Delta E + E(\Psi) - E(\Psi')|$ can now be made as small as $\widetilde{W}/D_h$, where $\widetilde{W}$ is the width in energy of all states that significantly couple to the original bubble states. It follows that the ratio (S11), and hence (S6), is reduced by a factor $W/\widetilde{W}$. If this effect rendered (S11) smaller than 1, the eigenstates would likely not hybridize across the subspaces, i.e. we would find *localization induced by coupling to further degrees of freedom*. However, the maximal conceivable value of $\widetilde{W}$ is $n\epsilon_m$, which is the local energy that is available for spreading. This yields $\widetilde{W}/W \lesssim n$, which is insufficient for localization, since the ratio (S6) is exponentially large in n. In contrast, such effects have been observed and discussed in the context of single particle problems coupled to extra degrees of freedom in Refs. [16, 21]. In those cases, there is no exponentially large factor that offsets the effect of an increased bandwidth $\widetilde{W}$, which makes coupling-induced localization possible.

Dynamic retardation

Even though the attachment of the states on the lower line of (S9) cannot prevent hybridization, it does of course increase the *timescale* necessary for transitions between the two bubble positions. The transition rates can be estimated from a simple Fermi Golden Rule calculation as

$$\tau^{-1} \sim \frac{|\langle \Psi\eta|H_1|\Psi'\eta'\rangle|^2}{|\Delta E + E(\Psi) - E(\Psi')|}, \qquad \frac{|\langle \widetilde{\Psi}\eta|H_1|\widetilde{\Psi}'\eta'\rangle|^2}{|\Delta E + E(\widetilde{\Psi}') - E(\widetilde{\Psi})|}, \tag{S14}$$

before and after attaching the extra states, respectively. The first rate is of order g^2/W, while the second is of order $(d_h/D_h)g^2/\widetilde{W}$. Hence, by adding the new states, we have increased the timescale by order D_h/d_h (keeping only terms exponential in n). This is very intuitive:

Transitions are now only possible from a fraction d_h/D_h of all states. One can also view this as follows: For a large bubble close to criticality (with structure *cciiiiiiiiicc*) the 'active' configurations of the type *cccchhhhcccc* manifest themselves as large deviations and they occur with exponential rarity. Yet, as shown above, they do lead to hybridization of eigenstates, and hence to delocalization.

ASSISTED HOPPING MODEL IN $d = 2$

Here we describe our numerical analysis for an assisted hopping model. The main aim is to show that delocalization on a resonant subgraph remains robust to adding additional terms that connect that subgraph to localized states. We also show coexistence of localized and delocalized states, a failure of Mott's argument, which is, however, a particularity of the zero density limit of the considered model.

Description of the model.

To reach the largest possible system sizes, we consider a Hamiltonian in $d = 2$ with spin-orbit coupling, which gives rise to weak anti-localization and thus allows for a genuine delocalized phase. To the best of our knowledge, this is the smallest system where delocalization can be expected, and is thus best suited for a numerical analysis. Here, 'smallest' means that the dimension of the Hilbert space grows at the slowest possible rate with growing linear size L.

Let H be the Hamiltonian of two indistinguishable hard-core bosons (with positions $q_{1,2}$) having a single spin$\frac{1}{2}$ degree of freedom, s, attached to them. We consider points $q = (x, y)$ on the lattice $(\mathbb{Z}/L)^2$ and we impose periodic boundary conditions. The full Hamiltonian is

$$H = H_0 + h_1 H_1 + h_2 H_2, \tag{S15}$$

where H_0 is the uniformly distributed on-site potential

$$H_0 = \sum_q \epsilon_q a_q^+ a_q, \quad -W \le \epsilon_q \le W. \tag{S16}$$

H_1 is the single-particle hopping Hamiltonian

$$H_1 = \sum_{q\sim q'} (a_q^+ a_{q'} + a_q a_{q'}^+), \tag{S17}$$

($q \sim q'$ denoting nearest neighbors) and H_2 is the assisted hopping, including a spin-orbit interaction. We describe H_2 by its matrix elements. Let

