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Does the flatness of the velocity derivative blow up at a finite Reynolds number?

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Abstract. A tentative suggestion is made that the flatness of the velocity derivative could reach an infinite value at finite (though very large) Reynolds number, with possible implications for the singularities of the Navier–Stokes equations. A direct test of this suggestion requires measurements at Reynolds numbers presently outside the experimental capacity, so an alternative suggestion that can be tested at accessible Reynolds numbers is also made.

Keywords. High-Reynolds-number turbulence; derivative flatness; Navier–Stokes singularities.

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0. Preamble

The first author of this paper presented an invited talk at STATPHYS 22. The title of the presentation was "Onsager and the theory of hydrodynamic turbulence". The paper was jointly authored with G L Eyink of Johns Hopkins University, and will be published in *Rev. Mod. Phys.* in 2005. In that article, we traced Onsager's contributions to hydrodynamic turbulence through his published papers, correspondence and unpublished notes. It is remarkable that Onsager anticipated essential elements of several important results that others published many years later. One of the several topics that interested Onsager was the subject of Euler singularities and dissipative anomaly. He stated – and had in his unpublished notes, calculations to support the statement – that there is a precise minimal Hölder singularity that would lead to finite dissipation in the inviscid limit. The subject of finite-time Euler singularities has been fascinating to many mathematicians and numerical simulators alike, but the conclusions are still far from being clear. Instead of repeating parts of the review article with Eyink, it was thought that it might be better to say something different on the subject of singularities. The present paper may have some vague connection with Navier–Stokes singularities, and is more specifically about the nature of the flatness factor of the velocity derivative in turbulent flows.

1. Introduction

The possibility that the Navier–Stokes (NS) equations could attain singularities at some finite though large Reynolds numbers is intriguing [1]. Though the subject has held long-standing interest (see [2–9]), and the regularity of the three-dimensional (3D) solutions is assured for sufficiently low Reynolds numbers [10], a complete theory for high Reynolds numbers is lacking at present. One principal result from [6] is that the 3D singularities of the NS equations, if they exist, possess a Hausdorff dimension of less than unity in four-dimensional space-time. We are thus assured that the singularities would be very rare, and cannot take the form of a simple vortex line because it possesses Hausdorff measure of unity in space-time. However, the result does not rule out the appearance of singular features at some point along a vortex line for brief periods of time. A related and important conclusion [9] is that the 3D Euler solutions remain regular if the maximum norm of the vorticity remains bounded. Further pedagogical discussion of this issue can be found in [11]. One of the latest efforts on the subject of NS singularities is [12].

Most experimentalists believe that the tendency to form 3D NS singularities is smoothed out by viscosity and that they do not occur in practice. However, the questions are quite subtle, and it would be useful if experiments could motivate further discussion on singularities. In the absence of direct information, a chosen velocity gradient, or combinations of velocity gradients, are the next best substitutes to consider from the experimental point of view. In particular, statistical quantities such as the flatness (i.e., the normalized fourth-order moment) of the so-called longitudinal velocity derivative might be expected to detect the occurrence of such singularities. We restrict ourselves to making a few tentative remarks from this perspective on the behavior of the derivative flatness.

2. Experimental data

Figure 1, taken from [13], shows the flatness of the longitudinal velocity derivative as a function of the Reynolds number $R_\lambda = u'\lambda/\nu$ based on the root-mean-square velocity fluctuation u', the Taylor microscale λ and the fluid viscosity ν. This Reynolds number varies as the square root of the Reynolds number, $\mathrm{Re} = u'L/\nu$, based on the large scale L. The graph covers low Reynolds numbers from direct numerical simulations and a variety of laboratory experiments, as well as atmospheric surface layer at high Reynolds numbers. The data were collected in 1997. Some new data obtained since then are compatible with this figure, but the scatter has remained comparable for them as well. It is not clear if this scatter is a reflection of the non-universality of the flatness, or simply a sampling error due to the increasing demands placed at increasingly high Reynolds number on the length of data required for statistical convergence. It is usually implied that the latter is the case.

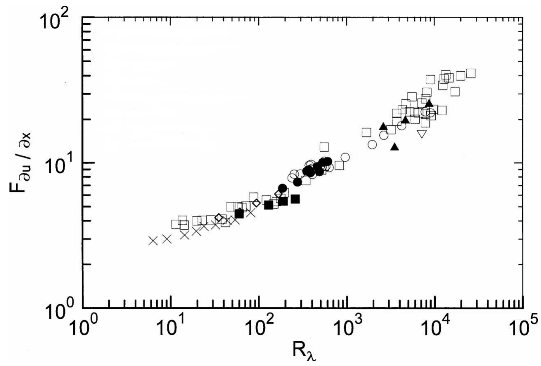

Figure 1. The figure is a compilation of the flatness F of the longitudinal velocity derivative $\partial u/\partial x$, taken from [13].

The interpretation is that the flatness increases as a power-law in R_λ, for $R_\lambda > 100$ or so. Another line of thinking [14] is that R_λ is not the proper Reynolds number to use for testing universality but we shall not consider it here: if there is a singular behavior in one Reynolds number, it is likely to show up in another.

