Taylor’s frozen-flow hypothesis in Burgers turbulence

A. Bahraminasab,1,2 M. D. Niry,3 J. Davoudi,4 M. Reza Rahimi Tabar,5,6,8 A. A. Masoudi,7 and K. R. Sreenivasan2

1Department of Physics, Lancaster University, Lancaster, LA1 4YB, United Kingdom
2ICTP, Stara Costiera 11, 34014 Trieste, Italy
3Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran
4Department of Physics, University of Toronto, 60 St. George Street, Toronto ON, Canada MSS 1A7
5CNRS UMR 6529, Observatoire de la Côte d’Azur, BP 4229, 06304 Nice Cedex 4, France
6Carl von Ossietzky University, Institute of Physics, D-26111 Oldenburg, Germany
7Department of Physics, Alzahra University, Tehran, 19834, Iran
8Department of Physics, Lancaster University, Lancaster, LA1 4YB, United Kingdom

(Received 26 November 2007; revised manuscript received 4 June 2008; published 30 June 2008)

By detailed analytical treatment of the shock dynamics in the Burgers turbulence with large scale forcing we calculate the velocity structure functions between pairs of points displaced both in time and space. Our analytical treatment verifies the so-called Taylor’s frozen-flow hypothesis without relying on any closure and under very general assumptions. We discuss the limitation of the hypothesis and show that it is valid up to time scales smaller than the correlation time scale of temporal velocity correlation function. We support the analytical calculation by performing numerical simulation of the periodically kicked Burgers equation.

DOI: 10.1103/PhysRevE.77.065302

PACS number(s): 47.27.Gs, 05.45.—a

In a 1938 paper G. I. Taylor introduced an assumption by which he deduced the spatial fluctuations of a turbulent velocity from the corresponding measurements of temporal fluctuations at a single point [1]. This hypothesis, know as Taylor’s frozen-flow hypothesis, relies on the existence of a large mean flow which translates the spatial structures past a stationary probe in a time smaller than the inherent evolution time of the fluctuations [2,3]. As remarked by Tennekes [4], the Eulerian temporal scaling is dominated by large scale energy containing structures which sweep inertial-range information past an Eulerian observer [7,8].

The importance of this hypothesis relies on the fact that most turbulence theories center on the scaling behavior of spatial structure functions of the velocity field [2,3]. In experimental assessments of spatial fluctuations in turbulent flows [6–8] this hypothesis has been a general guideline for extracting the information from one-point measurements with hot wires. Original measurements treated by Taylor were made in turbulence generated behind a stationary grid in a wind tunnel, and his hypothesis has become a standard technique employed in similar experiments which build our current view on turbulence.

Although Taylor’s hypothesis is significantly clear when there is a large scale average flow, its application in homogeneous isotropic flows has been a much more debated issue. The latter always is meant to reflect the qualitative picture that larger eddies randomly sweep the smaller eddies with their root-mean-square velocities.

Apart from a few theoretical attempts in verifying the hypothesis, a general quantitative framework for deriving it does not exist [9,10]. Such a quantitative understanding calls for characterization of the spatiotemporal fluctuations of small scale eddies responsible for random sweeping [10–13].

In retrospect the spatiotemporal structure functions are central objects in providing the required information. Such structure functions of arbitrary order \(q\) are defined as

\[
S_q(x,t) = \langle [\tilde{u}(x_2,t_2) - \tilde{u}(x_1,t_1)]^q \rangle,
\]

where \(\tilde{u}(x_2,t_2)\) are Eulerian velocities at two spatially distinct points that are separated by large \(x\) and \(t\). This hypothesis, know as Taylor’s frozen-flow hypothesis, states that for incremental time \(\Delta t = t_2 - t_1\) the scaling of the structure functions is dominated by the shock dynamics and set as

\[
S_q(x,t) \sim x^{\xi_q} F_q \left( \frac{t}{x^{\eta_q}} \right),
\]

where \(x = |x_2 - x_1|\) and \(t = |t_2 - t_1|\). \(F_q\)’s are homogeneous functions of their arguments and \(\eta_q\) are dynamic exponents. Two sets of exponents \(\xi_q\) and \(\eta_q\) are defined by casting two asymptotic limits \(\lim_{x \to 0} S_q(x,t) = |x_2 - x_1|^{\xi_q}\) and \(\lim_{t \to 0} S_q(x,t) = |t_2 - t_1|^{\eta_q}\), respectively.

