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Towards a dynamical theory of multifractals in turbulence

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Abstract

Making use of the exact equations for structure functions, supplemented by the equations for dissipative anomaly as well as an estimate for the Lagrangian acceleration of fluid particles, we obtain a main result of the multifractal theory of turbulence. The central element of the theory is a dissipation cut-off that depends on the order of the structure function. An expression obtained for the exponents s_n in the scaling relations

$$\overline{\left(\frac{\partial u}{\partial x}\right)^n} \Big/ \overline{\left(\frac{\partial u}{\partial x}\right)^{2^{n/2}}} \propto Re^{s_n},$$

between the velocity gradients $\partial u/\partial x$ and the Reynolds number Re, agrees well with experimental data.

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Questions of small-scale universality in fluid turbulence hover around the universality of the scaling exponents $\xi_{n,0}$ of velocity structure functions defined through relations such as

$$S_{n,0} = \overline{[u(x+r) - u(x)]^n} \equiv \overline{(\delta_r u)^n} \propto (\bar{\varepsilon}L)^{n/3}\left(\frac{r}{L}\right)^{\xi_{n,0}}, \tag{1}$$

where $u(x)$ is the velocity component along the separation distance r, measured at the position x and $\bar{\varepsilon}$ is the mean rate of energy dissipation. Here r lies in the inertial range given by $\eta \ll r \ll L$, where L is the large-scale at which the energy is being injected and $\eta \equiv (v^3/\bar{\varepsilon})^{1/4}$ is the dissipation scale, v being the fluid viscosity. The zero index in $\xi_{n,0}$ shows that no powers of the transverse velocity increments are involved in this particular definition (1). Kolmogorov [1] assumed that the velocity fluctuations in the inertial range are independent of both L and η, and that $\bar{\varepsilon}$, regarded as equal to the energy flux across scales, is the only relevant dynamical parameter. As is well known, Kolmogorov's proposal yields the linear relation $\xi_{n,0} = n/3$. Since, in the limit of vanishing viscosity (or, as $\eta \to 0$), Kolmogorov's scaling theory combines the exact expression [2] $S_{3,0} = -\frac{4}{5}\bar{\varepsilon}r$, it is reasonable to regard the theory loosely as dynamic. However, experimental and numerical data in three-dimensional turbulence have shown (see Ref. [3] for a recent account) that the scaling exponents $\xi_{n,0}$ depart from $n/3$, and that there exists a more complicated nonlinear spectrum of scaling exponents $\xi_{n,0}$. Its theoretical explanation for the velocity field has proved to be elusive, though considerable progress has been made for passive scalars [4].

In recent past, the problem of scaling exponents in turbulence has been analyzed within a general framework of the theory of multifractal (MF) processes reviewed in Ref. [5]. This approach has led to interesting interpretations and novel work (see Refs. [5,6] for incomplete list), but its shortcoming is the lack of connection with the dynamical equations. In this paper, a main relation of the MF theory is derived from dynamical equations, supplemented both by an order-of-magnitude estimate for the Lagrangian acceleration of a fluid particle, and the earlier work on dissipative anomaly [7,8].

For background, we review here the main ideas of the inertial-range MF theory, whose basis are the assumptions that (a) the velocity increments $\delta_r u$ have the form

$$\frac{\delta_r u(\mathbf{x})}{u'} \equiv \frac{u(\mathbf{x}+\mathbf{r}) - u(\mathbf{x})}{u'} \propto \left(\frac{r}{L}\right)^h, \tag{2}$$

where $u' \sim \delta_L u$ may be regarded as the root-mean-square value of u, and (b) there exists a spectrum of exponents h related to the fractal dimension of their support $D(h)$. Thus, $(r/L)^{3-D(h)}$ is proportional to the probability of the velocity increment falling within a sphere of radius r on a set of dimension $D(h)$. It is clear from (2) that

$$S_{n,0} = \overline{(\delta_r u)^n} \propto \left(\frac{r}{L}\right)^{\xi_{n,0}} = \int \mathrm{d}\mu(h)\left(\frac{r}{L}\right)^{nh+3-D(h)}, \tag{3}$$

where $\mathrm{d}\mu(h)$ is the weight of a local value of exponent h. Thus the scaling exponents $\xi_{n,0}$ are directly related to the spectrum $D(h)$. The goal of the theory is to find the functions $D(h)$ and $\mu(h)$. If one evaluates the integral in (3) in the steepest descent

approximation, as in the standard procedure, the precise form of $\mu(h)$ is irrelevant and only the spectrum $D(h)$ needs to be determined. However, this cannot be done within the MF theory itself.

