J Low Temp Phys (2009) 156: 84–94
DOI 10.1007/s10909-009-9903-1

The Decay of a Quantized Vortex Ring and the Influence of Tracer Particles

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Received: 15 April 2009 / Accepted: 12 June 2009 / Published online: 9 July 2009

Abstract We capture the decay of a quantized vortex ring in superfluid helium-4 by imaging particles trapped on the vortex core. The ring shrinks in time, providing direct evidence for the dissipation of energy in the superfluid. The ring with trapped particles collapses more slowly than predicted by an available theory, but the collapse rate can be predicted correctly if the trapping of the particles on the core is taken into account. We theoretically explore the conditions under which particles may be considered passive tracers of quantized vortices and estimate, in particular, that their dynamics on the large-scale is largely unaffected by the burden of trapped particles if the latter are spaced by more than ten particle diameters along the vortex core, at temperatures between 1.5 K and 2.1 K.

Keywords Superfluid helium · Quantized vortex

1 Introduction

One way of studying fluid dynamics is to follow the motions of particles suspended in the fluid. In classical fluids, the equations governing the particle motion are described in [1]. The technique is subject to several additional constraints in superfluid helium [2]. This paper, which considers the decay of quantized vortex rings, is a contribution to the understanding of particle motion in superfluid helium-4.

The interpretation of the motion of particles suspended in superfluid helium is in general complicated because the fluid itself has a complex nature [2, 3]. However,

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particles can be trapped on the cores of quantized vortices, according to the mechanism described by Parks and Donnelly [4]. When that happens, the particles make it possible to identify individual quantized vortices [5]. Furthermore, so long as a particle is trapped on the vortex core, its velocity normal to the tangent to the vortex is equal to the local velocity of the vortex itself. Bewley et al. [6] exploit this property to study the dynamics of quantized vortices undergoing reconnection; see also [7]. In this paper, we use particles to observe the decay of a quantized vortex ring and, more generally, explore the effect of the trapped particles on vortex dynamics.

We discuss our observations in the context of the two-fluid model of interpenetrating fluids [8], consisting of the superfluid which contains the quantized vortices and the normal fluid which is viscous. The mean velocity of one fluid is opposite that of the other in a counterflow. Turbulent kinetic energy in the superfluid is borne by quantized vortices, and at temperatures near T_λ, dissipation occurs when energy is transferred, by the so-called mutual friction, from the superfluid to the normal fluid, where the energy is dissipated through the familiar viscous mechanism.

Quantized vortex rings were detected in superfluid helium more than 40 years ago by Rayfield and Reif [9] and Careri, Cunsolo, and Mazzoldi [10]. These experimentalists studied the time of flight of ions through a sample of superfluid helium, under the influence of an electric field, and determined that the ions must be trapped on quantized vortex rings moving through the fluid. Their work helped to establish the diameter of a quantized vortex core, and the mutual friction parameters [11]. Recent work using numerical models has established that quantized vortex rings probably travel through liquid helium with companion vortex rings in the normal fluid [12].

The paper is in three parts. In the first part, we report the observation of a vortex ring and measure its decay. The ring decays with a different time dependence than is predicted by existing models. In the second part, we incorporate in the dynamics the hydrogen particles trapped on the vortex core and show that the model accurately reproduces the observed lifetime. In the last part, we extend the arguments to explore theoretically the circumstances under which particles can be regarded as passive tracers of vortex dynamics.

2 Experimental Observations

We observe the ring in a sample of superfluid helium containing a suspension of particles, prepared as in Bewley [13]. The particles, made of solid hydrogen, are of the order of 1 µm in diameter. Some particles become trapped on the cores of quantized vortices [5] and make the motions of the cores traceable. We observe the particles by shining in the fluid a light sheet of about 100 µm thickness, and imaging the sheet with a digital movie camera having a resolution of 16 µm per pixel. We cool the fluid steadily at a rate of 10 mK/min causing a thermal counterflow in the fluid, but there is no reason to think that the counterflow directly affects the dynamics of quantized vortices, as discussed in Bewley et al. [6].