Theoretical arguments resorting to the multifractal phenomenology support the existence of a hierarchy of dynamic exponents \(\eta_q\) if one compensates the sweeping effects by choosing a quasi-Lagrangian frame [14]. In an Eulerian frame for which the sweeping dominates the temporal fluctuations, \(\eta_q = \frac{\xi_q}{\eta_q} = 1\), at least up to leading order [2,14].

Here we calculate \(\eta_q\) for the one-dimensional Burgers equation stirred a forcing with large scale correlation in space and with a Wiener scaling in time. The exponents are derived from equations of motion without relying on any closure by which the dynamic exponents \(\eta_q\) are inferred. The crucial role of shocks in establishing the result is spelled out and the scaling solutions of the dynamic structure functions, i.e., \(\lim_{t \to 0} S_q(x,t)\), are obtained. Numerical simulations on the periodically kicked Burgers equation [15] are performed to support the analytical results for \(\eta_q\).

Our numerical and analytical calculations both indicate that for incremental time \(\Delta t = t_2 - t_1\) less than or at the order of the correlation time scale, i.e., \(\Delta t \leq t_{corr}\), the scaling of \(\lim_{t \to 0} S_q(x,t)\) are dominated by the shock dynamics and saturate, i.e., \(\xi_q = 1\) for \(q \geq 1\), while \(\eta_q = q\) for \(q < 1\). As a by-product the dynamical exponents are equal to unity, i.e., \(\eta_q = 1\), consistent with Taylor’s hypothesis. However when \(t\)
>t_{\text{corr}} the dynamics of lim_{\alpha \to 0} S_{\alpha}(x,t) are controlled by the scaling limit of the forcing. Hence we show that the Taylor frozen hypothesis does not hold for t > t_{\text{corr}} when the forcing scales as a Wiener process in time. The crossover from shock-dominated to forcing-dominated regime is also numerically verified.

We consider the Burgers equation in one dimension, i.e.,
\[ u_t + uu_x = \mu u_{xx} + f(x,t), \]
where \( f(x,t) \) is a zero-mean Gaussian statistically homogeneous, and white in time random process with covariance, \( \langle f(x_1,t_1)f(x_2,t_2) \rangle = C(x_2-x_1)\delta(t_2-t_1). \) The spatial correlation \( C(x_2-x_1) \), as a function of \( |x_2-x_1| \), is assumed to have a finite support \( \sigma = L \), where \( L \) is the system’s domain [16–18].

A convenient way to derive the dynamical evolution of \( S_{\alpha}(x,t) \) is via the standard generating function method [17,18]. Defining \( \Theta(t) = \exp(-\lambda x \bar{u}(t) - \lambda u^2(t)) \), the two-point generating function is given by \( \langle \cdots \rangle \), where \( \langle \cdots \rangle \) is an average over the forcing statistics. The Fourier transform of the generating function is the two point probability density function (PDF) \( P(t_1,t_2,x_1,t_2,x_2,t_2) \) defined at the points \( x_1 \) and \( x_2 \) with their related velocities \( u_{t_1} \) and \( u_{t_2} \). From Eq. (2) the dynamical equations of PDF at distinct times, say \( t_1 \) and \( t_2 \), are sought. Using the following change of variables \( t = t_2 - t_1, \quad T = \frac{x_{t_2} - x_{t_1}}{u_{t_1}} \), and transforming \( u_{t_1} \) and \( u_{t_2} \) with \( u = \frac{u_{t_2} - u_{t_1}}{u_{t_1}}, \) \( \omega = u_{t_2} - u_{t_1} \), one obtains the following equation for the PDF of velocity field difference as
\[
P_t = -uP_x - \frac{1}{2} \int K\left(u + \frac{\omega}{2}, u\right) P_x\left(u - \frac{\omega}{2}, u\right) du' - \frac{1}{2} \int K\left(u - \frac{\omega}{2}, u\right) P_x\left(u + \frac{\omega}{2}, u\right) du' + \frac{1}{2} C(0) P_{\omega \omega} - \frac{1}{2} C\left(\frac{1}{2}P_{\omega \omega} + \frac{1}{2}P_{\omega \omega \omega} \right) + \frac{G_2}{2} \frac{G_1}{2},
\]
where \( x = x_{t_2} - x_{t_1}, \) \( y = \frac{x_{t_2} - x_{t_1}}{u_{t_1}} \). The \( C(x) \) is the spatial correlation function of forcing, \( K(u) = \frac{\Pi(u)u^2}{\Pi(u)u^2 + \Pi(u)u^2} \), and \( H(u) \) is the Heaviside function. The terms \( G_i \) are defined as \( G_i = -\bar{u}u_t\partial_x \bar{u}(u_{t_{i+1}}|u_{t_i},u_{t_2},x_{t_1},x_{t_2},t_{i+1}-t_i,P_t) \), where \( \langle u_{t_{i+1}}|u_{t_i},u_{t_2},x_{t_1},x_{t_2},t_{i+1}-t_i,P_t \rangle \) is the average of \( u_{t_{i+1}} \) with the conditions that the velocity field has the values of \( u_{t_i} \) at points \( x_{t_i} \) and in times \( t_i \). Indeed the term \( G_i \) is the only term preventing Eq. (3) to be closed, which can be referred to a sort of dissipative anomaly [16].