$$\begin{aligned} \mathcal{S} \;=\; & \{q_1 = (x_1, y_1),\, q_2 = (x_2, y_2) \,:\, q_1 \neq q_2, \\ & \max\{|x_1 - x_2|, |y_1 - y_2|\} \le 1\} \end{aligned} \tag{S18}$$

be the set of pairs of spatially neighboring points. We then define $\langle q_1', q_2', s'|H_2|q_1, q_2, s\rangle$ to be

$$\mathbb{I}_{\mathcal{S}}(q_1', q_2')\mathbb{I}_{\mathcal{S}}(q_1, q_2)\, \langle q_1', q_2', s'|H_{\rm SO}|q_1, q_2, s\rangle, \tag{S19}$$

where the characteristic functions $\mathbb{I}_{\mathcal{S}}$ ensure that the initial and final pair configuration belong to $\mathcal{S}$. Further, $H_{\rm SO} = H_{\rm SO}^1 + H_{\rm SO}^2$ with

$$\begin{aligned} H_{\rm SO}^1 \;=\; & -i\left[\sigma^{(x)} T_{y_1} - \sigma^{(y)} T_{x_1}\right] \\ & - \; i\left[\frac{(\sigma^{(x)} - \sigma^{(y)})}{2} T_{x_1} T_{y_1} - \frac{(\sigma^{(x)} + \sigma^{(y)})}{2} T_{x_1} T_{y_1}^\dagger\right] + \text{h.c.} \end{aligned} \tag{S20}$$

Here $\sigma^{(x,y)}$ are Pauli matrices acting on the spin degrees of freedom, while the translation operators are defined by $T_{x_1}\big|(x_1, y_1), (x_2, y_2), s\big\rangle = \big|(x_1+1, y_1), (x_2, y_2), s\big\rangle$ and similarly for T_{y_1}. $H_{\rm SO}^2$ is defined analogously.

The Hamiltonian $H_{\rm SO}^1$ is a lattice version of the Rashba Hamiltonian $\sigma^{(x)} p_{y_1} - \sigma^{(y)} p_{x_1}$. We notice that restricting the definition of $H_{\rm SO}^1$ to the first term $-i\{\sigma^{(x)} T_{y_1} - \sigma^{(y)} T_{x_1}\}$ would lead to a degeneracy due to the lattice structure. This would prevent H from being a generic GSE Hamiltonian for any value of h_2.

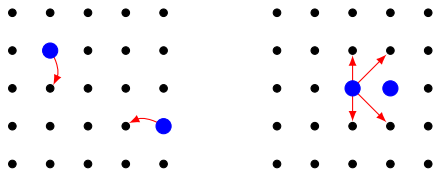

FIG. S1: Hopping for the Hamiltonian H. Left panel: single-particle hopping. Right panel: assisted hoppings of the left particle, allowed by the presence of the right particle.

Numerical results.

In all the simulations, we take $L = 9$ and $W = 1$. The analysis is divided into two parts.

(i) Delocalization via assisted hopping - First we take $h_1 = 0$ and $h_2 > 0$ (only assisted hopping). Since the majority of states (all configurations outside $\mathcal{S}$) are now trivially localized, we restrict ourselves to the subspace $\mathcal{H}_{\mathcal{S}}$ spanned by all the classical states in $\mathcal{S}$ (see (S18)), each coming with spin up/down. We aim at finding h_2 such that $H_0 + h_2 H_2$ can be considered a "typical" GSE matrix with truly delocalized eigenstates. For this, we evaluate numerically the parameter r defined as

$$r = \Big\langle \frac{1}{\dim(\mathcal{H}_{\mathcal{S}}) - 2} \sum_{n=2}^{\dim(\mathcal{H}_{\mathcal{S}})-1} \frac{\min\{\Delta E_n, \Delta E_{n-1}\}}{\max\{\Delta E_n, \Delta E_{n-1}\}} \Big\rangle, \tag{S21}$$

where $\langle\cdot\rangle$ is the disorder average and $\Delta E_n \equiv E_{n+1} - E_n$, with E_n being the ordered eigenenergies of the system. For the three classical ensembles, they take the values

$$r(\mathrm{GOE}) \simeq 0.53, \quad r(\mathrm{GUE}) \simeq 0.60, \quad r(\mathrm{GSE}) \simeq 0.67. \tag{S22}$$

For $h_2 = 0.7$, we find $r = 0.64 \pm 0.05$. This value is significantly larger than $r(\mathrm{GUE})$. The discrepancy with $r(\mathrm{GSE})$ presumably arises from the contributions from the more localized edges of the spectrum.