Figure 1 shows photographs of a ring whose diameter decays over time, while it remains nearly aligned with the illuminating light sheet. The temperature is 2.06 K. To measure the diameter of the ring, we refer to the diagram in Fig. 2, and provide the

Fig. 1 The photos are taken 1 s apart, as a ring vortex ring passes through the sheet of illuminating light. The vortex is made visible by a collection of hydrogen particles that are trapped on its core. The width of each photo is 2 mm

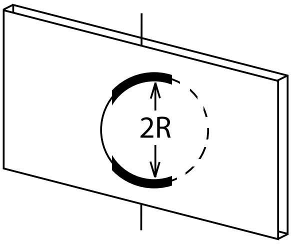

Fig. 2 In the early stages of the evolution of the ring shown in Fig. 1, the plane of the ring is inclined with respect to that of the illuminating light sheet. The plane of the ring shown is in the page, and the illuminating light sheet as it intersects the ring is shown in perspective. The portions of the ring that pass through the light sheet appear brighter and thicker because of blooming on the sensor. The portions of the ring that lie outside of the illuminating light sheet are also visible, but less brightly because they are illuminated by diffuse light scattered off other particles and surfaces inside the cryostat

following commentary. The camera projects the ring onto the image plane, which is the same as the plane of the light sheet. The ring is tilted with respect to the plane of the light sheet, and some part of the ring intersects the light sheet. The ring is brighter where it intersects the light sheet, and we mark this part of the ring in Fig. 2 with a thicker line. As can be seen in the figure, a circular ring could appear eccentric in the projection on the image plane, if it were rotated relative to the image plane. The pattern of illumination of the ring in Fig. 1 suggests that this is the case, and that the ring is rotated predominantly about the vertical axis, with respect to the image plane. Because of this, the projection of the ring onto the image plane only distorts the horizontal dimension, or the width, of the ring. Its height in the image is not modified by the projection and is, in fact, the diameter of the ring.

The measurements shown in Fig. 3 correspond to the distance between the centers of the thicknesses of the ring at the top and bottom of the ring image. The ring decays into a particle aggregate in about 6 seconds, with nearly linear decay over time.

If there is no counterflow in the helium, the lifetime of a vortex ring is approximately

$$\tau_t = \frac{2\pi R_0^2}{\alpha\beta_0\kappa}, \tag{1}$$

where R_0 is the initial radius of the ring, α is a mutual friction coefficient, and $\beta_0 = \ln(\frac{8R_0}{a}) - \frac{1}{2}$ [8]. Equation (1) predicts that a vortex ring with the same initial radius as the one observed would decay in about 2 seconds, or three times faster than the

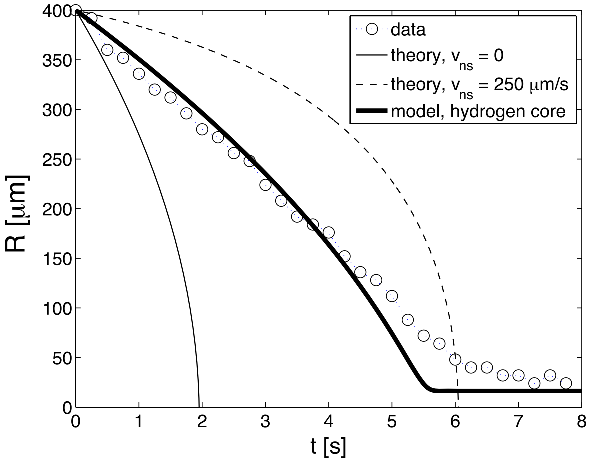

Fig. 3 The radius of the ring measured along its major axis. The error in measurement of the diameter is about 16 μm. The image of the ring closes after about 6 s. When the relative velocity between the normal fluid and superfluid, v_{ns}, is zero, theory predicts that the ring would decay in about 2 s. A counterflow velocity of $v_{ns} = 250\ \mu\text{m/s}$ causes a vortex ring to have the same lifetime as the one observed, but with a time dependence of the ring radius that does not agree with measurements. The present model, described in the text, reproduces the flattening of the decay of the radius, and accurately predicts the lifetime of the vortex ring

observed decay time. Furthermore, it would do so with nearly quadratic dependence of the lifetime on the ring radius, as can be seen in Fig. 3.

The large-scale relative motion between the superfluid and the normal fluid causes a quantized vortex ring to either become larger or smaller over time, depending on the relative velocity between the fluids, and the initial diameter of the vortex ring [8]. According to the theory, a counterflow of 250 μm/s opposing the vortex ring causes it to decay in about 6 seconds, equal to the lifetime of the observed vortex ring. However, we shall now show that the counterflow cannot account for the observations.

