Intermittency and the Passive Nature of the Magnitude of the Magnetic Field

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It is shown that the statistical properties of the *magnitude* of the magnetic field in turbulent electrically conducting media resemble, in the inertial range, those of passive scalars in fully developed three-dimensional fluid turbulence. This conclusion, suggested by the data from the Advanced Composition Explorer, is supported by a brief analysis of the appropriate magnetohydrodynamic equations.

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In recent years, the problem of turbulent advection and diffusion of scalar and vector fields, both passive and active, has received renewed attention (see, for instance, Refs. [1–4]). Here, we study one of the most interesting examples in this family, namely, the case of magnetohydrodynamics (MHD), in which the magnetic field fluctuation $\mathbf{B}$ is described by the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad (1)$$

$\mathbf{v}$ being the turbulence velocity and η the magnetic diffusivity. Equation (1) can be regarded as a vector analogue of the advection-diffusion equation

$$\frac{\partial \theta}{\partial t} = -(\mathbf{v} \cdot \nabla)\theta + D\nabla^2\theta \quad (2)$$

for the evolution of a passive scalar θ subject to molecular diffusivity D. Aside from the fact that $\mathbf{B}$ is a vector and θ a scalar, the equations are different also because $\mathbf{v}$ in Eq. (1) can be affected quite readily by the feedback of the magnetic field $\mathbf{B}$. Our interest here is to explore the extent of similarities, despite these obvious differences. While the idea of exploiting formal analogies between(1) and (2) is not new (see [5] and references cited there on the MHD/hydrodynamic analogy), the characterization of inertial-range similarities of the magnetic and passive scalar fields does not seem to have been attempted before.

Solar wind is an excellent natural ''laboratory'' for the MHD problem. It is known that the statistical properties of velocity fluctuations in solar wind are remarkably similar to those observed in fluid turbulence [6]. It is also known that the plasma power spectra of the magnetic field and velocity fluctuations often contain an ''inertial'' range with a slope of approximately $-5/3$ (see Refs. [6–8]. The spectrum for individual components of $\mathbf{B}$ varies from one component to another, and on the large scale features that vary across the 11 yr solar cycle and other *large*-scale anisotropies. The transverse variances of vector $\mathbf{B}$ can contain an order of magnitude more energy than the parallel variance. Such issues also enter discussions of the structure of large amplitude Alfvén waves and their effect on $\mathbf{B}$ [9–11].

Statistical properties of the *magnitude* $B = \sqrt{B_i^2}$ in the *inertial* range of scales are expected to be more universal (as is rather common for the inertial range properties [6]), and we shall study these properties from the data obtained from Advanced Composition Explorer (ACE) satellite magnetometers for the year 1998. The sun was quiet in this period, the data are statistically stable, and the scaling spectrum with $-5/3$ slope is quite typical (see Fig. 1). This slope is identical to that observed for passive scalar fluctuations in fully developed three-dimensional fluid turbulence [12]. Spurred by this inertial-range similarity between θ and B, we are motivated to explore further the properties of the magnitude B and compare them with those of the passive scalar.

For this purpose, we consider the scaling of structure functions

$$\langle |\Delta B_\tau|^p \rangle \sim \tau^{\zeta_p}, \quad (3)$$

where $\Delta B_\tau = B(t+\tau) - B(t)$. The exponent ζ_2 is directly related to the spectral exponent (for our case $\zeta_2 \approx 5/3 - 1 = 2/3$ [12]).

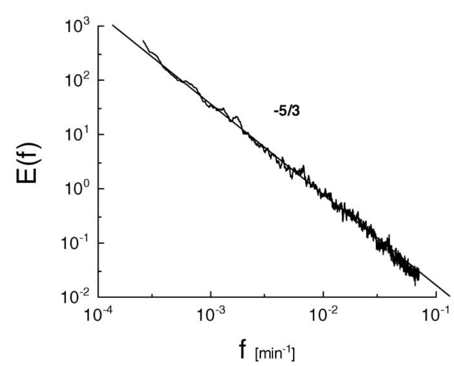

FIG. 1. Energy spectrum of the *magnitude* $B = \sqrt{B_i^2}$ of the magnetic field $\mathbf{B}$ in the solar wind plasma.