To characterize (de)localization we use the logarithm of the inverse participation ratio,

$$\mathrm{logIPR}(\psi) \equiv -\log_{10}\Big(\sum_{\eta} |\langle\psi|\eta\rangle|^4\Big), \tag{S23}$$

where the sum over η runs over the classical particle configurations.

Note that $\dim(\mathcal{H}_{\mathcal{S}}) = 648$, and thus $\mathrm{logIPR}(\psi) \sim 2.5$ for a fully delocalized state ψ. From the point of view of the parameter r, $h_2 = 0.7$ is rather optimal: The spectrum is mostly delocalized, but the Hamiltonian is still genuinely GSE. Indeed, when h_2 becomes significantly larger than 0.7, the localized tails of the spectrum are further suppressed, but the value of r starts bending down as an effect of approaching the integrable limit $h_2 \to \infty$.

(ii) Robustness of delocalization against addition of single particle hopping - Let us now fix $h_2 = 0.7$, but vary $h_1 > 0$. We determine numerically the statistics for the logIPR's of the eigenstates ψ of H. The results are shown in Fig. S2. The central message of that data is the following: Adding a finite h_1, which connects the resonant subspace $\mathcal{S}$ to its much larger localized complement, does not destroy the delocalization on the resonant subspace, as shown by the left and middle panel of Fig. S2. In particular, for $h_1 = 0.07$ (middle) we see delocalized states (inside the subspace $\mathcal{H}_{\mathcal{S}}$) coexisting with a majority of localized states. Obviously a relatively large h_1 leads to delocalization of almost all states, with logIPR's that start approaching the value $\log_{10}[\dim(\mathcal{H}) = 6480] \sim 3.5$ of fully delocalized wavefunctions, cf. the right panel. A comparison of histograms at the same values of h_1, but with $h_2 = 0$ (not shown) revealed that the histograms are significantly shifted to larger logIPR in the presence of the delocalized channel of mobile pairs.

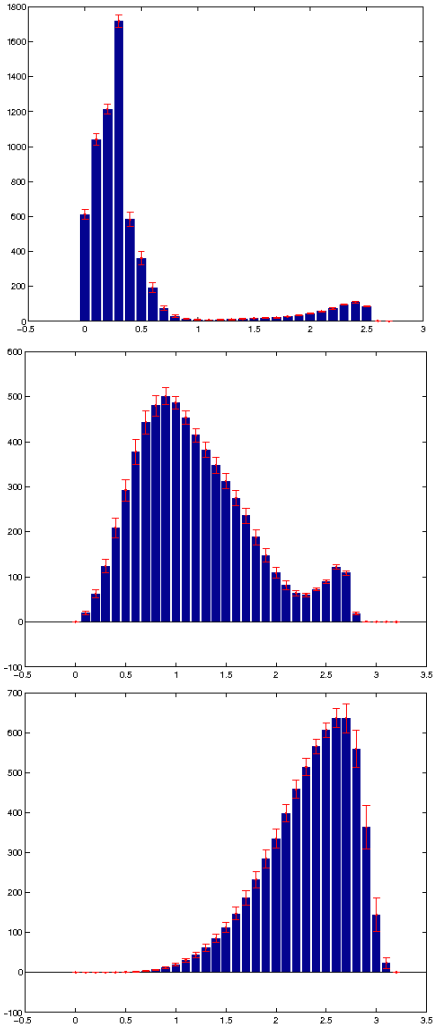

FIG. S2: Statistics for the logIPR's of all eigenstates of H for $h_2 = 0.7$ and different values of h_1. From left to right: $h_1 = 0.01$, $h_1 = 0.07$, $h_1 = 0.15$. Averages are taken over 500 realizations.