Multiplying both sides of Eq. (3) with \( \omega \delta \) and integrating it over \( u \) and \( \omega \) one obtains
\[
\frac{\partial S_{\omega \delta}}{\partial t} = -\delta \frac{\partial F_{\omega \delta}}{\partial x} + \frac{1}{2} q(q-1) C(x) S_{\omega \delta} + \langle \omega \delta (G_2 - G_1) \rangle.
\]

In principle the solutions of the above equation in the inviscid limit should have scaling forms of the type introduced in Eq. (1). However, Eq. (4) is not obviously tractable because the first and last set of terms on the right-hand side are not expressed in terms of \( S_{\alpha}(x,t) \) and the equation is not closed. In what follows we show that it is possible to treat the unclosed terms by means of shock representation and without resorting to any closure model.

In the inviscid limit both types of the terms are expressible in terms of operators localized on shocks when \( x \to 0 \). Recall the velocity \( \mu \) satisfying the Burgers equation develops shock solutions in the limit \( v \to 0 \). One may represent a shock locally as \( u(x,t) = u_0 \delta(x-x_0(t)) + u_0 \delta(x-x_0(t)) \), where \( H \) is a Heaviside function. The position \( x_0 \) is identified by two quantities, namely the velocity \( \mu \) in positions \( x_0, \mu_0 \) (set it, for example, to 0). In other words the velocity gradient (corresponding Burgers velocity) is not continuous at point \( x_0 \). At these singular points \( u_0 \) is defined as \( u_0(x_0,t) = u(x_0,t) \) keeping in mind that \( u_0 \geq u_+ \), while the shock strength \( s \) and the shock velocity \( \bar{u} \) are defined as \( s = u_+ - u_- \) and \( \bar{u} = \frac{1}{2}(u_+ + u_-) \).

Here we argue that the last term in the right-hand side of Eq. (4) vanishes in a particular space-time window. Indeed in the inviscid limit only small intervals around the shocks will contribute to the \( G_i \) terms. Each of \( G_i \)’s at space-time \((x_i,t_i)\) are nonvanishing only on the shocks [17,18]. As a near way of demonstrating the cancellation of \( G_i \) terms we return to the Fourier space and represent the related terms with \( \hat{G}_i \). The anomaly terms \( \hat{G}_2 - \hat{G}_1 \) in this space can then be written as
\[
\hat{G}_2 - \hat{G}_1 = -\lambda^2 (\bar{u}(1) e^{\lambda u_0} - e^{-\lambda u_0}),
\]
where \( x_i \) is the position of the \( i \)th shock and \( \lambda = \lambda_0 - \lambda_1 \) is the conjugate of \( \omega \) and \( e(\lambda,x,t) = \sum F(\hat{u}(x_i),t) \delta(x-x_i,t) \). The form factors \( F \) are read \( F(\hat{u},s) = -2e^{\hat{u}/s} \frac{w-1}{s^1/2} \sinh(s^{1/2}) \) [19]. This representation shows that the anomaly contribution is generally very complicated for arbitrary separation distance \( x \) and time difference \( t \). However it is possible to identify a space-time regime in which the following operator vanishes. In fact by Taylor expansion one easily sees that for \( |t_2 - t_1| \rightarrow 0 \) \( u_{t_{i+1}}|x_{t_{i+1}}(t_i) \rightarrow u_{t_i}|x_{t_i}(t_i) \), given \( x \to 0 \), \( \hat{G}_2 \) and \( \hat{G}_1 \) cancel out in the leading order. This condition physically means that the anomaly due to a shock at point \( x_i \) and in time \( t_i \) is the same in statistical sense at a point \( x_1 \) but with a time delay approximately equal to \( |t_2 - t_1| \approx \frac{|x_{t_2} - x_{t_1}|}{u_{t_1}} \), because the shocks move with their local velocity \( \bar{u} \). In this spatiotemporal window the time variations of velocities are mostly dominated by the random shocks which sweep the spatiotemporal fluctuations past a point [1,4].