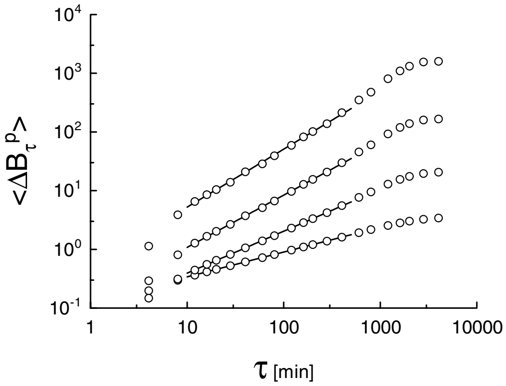

FIG. 2. Structure functions of magnetic field *magnitude* in the solar wind plasma as measured by the ACE magnetometers in nano-Tesla for the year 1998 (4 min averages).

Figure 2 shows the scaling of structure functions for the B in the solar wind data. Slopes of the least-square fits in the apparently scaling region provide us the exponents ζ_p; these are shown in Fig. 3 as circles. Triangles in the figure indicate experimental values obtained for temperature fluctuations in the atmosphere [13]. The other experimental data [14,15] are in agreement with each other to better than 5%. The $\star$ symbols are for the passive scalar field obtained by numerically solving the advection diffusion in three-dimensional turbulence [16]. It is clear that the exponents for the passive scalar data are in essential agreement with those for the *magnitude* fluctuations of the magnetic field.

One can analyze the solar wind data somewhat differently using the notion of the extended self-similarity (ESS). Since, empirically, the fourth-order exponent is quite closely equal to unity for magnitude fluctuations of the magnetic field, i.e.,

$$\langle|\Delta B_\tau|^4\rangle \sim \tau, \tag{4}$$

we can extend the scaling range (and consequently improve the confidence with which those exponents are determined) by redefining them as

$$\langle|\Delta B_\tau|^p\rangle \sim \langle|\Delta B_\tau|^4\rangle^{\zeta_p}. \tag{5}$$

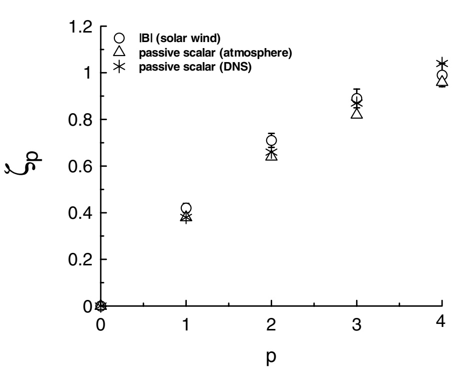

FIG. 3. Scaling exponents (3) calculated for B in the solar wind (circles) and for the passive scalar in the atmospheric turbulence (triangles [13]), and in the direct numerical simulation of three-dimensional fluid turbulence (stars [16])

Figure 4 shows the ESS dependence (5). The slopes of the least-square fits provide us with the ESS scaling exponents ζ_p, shown in Fig. 5 as circles. The other symbols have remained unchanged from Fig. 3. The shift of the exponents ζ_p in comparison to those from ordinary self-similarity is about 4%, but the scaling interval for ESS is considerably larger. This increased scaling range for ESS is well known in other contexts [17].

The results of Figs. 3 and 5 suggest that at least up to the level of the fourth-order the scaling exponents for the passive scalars and for the *magnitude* of the magnetic field are essentially the same. This is both surprising and thought provoking, and needs to be understood. To this end, let us return to Eq. (1) and specialize [18], for simplicity, to the incompressible case ($\nabla \cdot \mathbf{v} = 0$). Equation (1) can then be rewritten as

$$\frac{\partial \mathbf{B}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{v} + \eta \nabla^2 \mathbf{B}. \tag{6}$$

Let us now consider the equation for the magnitude B of the magnetic fluctuations given by $\mathbf{B} = B\mathbf{n}$, where $\mathbf{n}$ is the unit vector with its direction along $\mathbf{B}$: $n_i = B_i/B$. Multiplying both sides of Eq. (6) by the vector $\mathbf{n}$ and taking into account that $n_i^2 = 1$ we obtain

$$\frac{\partial B}{\partial t} = -(\mathbf{v} \cdot \nabla)B + \eta \nabla^2 B + \lambda B, \tag{7}$$

in which the "friction stretching" (or the production) coefficient λ in the last term has the form

$$\lambda = n_i n_j \frac{\partial v_i}{\partial x_j} - \eta\left(\frac{\partial n_i}{\partial x_j}\right)^2, \tag{8}$$

with the indices i and j representing the space coordinates, and the summation over repeated indexes is assumed. The first term on the right-hand side of Eq. (8) is crucial for any dynamo effect.