Multifractality in the inertial range will have consequences for dissipation scales as well. The authors of Ref. [9] used the local scaling (2) to construct eddy-turnover times that depended on h, equated them to diffusion times scales η^2/ν, and showed that a spectrum of h-dependent dissipation scales can be written as

$$\eta(h) \propto LRe^{-1/(1+h)} , \qquad (4)$$

where the large-scale Reynolds number $Re = u'L/\nu$. The exponents in (4) have to be related somehow to the spectrum of scaling exponents of structure functions. This can be done by assuming, for any h, that (2) is valid only for scales $r > \eta(h)$, with smoothness for smaller scales [5,10]. One can then evaluate (3) for scales larger than $\eta(h)$, using the steepest descent approximation up to the cut-off scale $r = \eta(h)$. It is easy to show [5] that

$$\overline{(\partial_x u)^n} \propto Re^{\zeta_{n,0}+n/2} \equiv Re^{s_n} , \qquad (5)$$

where $\zeta_{n,0} = p(n,0) - 3n/2$ and $p(n,0)$ is the solution of $p(n,0) = 2n - \xi_{p,0}$. Our specific goal is to obtain s_n theoretically.

A brief remark on our strategy may be helpful here. If the structure functions $S_{n,m} \equiv \overline{(\delta_r u)^n (\delta_r v)^m}$, where v now is the velocity component normal to the displacement vector $\mathbf{r}$, of the form $A_{n,m} r^{\xi_{n,m}}$, are non-analytic in the inertial range, and the viscous dissipation is the only mechanism for smoothing the singular nature of the structure functions in the inertial range, the balance between them occurs at the length scale $r \to \eta_{n,0}$, where $\eta_{n,0} \equiv \eta_n$ is an order-dependent length scale nominally separating the analytic and singular intervals. The analyticity of structure functions in the viscous range yields $S_{n,0} \propto \overline{(\partial_x u(0))^n} r^n$, so we have [11]

$$\eta_n \approx (\bar{\varepsilon}L)^{n/(3(n-3))} \overline{(\partial_x u)^n}^{1/(\xi_{n,0})-n} \approx \overline{(\partial_x u)^n}^{1/(\xi_{n,0})-n} . \qquad (6)$$

This equation defines the field $\eta(\mathbf{x}, t)$ through moments of velocity gradients, and picks out the strongest singularity of a chosen order dominating the inertial range asymptotics. Our strategy is based on the idea that if an n-th order structure function evaluated at the appropriate cut-off is $S_{n,0}(\eta_{n,0}) = A_n \eta^{\xi_{n,0}}$, with the Reynolds-number-independent proportionality coefficient A_n, then $S_{n,0}(r) \propto r^{\xi_{n,0}}$ for r in the inertial range. (Henceforth, to simplify notation, we will often set $\bar{\varepsilon} = L = 1$ and omit the second index in the ξ's and η's.)

As the first step in the theory, we write the exact equation for structure function of order $2n$ [11,12] (see also Refs. [3,13,14]) as

$$\frac{\partial S_{2n,0}}{\partial r} + \frac{d-1}{r} S_{2n,0} = \frac{(d-1)(2n-1)}{r} S_{2n-2,2}$$
$$+ (2n-1)\overline{\delta_r a_x U^{2n-2}} , \qquad (7)$$

where the increment of the *r*-component of Lagrangian acceleration of a fluid particle is given by

$$\delta_r a_x = -[\partial_{x'} p(x') - \partial_x p(x)] + v[\nabla^2_{\mathbf{x}'} u(\mathbf{x}') - \nabla^2_{\mathbf{x}} u(\mathbf{x})] \tag{8}$$

and $\mathbf{x}' = \mathbf{x} + \mathbf{r}$.

The second step requires the closure of Eq. (7), for which we need an expression for $\delta_r a_x$ in terms of velocity increments. The Laplacian in (8) can be represented in terms of finite differences on the dissipation cut off η and, by virtue of Eq. (6), the acceleration increment can be made a function of two fluctuating variables (operators) $\delta_\eta u$ and η. To make further progress, however, it is necessary to express η in terms of the velocity field itself.

