Energy spectrum of grid-generated He II turbulence

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A grid of bars towed through a sample of He II produces both superfluid turbulence and classical hydrodynamic turbulence. The two velocity fields—in the normal fluid and in the superfluid—have been observed to have the same energy spectral density over a large range of scales. Here, we introduce a characteristic scale $\ell_q = 2\pi(\varepsilon/\kappa^3)^{-1/4}$, where ε is the rate of turbulent energy dissipation per unit volume, and note that the energy spectrum in superfluid turbulence depends also on the quantum of circulation κ, for wave numbers $k > k_q \equiv 2\pi/\ell_q$. We propose that the spectral density in this range is of the form $\phi(k) = C\varepsilon\kappa^{-1}k^{-3}$, where C is the three-dimensional Kolmogorov constant in classical turbulence. This form is consistent with recent experiments in the temperature range 1.2 K$<T<$2 K on the temporal decay of the vortex line density in the grid-generated He II turbulence.

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Superfluid turbulence is usually viewed as a tangle of quantized vortex lines [1], and its possible relationship to hydrodynamic turbulence in classical fluids has not been given much thought. In most early studies, superfluid turbulence was created by applying a heat current that resulted in a counterflow of the normal and superfluid components of He II. This necessarily resulted in disconnected fields of hydrodynamic and superfluid turbulence, and so it seemed reasonable to think that the two fields were quite different in character. Recently, however, turbulence in He II has been generated in a manner similar to that in studies of classical turbulence, by either confining He II between rotating discs [2] or towing a grid of bars through a stationary sample of He II [3,4]. These experiments reveal a deep similarity between the two fields of turbulence. However, this similarity breaks down at small scales where the quantization effects of circulation have to be taken into account explicitly. The purpose of this paper is to propose, for superfluid turbulence, a form of spectral density at these small scales that is consistent with recent vortex line density measurements.

Two experimental observations are relevant to us. First, Maurer and Tabeling [2] produced turbulence in a flow of liquid helium confined between counterrotating discs, and obtained the energy spectral density at $T=2.3$ K, 2.08 K, and 1.4 K. From the standpoint of the two fluid model, one may think of liquid helium as being largely superfluid at 1.4 K and largely normal (or classical) at 2.08 K, and entirely normal at 2.3 K. Yet, the spectra obtained at all three temperatures were identical and possessed an inertial range of the classical Kolmogorov form. Second, in the experiments in Refs. [3,4], turbulence was produced by pulling a grid of bars through a stationary sample of He II, and the decaying average vortex line density (i.e., length of line per unit volume) over a measurement volume was obtained from the attenuation of the second sound. The character of the decay of the superfluid vortex line density did not change with temperature within the range 1.2 K$<T<$2.0 K, while the ratio of the superfluid to normal fluid density varied, over this temperature range, from near unity to almost zero.

Thus, even though the situation in He II turbulence is almost certainly quite complex, these two experimental observations suggest that the two fields of turbulence—the classical turbulence in the normal component and the superfluid turbulence—are closely similar to each other, and so, in particular, are their energy spectra. This view has been expanded quantitatively in Refs. [3–5]. Vinen [5] has argued that this similarity extends only for large enough length scales, and that at small length scales of order of intervortex spacing of the superfluid tangle, the two fields cannot be fully matched. For that range of scales, the superfluid spectral density would depend in addition on κ. Vinen used a dimensional argument similar to that of Kolmogorov to show that the spectral density in the wave-number space k would then be of the form

$$\phi(k) = C\varepsilon^{2/3}k^{-5/3}f(\varepsilon k^{-4}\kappa^{-3}), \tag{1}$$

where C is the three-dimensional Kolmogorov constant, ε is the rate of turbulent energy dissipation per unit volume, κ is the quantum of circulation of superfluid vortices, and f, which represents the substance of the mismatch between the two fields of vorticity, is an undetermined function of its argument.

We are interested here in the form of the superfluid energy spectrum [or the function f in Eq. (1)], in order to compare it with the temporal decay of the vortex line density known from experiments based on the second-sound attenuation. Consistent with the physical picture of superfluid turbulence as a tangle of quantized vortex lines, the spectral density must be finite up to very high k—loosely speaking up to the order of the inverse of the vortex core representing the ultimate cutoff scale for superfluid turbulence.

We first make the elementary observation that the argument of Vinen's function f in Eq. (1) allows us to define by dimensional considerations a characteristic quantum wavenumber, $k_q = (\varepsilon/\kappa^3)^{1/4}$, and the corresponding quantum length scale, $\ell_q = 2\pi(\varepsilon/\kappa^3)^{-1/4}$. The length ℓ_q represents the scale below which quantized circulation effects become important. The picture is that the two fields of turbulence have the same classical hydrodynamic spectral forms for all scales much larger than ℓ_q, but that quantization effects will modify the superfluid energy spectrum elsewhere. This modification comes about most likely by the action of the mutual friction force.

