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Taylor's frozen-flow hypothesis in Burgers turbulence

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By detailed analytical treatment of the shock dynamics in the Burgers turbulence with large scale forcing we calculate the velocity structure functions between pairs of points displaced both in time and space. Our analytical treatment verifies the so-called Taylor's frozen-flow hypothesis without relying on any closure and under very general assumptions. We discuss the limitation of the hypothesis and show that it is valid up to time scales smaller than the correlation time scale of temporal velocity correlation function. We support the analytical calculation by performing numerical simulation of the periodically kicked Burgers equation.

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In a 1938 paper G. I. Taylor introduced an assumption by which he deduced the spatial fluctuations of a turbulent velocity from the corresponding measurements of temporal fluctuations at a single point [1]. This hypothesis, know as Taylor's frozen-flow hypothesis, relies on the existence of a large mean flow which translates the spatial structures past a stationary probe in a time smaller than the inherent evolution time of the fluctuations [2,3]. As remarked by Tennekes [4], the Eulerian temporal scaling is dominated [5,6] by large scale energy containing structures which sweep inertial-range information past an *Eulerian* observer [7,8].

The importance of this hypothesis relies on the fact that most turbulence theories center on the scaling behavior of spatial structure functions of the velocity field [2,3]. In experimental assessments of spatial fluctuations in turbulent flows [6–8] this hypothesis has been a general guideline for extracting the information from one-point measurements with hot wires. Original measurements treated by Taylor were made in turbulence generated behind a stationary grid in a wind tunnel, and his hypothesis has become a standard technique employed in similar experiments which build our current view on turbulence.

Although Taylor's hypothesis is significantly clear when there is a large scale average flow, its application in homogeneous isotropic flows has been a much more debated issue. The latter always is meant to reflect the qualitative picture that larger eddies randomly *sweep* the smaller eddies with their root-mean-square velocities.

Apart from a few theoretical attempts in verifying the hypothesis, a general quantitative framework for deriving it does not exist [9,10]. Such a quantitative understanding calls for characterization of the spatiotemporal fluctuations of small scale eddies responsible for random sweeping [10–13].

In retrospect the spatiotemporal structure functions are central objects in providing the required information. Such structure functions of arbitrary order q are defined as $S_q(x,t)=\langle|[\vec{u}(\vec{x}_2,t_2)-\vec{u}(\vec{u}_1,t_1)]\cdot\frac{\vec{x}}{x}|^q\rangle$, where the $\vec{u}(\vec{x}_i,t_i)$ are Eulerian velocities at two spatially distinct points that are measured at two different times. In a statistically stationary state and in the inertial subrange, multiscaling assumption implies

$$S_q(x,t) \propto x^{\xi_q} F_q\left(\frac{t}{x^{z_q}}\right), \tag{1}$$

where $x=|\vec{x}_2-\vec{x}_1|$ and $t=|t_2-t_1|$ [2,3]. F_q's are homogeneous functions of their arguments and z_q are dynamic exponents. Two sets of exponents ξ_q and ζ_q are defined by casting two asymptotic limits $\lim_{t\to 0} S_q(x,t)\simeq|x_2-x_1|^{\xi_q}$ and $\lim_{x\to 0} S_q(x,t)\simeq|t_2-t_1|^{\zeta_q}$, respectively.

Theoretical arguments resorting to the multifractal phenomenology support the existence of a hierarchy of dynamic exponents z_q if one compensates the sweeping effects by choosing a quasi-Lagrangian frame [14]. In an Eulerian frame for which the sweeping dominates the temporal fluctuations, $z_q=\frac{\xi_q}{\zeta_q}=1$, at least up to leading order [2,14].

Here we calculate ζ_q for the one-dimensional Burgers equation stirred a forcing with large scale correlation in space and with a Wiener scaling in time. The exponents are derived from equations of motion without relying on any closure by which the dynamic exponents z_q are inferred. The crucial role of shocks in establishing the result is spelled out and the scaling solutions of the dynamic structure functions, i.e., $\lim_{x\to 0} S_q(x,t)$, are obtained. Numerical simulations on the periodically kicked Burgers equation [15] are performed to support the analytical results for ζ_q.