Suggestions have occasionally been made in the literature that, beyond a certain Reynolds number, the derivative flatness becomes a constant or develops some non-monotonic feature [15]. An interesting interpretation of this claim has been that a second transition of some kind occurs in turbulence – and who knows if there are more? – and that the nominal power-law range just discussed is not the asymptotic state. In another recent paper [16], however, it has been pointed out that the tendency for the flatness to reach constancy at some high Reynolds number is merely an experimental artifact; and that it disappears when the resolution of measurement is improved. At the least, this study suggests that there is a great need to resolve the velocity data better in order to know with any certainty the proper behavior of the flatness.

What might be the needed resolution? The conventional wisdom has been that the gradients can be computed from the velocity data reliably if the latter are measured with a resolution of the order of the Kolmogorov scale defined on the average dissipation rate [17]. Since this convention might well be reasonable at modest Reynolds numbers, by extension, the (average) Kolmogorov scale has been treated as the finest resolution required for all purposes at all Reynolds numbers (though, in practice, even this resolution has not been attained at high Reynolds numbers). For some time, however, there has been a growing realization [18,19] that the Kolmogorov scale is a fluctuating quantity due to the (approximate) multifractality [20] of the energy dissipation, and that scales significantly smaller than the average value do exist, especially at high Reynolds numbers. In another recent work [21], an explicit formula has been derived (subject to some elementary caveats) to show that the appropriate resolution depends on the moment order and the Reynolds number itself. The higher the order of the moment and the higher the Reynolds number, the more stringent is the required resolution in measurement and simulations. Direct numerical simulations of passive scalar [22], made with resolutions much smaller than the Batchelor scale (the nominally smallest scale for passive scalars), confirm that finer scales than the Batchelor scale do indeed exist. This result points to the essential plausibility of the arguments just made for the velocity data.

Given that we do not yet have the high Reynolds number flatness measurements with the required resolution, their behavior is still unclear. It follows that the existing data are open to different interpretations. One such interpretation is attempted below. The key to the interpretation is the fact that the flatness data of figure 1 – such as they are – appear to show a continually increasing tendency to curve upwards, and are not unambiguously described by a power-law (though a power-law may be a moderately good fit for some range of R_λ).

3. The new interpretation of the derivative flatness

We have argued elsewhere [23–25] that the natural parameter for assessing the behavior of small-scale turbulence is the logarithm of the Reynolds number, and that the inverse of $\log R_\lambda$ is the appropriate expansion parameter in turbulence. While this has no analytical justification that we are aware, several known features of turbulence point in this direction. For instance, the number of steps in the energy cascade is proportional – at least as we understand the situation presently – to the logarithm of the Reynolds number. In one interpretation [26], the number of 'particles' on which one does statistical mechanics of the inertial range is equal to $\frac{3}{2}\log_2 R_\lambda$. Further, it makes good sense to expand the Reynolds shear stress in pipe and channel flows in terms of the logarithm of the Reynolds number. Earlier, this same notion was put to extensive use by Barenblatt [27] for formulating the mean velocity distribution in wall-bounded flows; for an explicit formulation in more general contexts, see [28]. We thus examine here the consequences of the same line of thought for the derivative flatness. Since there is much scatter in the flatness data, we have fitted a local average for the data and considered it below.

Figure 2 shows the inverse flatness F^{-1} (from the local average fit) against $1/\ln(\mathrm{Re}_\lambda)$. New data from experiments performed in wind tunnels [29] and in atmospheric surface layer [30], are added. The straight line is drawn to indicate critical-like dependence

$$F(x) = c_1 \frac{x}{(x_\mathrm{c} - x)} \tag{1}$$

with

$$x = \ln(R_\lambda). \tag{2}$$

This fit is reasonable from $R_\lambda \approx 50$ (which is on the order of the minimum Reynolds number at which turbulence-like behavior sets in [31]) and describes all the measured data up to $R_\lambda \approx 20{,}000$ (which is about the highest Reynolds number at which measurements have been made so far [30]). Extrapolating this fit, perhaps with some peril, one can find the 'critical' value $x_\mathrm{c} = \ln R_\lambda^{(\mathrm{c})} \simeq 12.6 \pm 1.5$, as the intersection point of the fitting straight line with the horizontal axis. This gives a 'critical' value $R_\lambda^{(\mathrm{c})}$ of about 300,000 (corresponding to about 19 'particles'

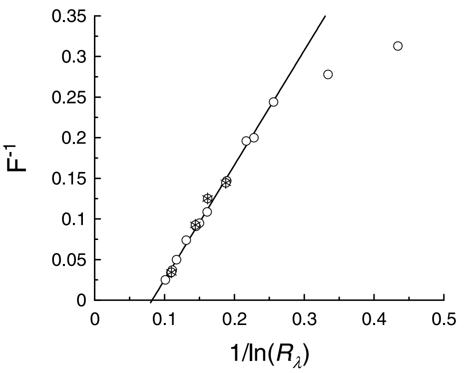

Figure 2. Inverse flatness data (circles) against inverse $\ln R_\lambda$. The solid straight line indicates the best fit with eqs (1) and (2).