The experimental observable in the experiments is the quantized vortex line density L. We are interested in being able to deduce the form of the superfluid energy spectrum that is consistent with the temporal decay of L. Our framework is Eq. (1) and the analytically solvable spectral decay model described in Refs. [3,6]. We particularly consider the influence of the functional form $f(\varepsilon k^{-4}\kappa^{-3})$ on the temporal decay of L. Let us assume the classical Kolmogorov spectral density in the entire spectral range between some large-scale wave number k_D and the quantum wave number $k_q \equiv (\varepsilon/\kappa^3)^{1/4}$, and the modified one beyond k_q, for which we choose a form

$$\phi(k) = C\varepsilon^{2/3}k^{-5/3}(\varepsilon k^{-4}\kappa^{-3})^{\alpha}. \tag{2}$$

The positive power α makes the spectrum more steep in k, while the negative α makes it less steep, compared to the classical roll off exponent of $-5/3$. By integration over $k_D < k < \infty$, for $\alpha \neq -1/6$, we get the energy E to be

$$E = \frac{3C}{2}\left[\varepsilon^{2/3}k_D^{-2/3} - \frac{6\alpha}{1+6\alpha}\varepsilon^{1/2}\kappa^{1/2}\right]. \tag{3}$$

Noting the definition of $\varepsilon = -dE/dt$, this is a differential equation for the decaying energy. However, for comparison with the second sound experiment it is useful to discuss the temporal decay in terms of the experimentally observed quantity, $L = L(t)$—which can be related in the case of homogeneous and isotropic turbulence to the energy decay rate by [5]

$$\varepsilon = \nu_{eff}\kappa^2 L^2. \tag{4}$$

Here ν_{eff} is the effective kinematic viscosity of turbulent He II [5,7,8], and is of the same order of magnitude as the kinematic viscosity defined as the dynamic viscosity over the total density of He II. The above differential equation can be solved analytically [9] for $L(t)$. Naturally, for $\alpha = 0$ we have a spectral density that remains classical for $k > k_D$ and corresponds to the usual universal power law with exponents -2 for decaying energy and $-3/2$ for decaying vorticity or vortex line density. For any $\alpha > 0$, $\phi(k)$ is steeper beyond k_q and L decays faster; the case of $\alpha \to \infty$ corresponds to a sharp cutoff at k_q, considered in Refs. [4,6]. For $-1/6 < \alpha < 0$, $\phi(k)$ becomes less steep beyond k_q and L decays more slowly. Note that the value of the roll-off exponent in the energy spectrum alone does not determine the character of the temporal decay of energy or vortex line density, but does so only when combined with the time evolution of the relevant length scales that is implicitly given by their dependence on ε.

Returning to the two fluid model of turbulent He II, the precise way in which the quantization effects enter the picture depends on whether k_q is comparable to k_η, where k_η is the inverse of the Kolmogorov scale η. Two possibilities arise.

First, let us assume that the Kolmogorov scale η is larger than the quantum scale. In this case, the quantized vortices lie entirely in the strain field set up by scales $O(\eta)$, by which they are stretched. It is possible then that an effective quantum subrange gets established within which the only important parameters are the quantum of circulation κ, and the strain rate at the Kolmogorov scale, $O((\varepsilon/\nu)^{1/2})$, where ν is the kinematic viscosity of the normal component. (In the above expression, an effective viscosity might replace the actual kinematic viscosity, see [7], but the two quantities are of the same order, as already noted.) Since the strain field set up by the Kolmogorov scales is proportional to the square root of the energy dissipation, that is the only power of ε that ought to appear in the spectral density. This corresponds to the case $\alpha = -1/6$ [see Eq. (2)]. The superfluid spectral density will then be of the form

$$\phi(k) = C\varepsilon^{1/2}\kappa^{1/2}k^{-1}. \tag{5}$$

It is possible to show, in the same spirit as above [9], that this special case of negative α also leads to the temporal decay of L that gradually slows down relative to the exponent of $-3/2$.

Let us apply this physical picture to the late stage of the decay, when the growing Kolmogorov scale becomes of the size of the channel D. Assuming the k^{-1} scaling in the entire spectral range between the large-scale wave number $k_D = k_\eta$ and k_q, we get, by integration, the energy E to be

$$E = C\kappa^{3/2}\varepsilon^{1/2}\ln(k_q/k_\eta). \tag{6}$$

Differentiating with respect of time and using the relation (4) leads to the inverse rate of decay of the vortex line density, consistent with numerical simulations [10] and also with the Vinen's original model [11], developed on the basis of experiments in *counterflow* He II turbulence.