While we cannot state with certainty the magnitude or direction of the local counterflow in the experiment, we can estimate it by measuring the heat removed evaporatively from the helium bath. Heat flows into the helium through the imperfect insulation of the cryostat, steadily heating it up when the cooling is switched off. From the rate of temperature rise of the helium, we estimate the total heat load on the helium is of the order of 100 mW. That amount of heat flux produces a counterflow of about 500 μm/s; in fact, this magnitude is comparable to the observed velocities of the particles. The heat flux enters the helium bath from the entire surface area, but leaves essentially from the free surface of the bath. Therefore, we should expect to see significant heat flux only at the free surface, where it is likely to be oriented perpendicular to the latter; this direction is in the plane of the vortex ring. This orientation of the heat flux leaves the evolution of the ring radius unaffected. More importantly, as shown in Fig. 3, the functional form of the ring decay predicted by this hypothesis is quite different from the data. Therefore, we seek another explanation for the retarded decay of the observed ring.

3 A Model for the Ring Decay

3.1 The Governing Equation

We review the derivation of the equation governing the evolution of the vortex ring, and add to it the effect of particles decorating the core. Following Barenghi et al. [11], we balance the Magnus force on an element of the ring against the external forces on the element, so that

$$\mathbf{f}_M + \mathbf{f}_v = 0. \tag{2}$$

We assume that the vortex core has a much smaller radius than the ring, and that the normal fluid is quiescent, as is the superfluid far from the ring. Furthermore, we neglect the influence of sound generation by pressure fluctuations, which is thought to be negligible at the temperature of the experiment, and at the observed scales [14].

The interaction between the circulation of the vortex and its movement through the fluid creates the Magnus force,

$$\mathbf{f}_M = \rho_s \kappa \mathbf{s} \times (\mathbf{v}_L - \mathbf{v}_i), \tag{3}$$

where ρ_s and κ are the density and circulation of the quantized vortex, respectively, $\mathbf{s}$ is a unit vector tangent to the ring, $\mathbf{v}_L$ is the velocity of an element of the ring, and $\mathbf{v}_i$ is the self-induced fluid velocity of the ring.

The self-induced velocity, $\mathbf{v}_i$, is the flow induced at any point on the ring by the rest of the ring. The vortex ring acts on itself because it is curved on itself, resulting in its motion through the fluid. In the absence of other forces, a vortex ring moves with the induced velocity

$$v_i = \frac{\kappa}{4\pi R}\left(\ln\frac{8R}{a} - \frac{1}{2}\right), \tag{4}$$

where R is the ring radius and a that of the core. The induced velocity depends strongly on the ring radius but only logarithmically on the core size.

Forces on the vortex core, $\mathbf{f}_v$, can be written generally as

$$\mathbf{f}_v = -D_D \mathbf{v}_L - D_L \mathbf{s} \times \mathbf{v}_L. \tag{5}$$

This equation includes a drag term, D_D, and a lift term, D_L (both of which are expressed in units of force per unit length per unit velocity). We do not include a component of the force parallel to the vortex core because there is no motion along it. In the case of a quantized vortex without particles, the coefficients, D_D and D_L are the temperature-dependent mutual friction parameters, γ_0 and γ_0', respectively.

Using $\mathbf{s} = (0, 1, 0)$, $\mathbf{v}_L = (\dot{R}, 0, v_z)$, and $\mathbf{v}_i = (0, 0, v_i)$ in polar coordinates, such that the unit vectors are $(\hat{R}, \hat{\theta}, \hat{z})$, with $\hat{z}$ as the axis of symmetry of the ring, (2) to (5) combine to form two scalar equations. The variable v_z can be eliminated to yield the governing equation for the ring radius as

$$\dot{R} = \frac{dR}{dt} = -\beta P \frac{\kappa}{4\pi R}. \tag{6}$$

The rate at which the ring collapses is controlled by the parameters

$$\beta = \ln\frac{8R}{a} - \frac{1}{2}, \tag{7}$$

$$P = \frac{\rho_s \kappa D_D}{D_D^2 + (D_L - \rho_s \kappa)^2}; \tag{8}$$

the larger the values of β and P, the more quickly does the ring decay. The parameter β contains the influence of the vortex core radius on the self-induced velocity of the vortex. The parameter P measures the influence of the drag and lift forces on the vortex core. The parameter P has a maximum as a function of the drag force, D_D, so the ring decays slowly when the drag is both large and small.