Our numerical and analytical calculations both indicate that for incremental time $t=t_2-t_1$ less than or at the order of the correlation time scale, i.e., $t\lesssim t_{corr}$, the scaling of $\lim_{x\to 0} S_q(x,t)$ are dominated by the shock dynamics and saturate, i.e., $\zeta_q=1$ for $q\geqslant 1$, while $\zeta_q=q$ for $q<1$. As a by-product the dynamical exponents are equal to unity, i.e., $z_q=1$, consistent with Taylor's hypothesis. However when t

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$>t_{corr}$ the dynamics of $\lim_{x\to 0} S_q(x,t)$ are controlled by the scaling limit of the forcing. Hence we show that the Taylor frozen hypothesis does not hold for $t>t_{corr}$ when the forcing scales as a Wiener process in time. The crossover from shock-dominated to forcing-dominated regime is also verified numerically.

We consider the Burgers equation in one dimension, i.e.,

$$u_t + uu_x = \nu u_{xx} + f(x,t), \tag{2}$$

where $f(x,t)$ is a zero-mean Gaussian statistically homogeneous, and white in time random process with covariance, $\langle f(x_2,t_2) f(x_1,t_1)\rangle = C(x_2-x_1)\delta(t_2-t_1)$. The spatial correlation $C(x_2-x_1)$, as a function of $|x_2-x_1|$, is assumed to have a finite support $\sigma \simeq L$, where L is the system's domain [16–18].

A convenient way to derive the dynamical evolution of $S_q(x,t)$ is via the standard generating function method [17,18]. Defining $\Theta = \exp(-i\lambda_1 u_1 - i\lambda_2 u_2)$ the two-point generating function is given by $\langle\Theta\rangle$, where $\langle\cdots\rangle$ is an average over the forcing statistics. The Fourier transform of the generating function is the two point probability density function (PDF) $P(u_1,u_2,x_1,t_1,x_2,t_2)$ defined at the points x_1, t_1 and x_2, t_2 with their related velocities u_1 and u_2. From Eq. (2) the dynamical equations of PDF P at distinct times, say t_1 and t_2, are sought. Using the following change of variables $t=t_2-t_1$, $T=\frac{t_1+t_2}{2}$, and transforming u_1 and u_2 with $u=\frac{u_1+u_2}{2}$, $\omega = u_2-u_1$, one obtains the following equation for the PDF of velocity field difference as

$$\begin{aligned} P_t = &- uP_x - \frac{1}{2}\int K\left(u + \frac{\omega}{2} - u'\right) P_x\left(u - \frac{\omega}{2}, u'\right) du' \\ &- \frac{1}{2}\int K\left(u - \frac{\omega}{2} - u'\right) P_x\left(u', u + \frac{\omega}{2}\right) du' + \frac{1}{2}C(0)P_{u\omega} \\ &- \frac{1}{2}C(x)\left[\tfrac{1}{4}P_{uu} - P_{\omega\omega}\right] + \frac{G_2}{2} - \frac{G_1}{2}, \end{aligned} \tag{3}$$

where $x=x_2-x_1$, $y=\frac{x_1+x_2}{2}$. The $C(x)$ is the spatial correlation function of forcing, $K(u)=\frac{H(u)-H(-u)}{2}$, and $H(u)$ is the Heaviside function. The terms G_i's are defined as $G_i = -\nu\partial/\partial u_i\{\langle u_{ix_ix_i}|u_1,u_2,x_1,x_2,t_1,t_2\rangle P\}$, where $\langle u_{ix_ix_i}|u_1, u_2,x_1,x_2,t_1,t_2\rangle$ is the average of $u_{ix_ix_i}$ with the conditions that the velocity field has the values of u_i's at points x_i and in times t_i. Indeed the term G_i is the only term preventing Eq. (3) to be closed, which can be referred to a sort of dissipative anomaly [16].

Multiplying both sides of Eq. (3) with ω^q and integrating it over u and ω one obtains

$$\frac{\partial S_q}{\partial t} = -\frac{\partial S_{q,1}}{\partial x} + \frac{1}{2}q(q-1)C(x)S_{q-2} + \langle\omega^q(G_2-G_1)\rangle. \tag{4}$$

The first term in the right-hand side of Eq. (4) is of the type of mixed structure functions of center of mass and incremental velocities $S_{q,r}=\langle\omega^q u^r\rangle$. The second term is the contribution of forcing and the last term is the combination of dissipative terms $\langle\omega^q(G_2-G_1)\rangle$ in Eq. (4) that are at variance with the usual anomaly terms in the equations of spatial structure functions at one time $\langle\omega^q(G_2+G_1)\rangle$ [17,18].

In principle the solutions of the above equation in the inviscid limit should have scaling forms of the type introduced in Eq. (1). However, Eq. (4) is not obviously tractable because the first and last set of terms on the right-hand side are not expressed in terms of $S_q(x,t)$ and the equation is not closed. In what follows we show that it is possible to treat the unclosed terms by means of shock representation and without resorting to any closure model.