We consider two scenarios. In the first, we have the option of expressing the acceleration terms through either a model for the conditional mean of the acceleration increment for a fixed value of the velocity increment, or through a direct relation between δa and δu—somewhat in the spirit of Kolmogorov's refined similarity hypothesis [15]. Choosing the former option, we model in the limit $r \to \eta$ where η is a generic local dissipation scale, the *x*-component of acceleration term as [1]

$$\overline{\delta_\eta a_x | \delta_\eta u} \approx \overline{\frac{\delta_\eta u}{\tau}} |\delta_\eta u \approx \overline{\frac{(\delta_\eta u)^2}{\eta}} |\delta_\eta u \approx \frac{(\delta_\eta u)^3}{v} , \tag{9}$$

where $\tau \approx \eta / \delta_\eta u$ is the life-time of a fluctuation on the scale η. In the last step in Eq. (9), we have used

$$v \approx \eta \delta_\eta u , \tag{10}$$

where η is to be regarded as a random field. This step reduces the number of random fields from 2 to 1. Expression (10), which is central for our theory, is proposed here on dimensional grounds but will be obtained below from a second scenario considering dissipative anomaly.

This second scenario follows Polyakov's work [7] on statistically steady turbulence due to the one-dimensional Burgers equation stirred by a large-scale random force. In that work, on the basis of the energy balance equation for $v = 0$, namely,

$$\frac{1}{2}\frac{\partial u^2}{\partial t} + \frac{1}{3}\frac{\partial u^3}{\partial x} = fu , \tag{11}$$

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[1]Due to the spatial homogeneity of turbulence $\overline{\delta a_x} = 0$. This constraint is satisfied if, in addition to (9), we account for the contribution from the Bernoulli's equation-based model for the pressure gradient, proposed in Refs. [14] and [11], as

$$\overline{\delta_\eta a_x | \delta_\eta u} \approx \frac{(\delta_\eta u)^3}{v} + a\frac{(\delta_\eta u)^2}{\eta} ,$$

where the constant *a* is chosen from the condition that $\frac{4}{5} Re = aA_2 \eta_2^{-2+\xi_2}$. Here A_2 is the amplitude in the relation $S_{2,0}(\eta) = A_2 \eta_2^{\xi_2}$ (remembering that $\bar{\varepsilon} = L = 1$). Since the Kolmogorov constant $A_2 \approx 2$ (see [18]), this gives $a \approx 0.4$. The first term on the right side of the above equation comes from viscous dissipation, while the second term describes pressure effects on the dissipation scale η. This term renormalizes the coefficients in front of the remaining contributions to (7) and thus can be neglected in our analysis.

Polyakov derived the dissipation anomaly, which is related to the local form of the Kolmogorov law [2] as

$$-\frac{d}{dt}\left(u\left(x+\frac{y}{2}\right)u\left(x-\frac{y}{2}\right)\right) \approx \frac{2}{3}\frac{\partial}{\partial x}u^3$$
$$+\lim_{y\to\eta}\frac{1}{6}\frac{\partial}{\partial y}\left(u\left(x+\frac{y}{2}\right)-u\left(x-\frac{y}{2}\right)\right)^3 = D. \qquad (12)$$

Here, $y \to \eta \to 0$, and

$$D \approx -F = f\left(x+\frac{y}{2}\right)u\left(x-\frac{y}{2}\right) + f\left(x-\frac{y}{2}\right)u\left(x+\frac{y}{2}\right)$$

when $v = 0$, while

$$D = v(u(x+y)\partial^2_{x-y}u(x-y) + u(x-y)\partial^2_{x+y}u(x+y))$$

when the forcing f is zero. Eq. (12) balances the singular contributions in the limit $y \to \eta \to 0$, while the regular contributions disappear by virtue of (11). The coordinate shift y in Eq.(12) is identical to Kolmogorov's displacement $r = y \to \eta$. If, as $\eta \to 0$, the velocity field is non-differentiable (i.e., singular), the left side of Eq. (12) does not approach zero <u>even in the limit</u> $v \to 0$. In a statistically steady state, Eq. (12) immediately gives $(u(x+y)-u(x))^3 = -12Fy$ for the inertial range $y = r \ll L$. We can see that the celebrated $(-4/5)$ths law of Kolmogorov [2] is not locally valid because of the $O(\partial_x u^3)$ term in (12); this term can, however, be eliminated by averaging (12) over the directions of velocity vector $\mathbf{u}/u$.