3.2 The Effect of Particles

To model the observed decay, we consider the viscous drag on the particles decorating the vortex ring and the fact that the particles impede circulating superfluid. The two phenomena affect the values of P and β, respectively. We shall see that the primary effect of the particles in determining the decay of the ring appears through β.

First, the particles add viscous drag to the term D_D. However, there is no reason to think that the particles contribute to the lift, D_L; they would do so only if they were rotating, but we know of nothing that produces a torque on the particles. To find the viscous drag, we treat the hydrogen particles on the core of the vortex as forming a circular cylinder, which is itself formed into a circular ring. This may be valid when the hydrogen particles are so small, and so closely collected, that the core behaves like a uniformly malleable aggregate in the shape of a cylinder. Where the rings are intersected by the light sheet, the images do indeed appear uniformly bright. Furthermore, as the vortex ring shrinks, the radius of the ring of particles does not shrink below a certain value, in order to respect the constancy of the volume of particles. Note that we ignore shear stresses in the core so that it takes no work to reshape it.

We assume that the drag per unit length on the circular ring of particles is the same as that on a straight cylinder, and use Batchelor's expression [15] for the latter,

$$\mathbf{f}_c = \frac{4\pi\mu}{\ln(\frac{3.7\nu}{r_c v_L})}\mathbf{v}_L, \tag{9}$$

where μ and ν are the dynamic and kinematic viscosities of the fluid, r_c is the radius of the cylinder, and v_L is the magnitude of $\mathbf{v}_L$. Equation (9) is applicable as long as the ring radius is much larger than the diameter of the cylinder of particles embedded on the vortex core; it is accurate for low Reynolds numbers, which we estimate to be between 0.2 and 0.5 during the observed collapse of the ring. By equating $\mathbf{f}_c$ and $\mathbf{f}_v$ in (5), we have

$$D_D = \frac{4\pi\mu}{\ln(\frac{3.7\nu}{r_c v_L})} \tag{10}$$

for the particle-decorated core.

When the core is decorated with hydrogen particles, its radius, a in (7), is the radius of the cylinder formed by the particles, r_c. Note that the hollow-core model for the vortex ring is justified since the hydrogen takes the place of the fluid. Furthermore, the radius of the core must evolve with time, since the mass of hydrogen is conserved, so that

$$r_c^2 = \frac{V_c}{2\pi^2 R}, \tag{11}$$

where V_c is the volume of the particle-laden core.

We estimate the volume of solid hydrogen in the core from the images early in the evolution of the ring, and of the particle aggregate remaining after the ring collapses. Both estimates give a value of $V_c \approx 2 \times 10^{-6}$ cm^3. Errors in the estimation of the volume of hydrogen is the main contributor to the error in predicting the lifetime of the modeled vortex ring but this contribution is still moderate: a factor of four in the volume changes the lifetime only by about 25%.

By the coating of particles, the effective value of the radius a in (7) becomes larger. This reduces the parameter β to a third of its value without particles, causing the vortex ring to decay more slowly. On the other hand, the effect of particle loading on the parameter P is smaller. This may seem strange because of the direct appearance of the drag in P. The explanation appears to be that for each particle that sticks to the vortex ring, there is a commensurate reduction in the induced velocity, making the drag smaller. In this respect, there seems to be an approximate compensation of the two effects due to particle loading: increased area exposed to the flow and the decreased velocity of motion.

Another point worth making is that the values of P with or without mutual friction are comparable. While we cannot say with certainty whether the existence of particles on the core makes mutual friction irrelevant, this observation appears to point in that direction.

3.3 Numerical Method

The model for the vortex ring can now be integrated numerically to obtain quantitative answers. We make a first order approximation of the time derivative in (6), and compute successive values of the radius of the ring, R, according to

$$R(t+\Delta t) = R(t) - P\beta \frac{\kappa}{4\pi R}\Delta t, \tag{12}$$

where Δt is the time increment.

A complication arises when the drag is given by (10). The speed, v_L, appears in the logarithmic term, and v_L is itself a function of $\dot{R}$. This means that (6) cannot be solved analytically for $\dot{R}$. However, according to (9), the primary variation in the drag force, $\mathbf{f}_c$, is due to its linear dependence on $\mathbf{v}_L$, and the variation due to the log term is comparatively small. On the basis of this observation, we solve (12) by using v_L from the previous iteration, at time t, to evaluate the logarithmic term in (10).