In the inviscid limit both types of the terms are expressible in terms of operators localized on shocks when $x\to 0$. Recall the velocity u satisfying the Burgers equation develops shock solutions in the limit $\nu\to 0$. One may represent a shock locally as $u(x,t)=u_+H[x-x_0(t)]+u_-H[x_0(t)-x]$, where H is a Heaviside function. The position x_0 is identified by two quantities, namely the velocity u in positions x_{0^+}, x_{0^-} (set it, for example, to 0). In other words the velocity gradient (corresponding Burgers velocity) is not continuous at points x_0. At these singular points $u_\pm$ is defined as $u_\pm(x_0,t) = u(x_{0\pm},t)$ keeping in mind that $u_- > u_+$, while the shock strength s and the shock velocity $\bar{u}$ are defined as $s=u_+-u_-$ and $\bar{u}=\frac{1}{2}(u_++u_-)$.

Here we argue that the last term in the right-hand side of Eq. (4) vanishes in a particular space-time window. Indeed in the inviscid limit only small intervals around the shocks will contribute to the G_i terms. Each of G_i's at space-time (x_i,t_i) are nonvanishing only on the shocks [17,18]. As a neat way of demonstrating the cancellation of G_i terms we return to the Fourier space and represent the related terms with $\mathcal{G}_i$. The anomaly terms $\mathcal{G}_2-\mathcal{G}_1$ in this space can then be written as

$$\mathcal{G}_2 - \mathcal{G}_1 = -\lambda^2\langle\epsilon_2(\lambda)e^{-\lambda u_1} - \epsilon_1(-\lambda)e^{\lambda u_2}\rangle, \tag{5}$$

where x_i is the position of the ith shock and $\lambda=\lambda_2-\lambda_1$ is the conjugate of ω and $\epsilon(\lambda;x,t)=\Sigma_i F(\bar{u}_i,s_i)\delta[x-x_i(t)]$. The form factors F are read $F(\bar{u},s)=-2e^{\lambda\bar{u}}\lambda^{-1}\partial_\lambda\lambda^{-1}\sinh(\frac{s\lambda}{2})$ [19]. This representation shows that the anomaly contribution is generally very complicated for arbitrary separation distance x and time difference t. However it is possible to identify a space-time regime in which the following operator vanishes. In fact by Taylor expansion one easily sees that for $|t_2-t_1|\,\frac{\partial u_2}{\partial s}|_{s=t_1} \simeq -|t_2-t_1|\bar{u}\,\frac{\partial u_1}{\partial x}|_{x=x_i(s)}$, given $x\to 0$, $\mathcal{G}_2$ and $\mathcal{G}_1$ cancel out in the leading order. This condition physically means that the anomaly due to a shock at point x_2 and in time t_2 is the same in statistical sense at a point x_1 but with a time delay approximately equal to $|t_2-t_1|\simeq\frac{|x_2-x_1|}{\bar{u}}$, because the shocks move with their local velocity $\bar{u}$. In this spatiotemporal window the time variations of velocities are mostly dominated by the random shocks which sweep the spatial fluctuations past a point [1,4].

The other set of terms on the right hand side of Eq. (4), $S_{q,1}=\langle\omega^q u\rangle$, are rooted in the uP_x term in Eq. (3). Using stationarity, i.e., $\partial_T P=0$, it is easy to reexpress $\lim_{x\to 0} P_x$ in terms of the measures describing the statistics of shocks. In the spatiotemporal regime where $x\lesssim\bar{u}t$ it is possible to show

$$\lim_{x,t\to 0} P_x(u,\omega,x,t) = N_x(\omega)M(u) = [\rho\langle s\rangle\delta^1(\omega) + \rho S(\omega,T)]M(u), \tag{6}$$

where $\delta^1(\omega)=d\delta(\omega)/d\omega$, and $N_x(\omega)=\int\lim_{x\to 0} P_x(u,\omega,x,t)du$. Here $\rho=\Sigma_i\delta[x-x_i(T)]$ and $S(\omega,T)$ are the shock number density in space and the PDF of $s(x_1,t_1)$ conditioned to x_1 being on a shock location, respectively.

To leading order the result is t *independent* and the averages on u and ω on the shocks *separate*. The functional dependence of $\lim_{x,t\to 0} P_x(u,\omega,x,t)$ on u appears as an unknown function $M(u)$, where $\int M(u)du=1$ (see also [14]). Although the precise form of $M(u)$ does not enter in later arguments the separation of u and ω is central for obtaining the later results.