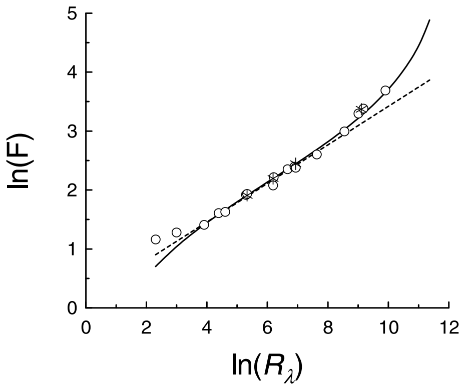

Figure 3. Same as in figure 2 but in the conventional log–log coordinates.

as mentioned above). While the error bar in $1/\ln R_\lambda^{(\mathrm{c})}$ is modest, it translates to huge uncertainty between 66,000 and 1,300,000 in the value of $R_\lambda^{(\mathrm{c})}$ itself.

A slightly different perspective on the topic is provided in figure 3, which compares the smoothed data (circles) with the 'critical' approximation (1) and (2) (solid curve) as well as conventional power law (dashed line) in log–log coordinates. Even though there is a slight suggestion that the data prefer the 'critical' line to the pure power-law (towards the upper end of R_λ), one needs an order of magnitude higher R_λ in order to determine unambiguously whether the data will follow the traditional power law or the critical behavior. Such Reynolds numbers are outside the limits of the present experiments.

Thus, the suggestion of 'criticality', or the singular behavior of the flatness, could remain only as an academic curiosity except for one possibility. As we have already discussed, the velocity measurements, from which the derivative flatness of figure 1 is computed, have been made typically with a resolution on the order of the Kolmogorov scale. Since those made with poorer resolution are known [16] to fall below those of figure 1, it is eminently possible that the flatness measured with better resolution will fall above those of figure 1. This behavior probably manifests itself already at R_λ values at the top end of figure 1. If the data at those same Reynolds numbers can be repeated with improved resolution, and they show a tendency to migrate more towards the 'critical' curve, one may infer that there is possibly an element of truth in our speculation.

4. Discussion and conclusion

On the basis of available evidence, we have tentatively argued that singular behavior of the flatness at some large but finite Reynolds number may not be ruled out. To show that this suggestion is on the right path, one needs measurements of the derivative flatness at the highest possible Reynolds number available with increasingly improved resolution. If such measurements migrate towards the 'critical' curve of figure 3, the suggestion would have acquired a status beyond mere speculation. Of course, to be certain, one needs data at much higher R_λ, which seems to be out of reach for the present.

A few further remarks might be useful. All available evidence [31,32] suggests that the energy dissipation rate, which is proportional at high Reynolds numbers to the second moment of the velocity derivative, becomes independent of the Reynolds number. There is thus no evidence that the singularities, if they exist, will contribute to the second moment. We have examined the evidence available for the (normalized) sixth moment. Though the scatter of the data is larger than that for the fourth moment, the data seem to fit the formula

$$F_6 = c_2 \left[\frac{x}{x_\mathrm{c} - x}\right]^{2.3 \pm 0.2}, \tag{3}$$

where x and x_c have the same meaning as before. It may thus be speculated that higher order moments will have the same form, with the same 'critical' Reynolds number, but with increasingly larger values of the exponent. Needless to say, forms such as (1) and (3) can be approximated adequately as power laws in some range of x.

Finally, we may ask: if the singularities do indeed arise in the flow, what does it say about the NS equations themselves? Can one ever say anything definitive about the NS singularities through experiments? Does one have to consider something else besides the NS equations once singularities arise in the flow? In the face of these difficult and unanswered questions, the spirit of this note should be taken to be that experiments may be able to sharpen mathematical questions. At present, we would be content if the note sharpened experimental questions about the flatness a bit better. We are well aware that there is in this subject no substitute for convincing mathematical demonstrations.

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$$\eta_h \sim L \left(\frac{\mathrm{Re}^*}{\mathrm{Re}} \right)^{1/(1+h)},$$

at a point where the velocity Hölder exponent is h, where Re^* is some reference Reynolds number. A singularity means a point where $\eta_h = 0$. This can never happen when $\mathrm{Re} < \mathrm{Re}^*$, which agrees with Leray's result [2] that the NS solutions are regular at a sufficiently low Re. Furthermore, a singularity can occur for $\mathrm{Re} > \mathrm{Re}^*$ only if there is a point with $h \leq -1$
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$$F_n = \mathrm{Re}^{s_n}$$

where F_n is the normalized nth order moment of the velocity derivative (for instance, the flatness corresponds to $n = 4$), $s_{2n} = \frac{\zeta_{2n} - 2n}{\zeta_{2n} - \zeta_{2n+1} - 1} - n$, and ζ_n is the scaling exponent of the nth order structure function
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