In order to test that the solution depends only weakly on the logarithmic term, we also solve (12) under the assumption that the logarithmic term is constant. We choose constant values that are a factor of two larger and smaller than those that result when the log term is evaluated using v_L from the previous iteration. Solving (12) gives lifetimes of the ring that are different by less than 10% from those that include the log variation described above. We conclude that assuming slow variation of the log term is quite reasonable.

In Fig. 3, we plot the result of solving (12), using a step size of $\Delta t = 1$ ms. We check the method by confirming that the work done by the drag force during the collapse of the ring is equal to the energy lost. That is, $\Delta E = 2\pi \sum R\mathbf{f}_c \cdot \mathbf{v}_L \Delta t$, where ΔE is given

$$\Delta E = \frac{1}{2}\rho\kappa^2 R_0\left(\ln\frac{8R_0}{a} - 2\right). \tag{13}$$

The model accurately depicts the evolution of the ring.

The lifetime of the vortex ring can be defined as the time taken for the ring radius to equal the core radius. Without particles, under the influence of only mutual friction, this occurs after 2 s according to (1), or at about a third of the observed value. With particles, the lifetime of the ring is 5.3 s, comparable to the observed time of about 6 s.

Although including such effects as the inertia of the particles, motion of the normal fluid, and counterflows in the fluid might produce a more accurate depiction of the decay, the present model seems to be sufficient to demonstrate the influence of particles. We have also neglected the mechanical strength of the hydrogen particles in the core, and the close correspondence of the model and the observation suggests that hydrogen particles do not make the quantized vortices rigid and immobilized.

4 Particles with Space Between Them

Although the above results show that a bare quantized vortex behaves differently from that with continuous particle coating, there are circumstances in which it is possible to treat particles as passive markers. To understand these circumstances, we consider a vortex with regularly spaced particles, as in Fig. 4. Lets us denote the ratio of the distance between particles, l, to the particle diameter, d, as

$$\sigma = \frac{l}{d}. \tag{14}$$

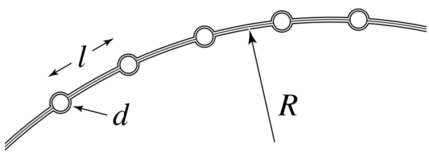

Fig. 4 For the observed ring, particles on the core form a malleable aggregate and modify the evolution of the vortex. To explore the effect of the particles if they were spaced far apart along the core of the quantized vortex, we model hydrogen particles as spheres with diameter d trapped at intervals l along the core of a vortex whose radius of curvature is R

In fact, in experiments on vortex lines, particles seem to be spaced sparsely along the core [6] rather than continuously coating them as seen to be the case for vortex rings. As described in Bewley [16], the mean spacing of particles under certain conditions is about 130 µm, or of the order of 100 particle diameters. The distribution is essentially Gaussian with a standard deviation of about 25 µm.

In the case of a vortex ring, the primary effect of particles on the lifetime of the ring is due to the enlargement of the effective core radius, which lowers the self-induced velocity, as reflected in the logarithmic term, β, in (7). In order for the dynamics of the vortex to be similar with and without particles, the ratio

$$S_1 = \left(\ln\frac{8R}{r_e} - \frac{1}{2}\right)\Big/\left(\ln\frac{8R}{a_0} - \frac{1}{2}\right) \tag{15}$$

should be close to 1, where r_e is the effective radius of the core with particles on it, and a_0 is the radius of the core without particles.

To estimate the effect of the particles that are spaced apart along the core, as in Fig. 4, we make the approximation that the effective core radius, r_e is equal to the average core radius over a long length of line, $r_e = \frac{\pi}{8}\frac{d}{\sigma}$. This assumption may be valid as long as the particle spacing, l, is much smaller than the radius of curvature of the vortex, R. Equation (15) gives a value of r_e for which S_1 is within agreeable bounds. For an R of 100 µm, $S_1 < 1.1$ when $r_e < 1.5 \times 10^{-3}$ µm. This, in turn, specifies a minimum spacing of the particles, σ,

$$\frac{\sigma}{d} = \frac{\pi}{8r_e} \approx 260\ \mu\text{m}^{-1}. \tag{16}$$

In other words, 1 µm particles should be spaced by about 260 diameters along the core of the vortex in order for the self-induced velocity to be within 10% of its value with no particles. Note that smaller particles can be closer together than larger ones, for the same effect on the self-induced velocity.