In order to calculate ζ_q we use the representations in Eqs. (5) and (6) at leading order and substitute them back into Eq. (4). Therefore we obtain

$$\frac{\partial S_q}{\partial t} = -\rho\langle\bar{u}\rangle\langle s^q\rangle + \frac{1}{2}q(q-1)C(x)S_{q-2} + o(t). \qquad (7)$$

Solving the analytical scaling solutions in Eq. (7) we find the moments $q\geqslant 2$ of the velocity increments in time behave as

$$\langle|\omega|^q\rangle = |t|[-\rho\langle\bar{u}\rangle\langle|s|^q\rangle + C(0)\delta_{q,2}], \quad q\geqslant 2, \qquad (8)$$

where $\omega=\lim_{x\to 0}[u(x_1+x,t_1+t)-u(x_1,t_1)]$ and ρ is the average density of shocks in time. We remark that the leading term in $S_2(t)$ has a forcing contribution proportional to $C(0)$. The higher order moments, i.e., $q>2$ do not obtain contributions from forcing because they are always subleading in time.

For the lower order moments the competition of the regular part of the velocity increment in time dominates the contribution of shocks. One writes $[u(x,t_1+t)-u(x,t_1)]\simeq -\int_{t_1}^{t_1+t}\bar{u}(\partial_x u)ds+\int_{t_1}^{t_1+t}dW(s)$, where the first integral in the right-hand side resembles the sweeping of the spatial increment and hence scales as t. The second integral however scales as $t^{1/2}$ because the integral of the forcing is a Wiener process in time by definition. Consequently in leading order one may easily write

$$\lim_{x\to 0} S_q(x,t) \simeq A_q|t|^{q/2} + B_q|t|, \qquad (9)$$

where A_q and B_q are unknown amplitudes. Therefore, the low order moments scale as $t^{q/2}$ and the saturation should begin for orders $q\geqslant 2$.

To test the predictions of the analytical calculation, we have carried out numerical simulations of the Burgers turbulence by means of the so-called particle method [15]. The numerical experiments reported hereafter have been made with the kicking force and a kicking period $t_{kick}=0.001$. The number of collocation points chosen for our simulations is generally $N_x=10^5$. In order to perform temporal averages, since we need a large sequence of velocity time series to calculate intermittency in time, we run the algorithm for about $10^3 t_{kick}$. For time integration we adapt a time discretization of order $t_d=10^{-5}$. Between two subsequent kicks we advance the minimizers in time for $100dt$. We then construct the Eulerian velocity by means of the particles velocities after reaching to statistically stationary state. The results are shown in Fig. 1, in which the log-log plot of $\langle|u(x,t_1+t)-u(x,t_1)|^q\rangle$ is depicted as a function of the time increment t for different q's specified in the inset. The time span t in the measurements of velocity increments lies in the extended range of $t\in[10^{-5},2\times 10^{-1}]$. Therefore, we have the information about scaling both above and below the characteristic correlation time. The typical correlation time is about $t_{corr}\simeq 3\times t_{kick}$.

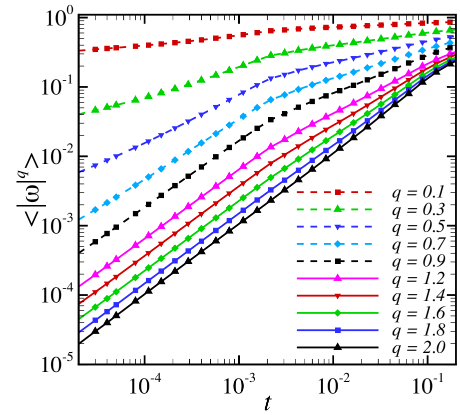

FIG. 1. (Color online) The scaling of the time increments of velocity $\langle\omega^q\rangle$ verses $t_2-t_1=t$ is sketched for different moments $q\in(0,2]$. The kicking time is 10^{-3} and the measured time span is showing the scaling both below and above it.

Although all high order moments $q\geqslant 2$ display a slope of approximately 1 the moments of orders $q\leqslant 2$ exhibit a crossover in their scaling. For $t\lesssim t_{corr}$ the scaling exponent is read to be $\zeta_q=q$, while for $t\gg t_{corr}$ one observes $\zeta_q=\frac{q}{2}$. When $t\lesssim t_{corr}$ the regular part of the velocity increment is dominated by $u_{reg}(t_1+t)-u_{reg}(t_1)\simeq(\partial_t u)t$. Similar to the decaying problem the latter gives the scaling $\langle\omega_{reg}^q\rangle\sim t^q$. We have plotted the behavior of ζ_q in terms of order q in Fig. 2. The figure shows the saturation for time scales $t\lesssim t_{kick}$ and $t\gg t_{kick}$, respectively.