TIME AVERAGED ENERGY IMBALANCE BETWEEN AN INITIALLY COLD SPOT IN A HOT SURROUNDING

As explained in the main text, we studied the time average of the energy imbalance $\Delta\varepsilon$, between a cold region of $L_c = 2$ sites and a hot region of $L - L_c$ sites. The initial state is a product between the ground state in the cold region and an eigenstate near the middle of the band in the hot region. The (time-averaged) imbalance

$$\Delta\varepsilon = (L-3)^{-1} \sum_{i \notin \{c, c\pm 1\}} (\varepsilon_i - \varepsilon_c) \tag{S24}$$

was studied as a function of system size L and disorder strength δJ. The results are shown in Fig. S3, where each plot corresponds to a different disorder value. The results were averaged over 5000 disorder realizations. The data show that $\Delta\varepsilon$ decreases slowly with system size. The dependence is consistent with exponential decay. We have fitted the associated characteristic length $\xi_{\delta J}$, which grows with increasing δJ. These lengths $\xi_{\delta J}$ are of the same order as system sizes achievable in current numerical studies.

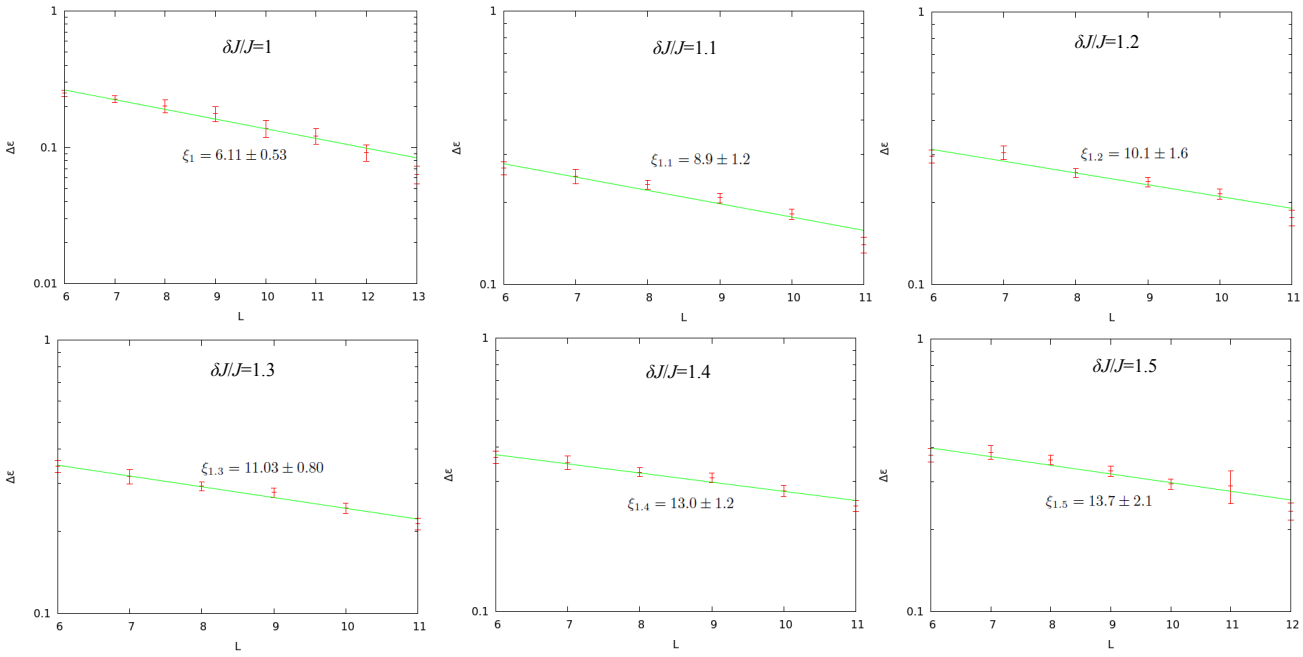

FIG. S3: Energy imbalance $\Delta\varepsilon$ as a function of system size, for disorder strengths $J \leq \delta J \leq 1.5J$.