Now using the finite difference definition of all derivatives on a dissipation scale $\eta \to 0$, we can write, after some algebra, that

$$\frac{1}{3}\frac{u^3(x^+)-u^3(x^-)}{\eta} + \frac{(u(x+2\eta)-u(x))^3}{6\eta}$$
$$\approx [2v/\eta^2] \times [u^2(x^+) + u^2(x^-) + u(x^+)u(x-3\eta)$$
$$+ u(x^-)u(x+3\eta) - 4u(x^+)u(x^-)], \qquad (13)$$

where $x^\pm = x \pm \eta$. This equation is correct up to $O(\eta^2)$. As mentioned earlier, the single-point contribution to this relation disappears when averaged over the "directions" of η. While the left side of (13) involves two-point differences, the right side includes contributions from four shifted points. To proceed further, we assume that η plays the same role as the width of typical shock structures, and conclude that $u(x+3\eta) - u(x) \approx u(x^+) - u(x^-) \approx u(x+2\eta) - u(x)$, as a result of which the right side of (13) is $O[v(\delta_\eta u)^2/\eta^2]$. This leads to (10).

In three dimensions, however, additional terms appear due to the pressure gradient–velocity product. The relevant extensions have been made in Ref. [8]. The finite-difference representation of the equations from Ref. [8] on the dissipation scale $\eta \to 0$ yield the estimate

$$\frac{(\delta_\eta u)^3}{\eta} \approx -\delta_\eta\left(\frac{\partial p}{\partial x}u\right) - v\frac{(\delta_\eta u)^2}{\eta^2}. \qquad (14)$$

Since on the dissipation scale the pressure and dissipation contributions are of the same order [16], expression (14) gives the same balance relation, $v \approx \eta \delta_\eta u$, obtained above. Thus, the relation (10) applies also to three-dimensional turbulence.

Two comments are in order. First, the model for acceleration should include the $O((\delta_\eta u)^2/\eta)$ quadratic contribution coming from the pressure terms [16]. However, the pressure term simply renormalizes the coefficients in front of the remaining contributions to (7) and, as a result, does not alter the steps presented above (see also Ref. [11]). Second, to our knowledge, there are no experimental or numerical data that directly address the conditional acceleration term of Eq. (9), though related conditional data are accumulating rapidly [16,17].

Substituting Eq. (9) (after using Eq. (10) into Eqs. (7) and (8) we obtain an infinite set of equations coupling the structure functions $S_{2n}(r)$ and $S_{2n+1}(r)$. These equations are valid for all magnitudes of displacement $r \ll L$, including $r \to \eta_{2n}$. It will become clear below that $\eta_{2n} \geqslant \eta_{2n+1}$ and, as a result both $S_{2n}(\eta_{2n})$ and $S_{2n+1}(\eta_{2n})$ are in their respective algebraic ranges, i.e., $S_{2n}(\eta_{2n}) \propto \eta_{2n}^{\xi_{2n}}$ and $S_{2n+1}(\eta_{2n}) \propto \eta_{2n}^{\xi_{2n+1}}$. Thus, on the scale η_{2n}, Eqs. (9) and (7) give

$$\eta_{2n}^{\xi_{2n}-1} \approx Re\, \eta_{2n}^{\xi_{2n+1}} \ . \tag{15}$$

We thus have

$$\eta_{2n} \approx Re^{1/(\xi_{2n}-\xi_{2n+1}-1)} \ , \tag{16}$$

giving, for $n = 1$, the well-known relation [19] for the dissipation scale $\eta_2 \approx Re^{1/(\xi_2-2)}$. Since by Hölder inequality, for all $q \geqslant p$, the exponents $\xi_p \geqslant \xi_q$, it follows from (16) that $\eta_{2n} \geqslant \eta_{2n+1}$, which justifies the derivation of expression (15). By virtue of Eq. (6) we have

$$\overline{\left(\frac{\partial u}{\partial x}\right)^{2n}} \propto Re^{(\xi_{2n}-2n)/(\xi_{2n}-\xi_{2n+1}-1)} \ . \tag{17}$$

To compare Eq. (17) with the outcome of the multifractal formula (5), we notice that both are the same in the limit $n \to 1$. In the limit $n \to \infty$, if the exponents $\xi_n \to \xi_\infty = \text{const}$, or $\xi_n \to \alpha n$, Eq. (17) and the outcome of mutifractal formula (5) are identical.