The other set of terms on the right hand side of Eq. (4), \( S_{\alpha > 0}(\hat{u}(\omega)) \), are rooted in the \( P_\alpha \) term in Eq. (3). Using stationarity, i.e., \( \omega \partial \hat{P}_\alpha = 0 \), it is easy to reexpress \( \lim_{\alpha \to 0} \hat{P}_\alpha \) in terms of the measures describing the statistics of shocks. In the spatiotemporal regime where \( x < \bar{u}t \) it is possible to show
\[
\lim_{\alpha \to 0} P_\alpha(u,\omega,x,t) = N_\alpha(\omega)M(\omega) = [\rho(s)\delta(s) + \rho S(\omega,T)]M(\omega),
\]
where \( \delta(s) = d\delta(\omega)/d\omega \) and \( N_\alpha(\omega) = \lim_{\alpha \to 0} P_\alpha(u,\omega,x,t)du. \) Here \( \rho = \Sigma \delta(x-x(T)) \) and \( \delta(x-x(T)) \) are the
shock number density in space and the PDF of \( s(x_1, t_1) \) conditioned on \( x_1 \) being on a shock location, respectively.

To leading order the result is \( t \) independent and the averages on \( u \) and \( \omega \) on the shocks separate. The functional dependence of \( \lim_{t \to 0} P_t(u, \omega, x, t) \) on \( u \) appears as an unknown function \( M(u) \), where \( \int M(u) du = 1 \) (see also \([14]\)). Although the precise form of \( M(u) \) does not enter in later arguments the separation of \( u \) and \( \omega \) is central for obtaining the later results.

In order to calculate \( \xi_q \) we use the representations in Eqs. (5) and (6) at leading order and substitute them back into Eq. (4). Therefore we obtain

\[
\frac{\partial S_q}{\partial t} = -\rho(\bar{u})(s^q) + \frac{1}{2} q(q-1)C(x)S_{q-2} + o(t) \tag{7}
\]

Solving the analytical scaling solutions in Eq. (7) we find the moments \( q \approx 2 \) of the velocity increments in time behave as

\[
\langle |\omega|^q \rangle = |r|^{q/2} + B_q |r|, \tag{8}
\]

where \( \omega = \lim_{t \to -0} [u(x_1 + x + t, t) - u(x_1, t)] \) and \( \rho \) is the average density of shocks in time. We remark that the leading term in \( S_2(t) \) has a forcing contribution proportional to \( C(0) \). The higher order moments, i.e., \( q > 2 \) do not obtain contributions from forcing because they are always subleading in time.

For the lower order moments the competition of the regular part of the velocity increment in time dominates the contribution of shocks. One writes \([u(x_1 + x + t, t) - u(x_1, t)]=-f_1^{\text{reg}}(\bar{u}) dt+\int_{t_1}^{t} dW(s)\), where the first integral in the right-hand side resembles the sweeping of the spatial increment and hence scales as \( t \). The second integral however scales as \( t^{1/2} \) because the integral of the forcing is a Wiener process in time by definition. Consequently in leading order one may easily write

\[
\lim_{x \to 0} S_q(x, t) = A_q |r|^{q/2} + B_q |r|, \tag{9}
\]

where \( A_q \) and \( B_q \) are unknown amplitudes. Therefore, the low order moments scale as \( r^{q/2} \) and the saturation should begin for orders \( q \approx 2 \).