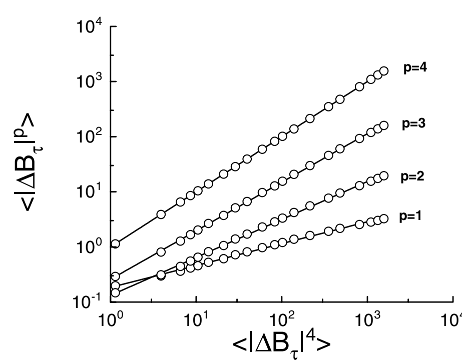

FIG. 4. Extended self-similarity (ESS) of the magnetic field *magnitude* in the solar wind plasma.

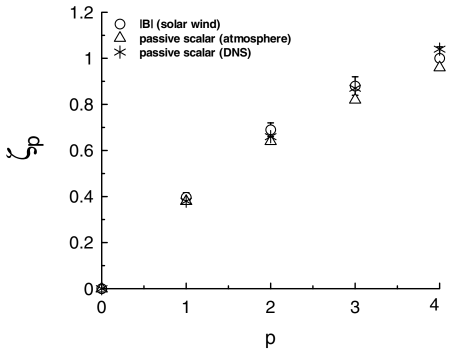

FIG. 5. The same as in Fig. 3 but using the ESS method (5) for B.

If the statistical behaviors of $\Delta\theta_\tau$ and ΔB_τ are to be similar, as suggested by Figs. 3 and 5, we should be able to observe the underlying similarity in Eqs. (2) and (7). The major difference is the presence in Eq. (7) of the production term λB. However, given the empirical indications that ΔB_τ and $\Delta\theta_\tau$ possess the same scale-similarity in the inertial range, it would seem that there must be circumstances under which the λ term in Eq. (7) is small. What are those circumstances?

The second term in λ is assured to be small because the smallness of the magnetic diffusivity η, but difficulties may arise from the first term on the right-hand side of Eq. (8). As already remarked, the MHD turbulence is usually anisotropic with respect to the large scale magnetic field [19] and that the parallel gradients are suppressed relative to the perpendicular gradients. While this feature can significantly diminish the first term on the right-hand side of Eq. (8) in the large scales, the real issue for us here is the effective cancellation in the first term in the inertial range.

To address this issue, let us consider the following conditional average of Eq. (7): fix the magnitude B in the vector field $\mathbf{B} = B\mathbf{n}$ and perform the average over all realizations of the direction vector field $\mathbf{n}$ permitted by the vector Eq. (6). Let us denote this ensemble average as $\langle \ldots \rangle_\mathbf{n}$, and call this *directional* averaging. From the definition, this averaging procedure does not affect B itself, but affects the velocity field $\mathbf{v}$ and the coefficient λ in Eq. (7). We thus obtain

$$\frac{\partial B}{\partial t} = -(\langle \mathbf{v} \rangle_\mathbf{n} \cdot \nabla) B + \eta \nabla^2 B + \langle \lambda \rangle_\mathbf{n} B. \tag{9}$$

Two comments are needed. First, the conditional average indicated by $\langle \ldots \rangle_\mathbf{n}$ and the global average indicated by $\langle \ldots \rangle$ are quite different; for this reason, the quantity B in (9) remains a *fluctuating* variable. Second, the solutions of the original Eq. (6) satisfy Eqs. (7) and (9), but not all possible formal solutions of Eqs. (7) and (9) satisfy Eq. (6); similarly, not all formal solutions of Eq. (9) satisfy Eq. (7) while all solutions of Eq. (7) do satisfy Eq. (9). Restricting comments to the relationship between Eqs. (7) and (9), the solutions of the two equations are the same only if the initial conditions are the same and if realizations of $\langle \mathbf{v} \rangle_\mathbf{n}$ and of $\langle \lambda \rangle_\mathbf{n}$, related to these initial conditions by the conditional average procedure, are obtained from solutions applicable to Eq. (7) (cf. Ref. [20] where the author used particular ensemble averages for the evolution of $\mathbf{B}$).