We remarked earlier that particle loading does not have large impact on the decay rate through the parameter P. In other words, the viscous drag on particles does not play a significant role in modifying the lifetime of the vortex ring of Fig. 1. However, the drag does affect other aspects of vortex dynamics. To expand on this point, we compare the mutual friction force, per unit length of vortex line, $\mathbf{f}_m$, to the drag force on the particles, per unit length of line, $\mathbf{f}_p$. In doing so, we distribute continuously over the vortex the forces applied locally by the particles. In effect, this assumes that the vortex is stiff on scales smaller than l, which is probably not true, but the assumption allows us to make a rough estimate of the influence of the particles. In general, the mutual friction force can point in a direction that is different from that of the drag, but it is instructive to compare their magnitudes,

$$S_2 = \frac{|\mathbf{f}_m|}{|\mathbf{f}_p|} \approx \sigma\frac{\rho_s\kappa}{3\pi\mu}(\alpha^2 + \alpha'^2)^{1/2}. \tag{17}$$

Here, we use the drag on a spherical particle, $3\pi\mu d v_L$ [15], and neglect the interaction between particles, which makes the formula valid for $\sigma \gg 1$. The temperature dependent parameters, α and α', describe the strength of the mutual friction. The

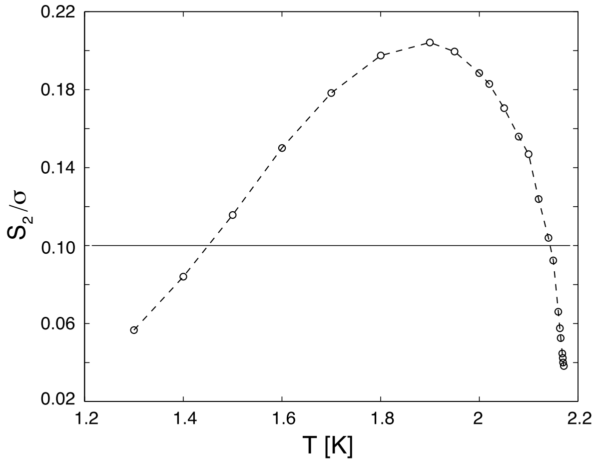

Fig. 5 The ratio of the magnitude of the force on a quantized vortex core due to mutual friction to the drag caused by particles trapped on the core of the vortex. We incorporate the factor σ, which is the ratio of the particle spacing to their diameter. As described in the text, the particle drag for $\sigma = 1$ is at least 5 times larger than mutual friction. The *horizontal line* separates the region where viscous drag dominates from that where mutual friction dominates when the particles are spaced by 10 diameters

magnitude of the geometric mean of their squares is of order one between 1.4 K and 2.1 K. Note that the size of the particle plays no direct role in determining the influence of the viscous drag.

In Fig. 5, we evaluate S_2/σ. When $\sigma = 1$, which approximately describes the case for the vortex ring presented above, S_2 is always less than unity and viscous drag dominates over the mutual friction. When $\sigma > 10$, or the particles are spaced by more than ten diameters, there is a range of temperatures between 1.5 K and 2.1 K for which the mutual friction may play a dominant role in determining the evolution of the quantized vortex. Note however, that we consider the action of the particles as being averaged over long lengths of line, and that the particles could still have a strong local effect. Instances of these effects are given in [16].

5 Conclusions

In summary, we have observed the decay of vortex rings by using solid hydrogen particles to mark them. The very occurrence of decay validates the phenomenology of dissipation in quantized vortex turbulence through the exchange of energy between the superfluid and the normal fluid. However, we observe that hydrogen particles substantially modify the behavior of quantized vortices on which they are trapped by effectively expanding the diameter of the vortex core. A model which treats the core of the vortex as a cylinder passing through a viscous fluid accurately predicts the lifetime of the observed vortex ring, including the flattening of the curve that the data follow for long times. These considerations provide a warning about the effect of particles on the behavior of quantized vortices. We extend the model to predict theoretically when the vortex dynamics would remain unaffected by the particles. We propose two parameters for gauging the effects of the particles, and find, as an example, that the effects may be negligible for a vortex with a 100 μm radius of curvature when 1 μm particles are spaced along it by more than 260 particle diameters. Between 1.5 K and 2.1 K, one can relax this condition and state that the trapped particles could experience smaller drag than the mutual friction force, in a global sense, when they are spaced by more than ten diameters. Under these conditions, the particles could be used as passive tracers of vortex dynamics.

Acknowledgements The data were acquired at the University of Maryland, as part of collaborative research with D.P. Lathrop. We are grateful to Jürgen Vollmer for discussion.

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