To test the predictions of the analytical calculation, we have carried out numerical simulations of the Burgers turbulence by means of the so-called particle method \([15]\). The numerical experiments reported hereafter have been made with the kicking force and a kicking period \( t_{kick} = 0.001 \). The number of collocation points chosen for our simulations is generally \( N_c = 10^3 \). In order to perform temporal averages, since we need a large sequence of velocity time series to calculate intermittency in time, we run the algorithm for about \( 10^3 t_{kick} \). For time integration we adapt a time discretization of order \( t_s = 10^{-5} \). Between two subsequent kicks we advance the minimizers in time for \( 100dt \). We then construct the Eulerian velocity by means of the particles velocities after reaching to statistically stationary state. The results are shown in Fig. 1, in which the log-log plot of \( \langle |\omega(x, t_1 + t) - u(x, t_1)|^q \rangle \) is depicted as a function of the time increment \( t \) for different \( q \)'s specified in the inset. The time span \( t \) in the measurements of velocity increments lies in the extended range of \( t \in [10^{-5}, 2 \times 10^{-1}] \). Therefore, we have the information about scaling both above and below the characteristic correlation time. The typical correlation time is about \( t_{corr} \approx 3 \times t_{kick} \).

Although all high order moments \( q \approx 2 \) display a slope of approximately 1 the moments of orders \( q \approx 2 \) exhibit a crossover in their scaling. For \( t \leq t_{corr} \), the scaling exponent is read to be \( \xi_q = 1 \), while for \( t > t_{corr} \) one observes \( \xi_q = 2 \). For \( t \leq t_{corr} \) the regular part of the velocity increment is dominated by \( u_{reg}(t_1 + t) - u_{reg}(t_1) \cong (\bar{u}) dt \). Similar to the decaying problem the latter gives the scaling \( \langle |\omega_{reg}|^q \rangle \sim t^{q-2} \). We have plotted the behavior of \( \xi_q \) in terms of order \( q \) in Fig. 2. The figure shows the saturation for time scales \( t \leq t_{kick} \) and \( t > t_{kick} \), respectively.

We also checked the statistical independence of the center of mass velocity \( 2\bar{u} = u_1 + u_2 \) and velocity increments \( u = u_1 - u_2 \). As an instance we numerically measure the quantity \( \langle u_{1/2}^q \rangle \) \( \omega(x, t) \) for the range of space-time where statistical convergence are accessible. Indeed Fig. 3 demonstrates that for the time span of the order \( t \in [10^{-3}, 10^{-1}] \) the conditional averages display a regime of independency for \( u \) and \( \omega \). The measurements shown here are done for a fixed separation.
The conditional moment $\langle u^2 \rangle / \omega$ as a function of $\omega$ at different time spans $|t_2 - t_1|$ below and above the kicking time. For any given time increment $t$ the conditional average is normalized by the corresponding $\langle u^2 \rangle$.

In summary, scaling exponents $\zeta_q$ of the moments of velocity increment in time are derived from randomly forced Burgers equation. When the incremental time $t = t_2 - t_1 \leq t_{corr}$ they are analytically shown to saturate to unity, i.e., $\zeta_q = 1$ for $q \geq 1$, while $\zeta_q = q$ for $q < 1$. Our numerical simulations of the Burgers equation stirred by a kicking force support the results. At large times when $t \gg t_{corr}$, our numerical results reveal a non-universal scaling regime in which the higher order moments still saturate $\zeta_q = 1$, but for $q > 2$ while the low order moments display a normal scaling $\zeta_q = q/2$ for $q < 2$. The later nonuniversal scaling of low order moments $\zeta_q = q/2$ for $q < 2$ is a direct consequence of the forcing which is Wiener in time at every point of space.

Although the advective terms in the Navier-Stokes and the Burgers equations are similar the nonlocal pressure contribution prevents the formation of shocks in the Navier-Stokes turbulence. Closed analytic forms of the pressure and dissipative terms can only be accomplished by means of phenomenological closure approximations [10,20]. However, an analytic treatment of the relevant small-scale singularities in the Navier-Stokes turbulence is yet an open problem and, moreover, the closures are typically not based on the geometry of the flow singularities. To conclude, our work highlights the importance of a detailed knowledge of singularity dynamics in deriving Taylor’s hypothesis. Generalizations to the Navier-Stokes turbulence requires further investigations.

We thank Uriel Frisch and Jeremie Bec for useful comments and discussions. J.D. thanks the ICTP and the Max Planck Institute for Complex Systems for partial support.

---