The moments of velocity derivatives are given from Eq. (17) to be

$$S_n = \overline{\left(\frac{\partial u}{\partial x}\right)^n} \Bigg/ \overline{\left(\frac{\partial u}{\partial x}\right)^2}^{n/2} \propto Re^{s_n} \ , \tag{18}$$

with

$$s_{2n} = \frac{\xi_{2n} - 2n}{\xi_{2n} - \xi_{2n+1} - 1} - n \ . \tag{19}$$

This expression can be evaluated readily if we know ξ_n. Using the result obtained in Refs. [11,12] for ξ_n, we get

$$\xi_n \approx \frac{1.15n}{3(1 + 0.05n)} \ . \tag{20}$$

The exponents s_2–s_6 from (19), after using (20), are listed in Table 1. Almost the same results are obtained if experimental values [3] are chosen for ξ_n instead of (20). These values also agree very well with results from several phenomenological MF models.

Several experimental measurements of s_n are available in the literature. For an incomplete list, see Refs. [17,19,20]. We compare in Table 1 the theoretical numbers above with the data of [20] in the atmospheric boundary layer at very high Reynolds numbers and the latest wind tunnel measurements [17] in grid turbulence. The differences between the two sets of experimental numbers are a measure of uncertainty in the data. Keeping this in mind, we may regard the agreement with the theoretical values to be very good. The conclusions from other data sets Ref. [19] are quite similar.

As an aside, we note that the theory can also be used to show that the results of Kolmogorov's Refined Similarity Hypotheses (RSH) [15] are at least numerically close to the ones derived above. We can evaluate the moments of velocity derivative $(\partial_x u)^{2n}$ by extrapolating RSH to η. The dissipation averaged on this scale is of the same order as the unaveraged dissipation. This assumption, commonly adopted in the literature, appears reasonable if there is no structure for scales smaller than η. From dissipation anomaly, we obtain

$$\varepsilon_\eta \approx \varepsilon \propto \frac{(\delta_\eta u)^3}{\eta} \ , \tag{21}$$

where

$$\varepsilon_\eta = \frac{1}{\eta^3} \int_\eta \varepsilon(x)\, d^3 x \tag{22}$$

is the dissipation rate averaged over a "ball" of a radius η with the center at $\mathbf{x}$. Eq. (21) is an order of magnitude estimate averaged over the "universal" Kolmogorov noise $V = \eta \varepsilon_\eta / (\delta_\eta u)^3$. Combining Eqs. (21) and (18) for the dissipation

Table 1
The scaling exponents s_n from theory and experiment

Exponent	s_2	s_3	s_4	s_5	s_6
Theory	0	0.066	0.176	0.34	0.49
Experiment	0	0.05	0.20	0.32	0.54
	0	(0.06)	(0.16)	(0.31)	(0.50)

The numbers in parentheses are from Ref. [17].

scale of the $3n$th moment of velocity difference we obtain

$$G_{2n} = \overline{\left(\frac{\partial u}{\partial x}\right)^{2n}} \propto Re^{g_{2n}} \,, \tag{23}$$

instead of Eq. (19), where

$$g_{2n} = n + \frac{\xi_{3n} - n}{\xi_{3n} - \xi_{3n+1} - 1} \,. \tag{24}$$

As we see, the two sets of formulae (18)–(19) and (23)–(24) are identical when $n = 1$ leading to the exact relation $\overline{(\partial_x u)^2} \propto Re$. With the relation (20) for the scaling exponents, both relations have the same asymptotics $G_{2n} \rightarrow Re^{2n}$ in the limit $n \rightarrow \infty$. In the interval $n \geqslant 1$, the formulae (18)–(19) and (23)–(24) differ by no more than a few percent.

In summary, the theory developed here combines the exact Eqs. (7) and (8) with relations (9) and (10). Together, they lead to Eqs. (18)–(19), which form the main result of the paper. While this form is known from the MF theory, the present paper obtains the exponents theoretically and the results agree well with experiments.

It is useful to restate here the approximations involved in derivation of (9) and (10). Expression (10) was obtained through the model (9) for the acceleration terms in (7), and also as an order-of-magnitude estimate from the equations for dissipation anomaly. Since, at the dissipation scale η, the pressure contribution simply renormalizes the coefficients in the left side of equation (7), the expression (9) for the viscous friction force introduces Re-dependence into (7). This is the reason why, for the fixed magnitude of the inertial range displacement r, the structure functions $S_{n,m}(r)$ are Re-independent, while the moments of derivatives are strongly Re-dependent. Though we cannot prove that either scenario leading to (10) is rigorous, the good agreement observed with experimental data gives us some confidence that the theory is a step in the right direction.

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