We also checked the statistical independence of the center of mass velocity $u=2\bar{u}=u_1+u_2$ and velocity increments $\omega=u_1-u_2$. As an instance we numerically measure the quantity $\langle(u_1+u_2)^2|\omega;x,t\rangle$ for the range of space-time where statistical convergence are accessible. Indeed Fig. 3 demonstrates that for the time span of the order $t\in[10^{-3},10^{-1}]$ the conditional averages display a regime of independency for u and ω. The measurements shown here are done for a fixed sepa-

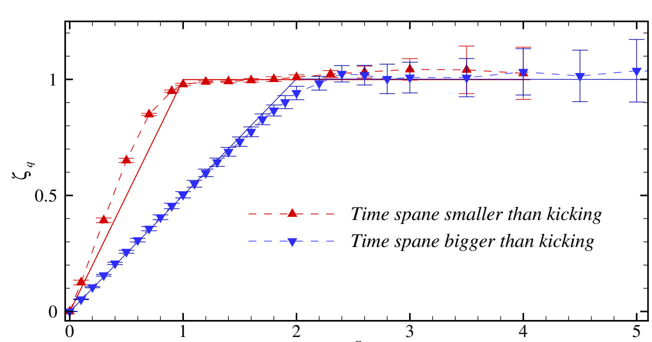

FIG. 2. (Color online) The saturation of ζ_q for different q's, where ζ_q is the scaling exponents $\langle|\omega|^q\rangle\sim|t|^{\zeta_q}$. The upper and lower figures are the saturation of scaling exponents for the time span smaller and bigger than kicking time, respectively. The correlation time scale is about $t_{corr}\simeq 3\times t_{kick}$. The results shows that Taylor frozen hypothesis is preserved just for time scales less than t_{corr}.

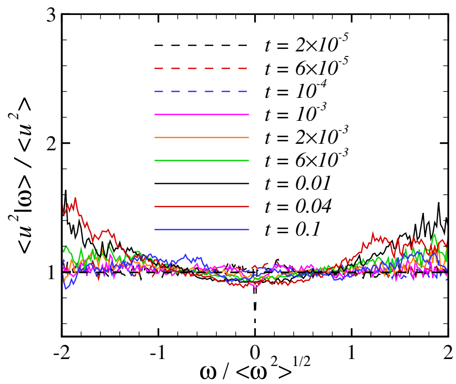

FIG. 3. (Color online) The conditional moment $\langle u^2|\omega\rangle$ as a function of ω at different time spans $|t_2-t_1|$ below and above the kicking time. For any given time increment t the conditional average is normalized by the corresponding $\langle u^2\rangle$.

ration $x\to 0$, but at different time windows t. As time increment t grows the domain of velocity increments ω where the independency is observed extends to larger values.

In summary, scaling exponents ζ_q of the moments of velocity increment *in time* are derived from randomly forced Burgers equation. When the incremental time $t=t_2-t_1 \lesssim t_{corr}$, they are analytically shown to saturate to unity, i.e., $\zeta_q=1$ for $q\geqslant 1$, while $\zeta_q=q$ for $q<1$. Our numerical simulations of the Burgers equation stirred by a kicking force support the results. At large times when $t\gg t_{corr}$, our numerical results reveal a non-universal scaling regime in which the higher order moments still saturate $\zeta_q=1$, but for $q>2$ while the low order moments display a normal scaling $\zeta_q=q/2$ for $q<2$. The later nonuniversal scaling of low order moments $\zeta_q=q/2$ for $q<2$ is a direct consequence of the forcing which is Wiener in time at every point of space.

Although the advective terms in the Navier-Stokes and the Burgers equations are similar the nonlocal pressure contribution prevents the formation of shocks in the Navier-Stokes turbulence. Closed analytic forms of the pressure and dissipative terms can only be accomplished by means of phenomenological closure approximations [10,20]. However, an analytic treatment of the relevant small-scale singularities in the Navier-Stokes turbulence is yet an open problem and, moreover, the closures are typically not based on the geometry of the flow singularities. To conclude, our work highlights the importance of a detailed knowledge of singularity dynamics in deriving Taylor's hypothesis. Generalizations to the Navier-Stokes turbulence requires further investigations.

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