Returning now to Eq. (9), the conditionally averaged velocity field $\langle \mathbf{v} \rangle_\mathbf{n}$ may possess statistical properties that are different from those of the original velocity field $\mathbf{v}$, and there can be circumstances under which $\langle \lambda \rangle_\mathbf{n} = 0$, or small. If so, the similarity between Eqs. (2) and (9) [and, consequently, between Eqs. (2) and (7)] can be the basis for the similarity in statistical properties of their solutions. Let us consider a generic set of conditions, presumably for the inertial range, which can result in $\langle n_i n_j \partial v_i / \partial x_j \rangle_\mathbf{n} = 0$. This can be a combination of isotropy, which yields

$$\langle n_i n_j \rangle_\mathbf{n} = 0 (i \neq j), \qquad \langle n_1^2 \rangle_\mathbf{n} = \langle n_2^2 \rangle_\mathbf{n} \langle n_3^2 \rangle_\mathbf{n}, \tag{10}$$

and the statistical independence

$$\langle n_i n_j \varphi \rangle_\mathbf{n} = \langle n_i n_j \rangle_\mathbf{n} \langle \varphi \rangle_\mathbf{n}, \tag{11}$$

where $\varphi = \partial v_k / \partial x_l$ for arbitrary k and l.

We now use conditions (10) and (11) in the presence of the incompressibility condition $\partial v_i / \partial x_i = 0$ and obtain

$$\langle \lambda \rangle_\mathbf{n} = -\eta \left\langle \left(\frac{\partial n_i}{\partial x_j} \right)^2 \right\rangle_\mathbf{n}. \tag{12}$$

That is, the difference between the passive scalar equation (2) and the conditionally averaged equation (9) for B is reduced to pure "friction" with the friction coefficient given by (12). For small values of η this term does not affect the scaling properties of B in the inertial range [21].

One can also seek more general conditions under which the stretching part from the conditionally averaged coefficient $\langle \lambda \rangle_\mathbf{n}$ becomes negligible. For instance, it is not necessary for conditions (10) and (11) to be satisfied for all realizations of the magnetic field $\mathbf{B}$, but only for the subset of realizations that gives the main statistical contribution to the structure functions (3). Let us name this subset of realizations as I. The structure functions (3) depend on the statistical properties of the *increments* with respect to τ, namely ΔB_τ, belonging to the inertial range of scales. One of the consequences of intermittency is that the statistical properties of the increments are essentially different from those of the field $\mathbf{B}$ itself. Therefore, the subset I need not generally coincide with the subset G, say, that gives the main statistical contribution to the *global* average $\langle n_i n_j \partial v_k / \partial x_l \rangle$. This means, in particular, that the conditions (10) and (11) can be valid for the inertial range (i.e., for subset I), while globally (i.e., for subset G) these conditions could well be violated. For instance, the probability density function (PDF) of passive scalar fluctuations (not increments) are generically Gaussian [22], while the PDF of the magnetic fluctuations in the solar wind to be nearly log-normal [23,24]. In fact, this last observation goes well with the theoretically obtained result [25] that if the multiplicative term λB becomes significant (in a global sense) in equations such as (7) and (9) and (the term is locally small for our case anyhow), the effect is to transform a Gaussian distribution of the solution into a log-normal.

Another useful remark is that the properties of the spectrum of B can be deduced by proper reductions two-point PDF of the vector $\mathbf{B}$ (for one point statistics see, for instance, Ref. [26]). Therefore, the statistics of B cannot be considered an issue independent of $\mathbf{n}$ in the global sense (also locally $\langle \mathbf{v} \rangle_{\mathbf{n}}$ in Eq. (9) and $\langle \lambda \rangle_{\mathbf{n}}$ in Eq. (12) are still significantly dependent on $\mathbf{n}$). It should be stressed that we do not use the kinematic approach to the induction equation; in fact, $\langle \mathbf{v} \rangle_{\mathbf{n}}$ in Eq. (9) can be affected by the feedback of the vector magnetic field $\mathbf{B}$.

Finally, although the Letter does not deal with dynamo effects it is useful to consider very briefly a question of how and where the analysis would break down if a derivation of the α effect, for instance, was being sought. Introduction of a significant helical component to the assumed velocity statistics can result in the breaking of mirror symmetry. Therefore, the first (nondiagonal) condition in (10) will be violated, while the second (diagonal) one in (10) will remain intact.

In summary, we have shown that remarkable scaling similarities exist *in the inertial range* between the passive scalar and the *magnitude* of the magnetic field in MHD flows. Motivated by this observation, we have derived the dynamical equation for B and argued that, under circumstances governed by Eqs. (10) and (11), dynamical similarity exists between the equations governing the passive scalar and the *magnitude* of the magnetic fluctuations.

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