However, this is not what the experiments on grid generated He II turbulence show [4,12], and one must look for alternatives to Eq. (5). The grid experiments demonstrate that in the late stage of decay of the grid turbulence in He II, the vortex line density decays exponentially [3,4]. Measurements show that the effective kinematic viscosity is about 3–8 times less than κ [8], which in turn suggests that the Kolmogorov scale and the quantum scale are of the same order. One form of the spectral density (1) that is continuous at k_q and consistent with the exponential decay is obtained for $\alpha = 1/3$, and corresponds to

$$\phi(k) = C\varepsilon\kappa^{-1}k^{-3} \tag{7}$$

for $k > k_q$ [13]. The implication of the above equation is that there is only one time scale in the problem and it is given by $(\kappa/\varepsilon)^{0.5}$. This time scale represents a balance of the turbulent kinetic energy that is now almost entirely due to the quantized vortex lines, and the rate of energy "dissipation" presumably in the form of energy radiation by Kelvin waves, as visualized by Vinen [5]. Other likely mechanisms of dissipation are mutual friction and the shortening of the vortex cores during reconnection. Needless to say, the above spectral form is likely to be truncated sharply when the wave number is of the order of the core size of quantized vortices, so the divergences naively suggested by the k^{-3} form are not of concern.

The k^{-3} form is not the only possibility for explaining the observed exponential decay in late stages. One may potentially explain it, for example, by assuming the existence of big eddies (due to mean flow associated with inhomogeneities of the towed grid, etc), which survive to this last stage and decay exponentially due to finite viscosity. However, with characteristic size of the channel $D=1$ cm and effective kinematic viscosity ν of order 2×10^{-4} cm^2/s, one gets the characteristic viscous decay time D^2/ν of order of an hour or so, about two orders of magnitude larger than that measured experimentally.

Assuming again that during the late stages of decay the quantum scale is limited by the size of the channel, $k_D=2\pi/D$, the energy content of the turbulent flow is given by integration of Eq. (7) as

$$E=\frac{C\varepsilon}{2\kappa}\left(-\frac{2\pi}{D}\right)^2, \quad \text{and} \quad L=L_0\exp\left(-\frac{t}{t_0}\right), \qquad (8)$$

where the temperature independent characteristic decay time $t_0\cong(C/\kappa)(d/2\pi)^2$. For $C\cong1.5$ [14] and $D=1$ cm, Eq. (7) yields 38 s for t_0, close to experimental value (27 ± 5) s reported in Ref. [4] for the temperature range 1.3 K$\leqslant T\leqslant2$ K [15].

In these calculations leading to the exponential decay of L we have neglected any possible cutoff or other possible change of the spectral form for wave numbers larger than k_q. The experimental resolution at present does not allow us to make any statements on the form of the spectrum in this range. An intriguing possibility therefore remains that the k^{-3} scaling holds only in the range $(\varepsilon/\kappa^3)^{1/4}<k<(\varepsilon/\nu_{eff}^3)^{1/4}$, beyond which the the spectral density might, for reasons explained earlier, display the k^{-1} form before being truncated sharply; this form is also expected for a single straight vortex line. More experiments, including those at very low temperature are needed to shed more light on this issue. We note that a k^{-1} spectrum is found in the simulations of [10], but in the absence of any dissipation.

In summary, based on available experimental results [2,4] on He II isothermal turbulence, we have suggested a form of a three-dimensional energy spectrum in developed He II turbulence. The spectrum is classical for large scales. The quantization of circulation leads to the existence of the characteristic quantum length scale ℓ_q of order $2\pi(\varepsilon/\kappa^3)^{-1/4}$. For wave numbers $k>k_q=2\pi/\ell_q$, the spectrum depends also on the circulation quantum and is likely to be of the form $E(k)=C\varepsilon\kappa^{-1}k^{-3}$. This form of the energy spectrum is consistent with the experiments on temporal decay of the vortex line density in grid generated He II turbulence in the temperature range 1.2 K$<T<$2.0 K, but remains to be confirmed by direct measurement.

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resulting solution $L(t;\alpha=1/3)$ [9] describes experimental data on the temporal decay of the vortex line density after saturation of the energy containing length scale of [4] equally well, using classical value for C and the measured values of ν_{eff} [8], with t_{vo} as the only parameter. However, this form of decay cannot occur beyond $t_B/2$, when the growing quantum scale approaches the size of the channel and becomes saturated by it for the rest of the decay. It is easy to show that this happens when $L=L_q=\sqrt{\kappa/\nu_{eff}}(2\pi/D)^2$, which for the geometry of Ref. [4] corresponds roughly to L_q about 100 cm^{-2}, or to the average intervortex distance of about 1 mm.

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