Fluid turbulence

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The swirling motion of fluids that occurs irregularly in space and time is called turbulence. However, this randomness, apparent from a casual observation, is not without some order. Turbulent flows are as abundant in nature as life itself, and are pervasive in technology. They are a paradigm for spatially extended nonlinear dissipative systems in which many length scales are excited simultaneously and coupled strongly. The phenomenon has been studied extensively in engineering and in diverse fields such as astrophysics, oceanography, and meteorology. A few aspects of turbulence research in this century are briefly reviewed, and a partial assessment is made of the present directions.
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I. INTRODUCTORY REMARKS

The fascinating complexity of turbulence has attracted the attention of naturalists, philosophers, and poets alike for centuries, and ubiquitous allusions have been made to the turbulence of agitated minds and disturbed dreams, of furious rivers and stormy seas. Perhaps the earliest sketches of turbulent flows, capturing details with some degree of realism, are those of Leonardo da Vinci. A serious scientific study has been in progress for more than a hundred years, but the problem has not yet yielded to our efforts. As Liepmann (1979) has pointed out, the outlook, optimism, and progress have waxed and waned over time.

Few would dispute the importance of turbulence. Without it, the mixing of air and fuel in an automobile engine would not occur on useful time scales; the transport and dispersion of heat, pollutants, and momentum in the atmosphere and the oceans would be far weaker; in short, life as we know would not be possible on the earth. Unfortunately, turbulence also has undesirable consequences: it enhances energy consumption of pipe lines, aircraft and ships, and automobiles; it is an element to be reckoned with in air-travel safety; it distorts the propagation of electromagnetic signals; and so forth. A major goal of a turbulence practitioner is the prediction of the effects of turbulence and control them—suppress or enhance them, as circumstances dictate—in various applications such as industrial mixers and burners, nuclear reactors, aircraft and ships, and rocket nozzles.

Less well appreciated is the intellectual richness of the subject and the central place it occupies in modern physics. Looking into the problem, we are immediately faced with an apparent paradox. Even with the smoothest and most symmetric boundaries possible, flowing fluids—except when their speed is very low—assume the irregular state of turbulence. This feature, though not fully understood, is now known to bear *some* connection with the occurrence of dynamical chaos in nonlinear systems. Turbulence has constantly challenged and expanded the horizons of modern dynamics, the theory of differential equations, scaling theory, multifractals, large-scale computing, fluid mechanical measurement techniques, and the like.

Until the 1960s, turbulence was *the* paradigm system in which the excitation of many length scales was recognized as important. The powerful notions of scaling and universality, which matured when renormalization group theory was applied to critical phenomena, had already manifested in turbulence a couple of decades earlier. Turbulence and critical phenomenon share the feature that a continuous range of scales is excited in both; however, they are different in that the fluctuations in turbulence are strong and there exists no small parameter. Thus, turbulence is a paradigm in non-equilibrium statistical physics, in which fluctuations and macroscopic space-time structure coexist. It is an example like no other of spatially extended dissipative systems.

An excellent case can thus be made that turbulence is central to flow technology as well as modern statistical and nonlinear physics. The reader wishing to learn about the subject should begin with Monin and Yaglom (1971, 1975), and move on, for different specialized perspectives, to the books of Batchelor (1953), Townsend (1956), Bradshaw (1971), Leslie (1972), Lesieur (1990), McComb (1990), Chorin (1994), Frisch (1995), and Holmes *et al.* (1998). There are many useful review articles, each emphasizing a different aspect. Some examples are Corrsin (1963), Saffman (1968), Roshko (1976), Cantwell (1981), Narasimha (1983), Hussain (1983), Frisch and Orszag (1990), Lumley (1990), Sreenivasan (1991), Nelkin (1994), Siggia (1994), L'vov and Procaccia (1996), Sreenivasan and Antonia (1997), Zhou and Speziale (1998), Smith and Woodruff (1998), and Canuto and Christensen-Dalsgaard (1998). The two volumes of Monin and Yaglom, covering the subject only until the early seventies, contain more than 1600 pages. Several hundred papers have appeared on the subject since then. Discussing this vast subject in any depth and completeness would be a herculean task. This article makes no such pretensions; instead, it makes a few isolated and qualitative observations to suggest the nature of progress made: slow, multi-faceted, useful, insightful—but often soft. While the importance of turbulence has long made its study imperative, all the tools needed for such a complex undertaking are not fully in place. In this sense, despite its age, turbulence is a frontier subject.

FIG. 1. A turbulent jet of water emerging from a circular orifice into a tank of still water. The fluid from the orifice is made visible by mixing small amounts of a fluorescing dye and illuminating it with a thin light sheet. The picture illustrates swirling structures of various sizes amidst an avalanche of complexity. The boundary between the turbulent flow and the ambient is usually rather sharp and convoluted on many scales. The object of study is often an ensemble average of many such realizations. Such averages obliterate most of the interesting aspects seen here, and produce a smooth object that grows linearly with distance downstream. Even in such smooth objects, the averages vary along the length and width of the flow, these variations being a measure of the spatial inhomogeneity of turbulence. The inhomogeneity is typically stronger along the smaller dimension (the ''width'') of the flow. The fluid velocity measured at any point in the flow is an irregular function of time. The degree of order is not as apparent in time traces as in spatial cuts, and a range of intermediate scales behaves like fractional Brownian motion.

II. THE PHENOMENON AND THE GOAL

Water flowing from a slightly open faucet is smooth and steady, or laminar. As the faucet opens up more, the flow becomes erratic. Figure 1 illustrates that a seemingly erratic turbulent flow is actually a labyrinth of order and chaos. Swirling flow structures—or patterns—of various sizes are intertwined with fluid mass of indifferent shape. Being static, however, the picture does no justice to the dynamical interaction among the constituent scales of the flow. Casual observations suggest that the patterns get stretched, folded and tilted as they evolve, losing shape by agglomeration or breakup—all in a manner that does not repeat itself in detail. Unlike patterns in equilibrium systems, which are associated with phase transitions, those in fluid flows are intimately related to transport processes. The patterns in fluid systems exhibit varying sensitivity to initial and boundary conditions, and are rich in morphology (see, e.g., Cross and Hohenberg, 1994).

The key to the onset of turbulence has long been believed to be the successive loss of stability that occurs with ever increasing rapidity as a typical control parameter in a flow problem is increased (e.g., Landau and Lifshitz, 1959). The most familiar control parameter is the Reynolds number[1] Re, which expresses the balance between the nonlinear and dissipative properties of the flow. This scenario is thought to be relevant especially for flows whose vorticity attains a maximum in the interior, instead of at the boundary. Linear and nonlinear stability theories have been successful in describing the initial stages of the transition to turbulence (e.g., Drazin and Reid, 1981), but the later stages seem quite abrupt (e.g., Gollub and Swinney 1975), and not amenable to stability analysis. This abruptness is qualitatively in the spirit of the modern theory of deterministic chaos (Ruelle and Takens 1971), and is especially characteristic of boundary layers[2] (Emmons, 1951). While this situation is reminiscent of second-order phase transitions in condensed matter, it is unclear if the analogy is helpful in a serious way.

In any case, at high enough Reynolds numbers, nonlinear interactions produce finer and finer scales, and the scale range in developed turbulence is of $O(Re^{9/4})$. The Reynolds number could be several million in the earth's atmosphere a few meters above the ground or in the boundary layer of an aircraft fuselage. Clearly, in such instances, only a statistical description of turbulence and the prediction of its consequences—such as increased mixing, transport, and energy loss—are of practical value. The discovery of an efficient procedure to do this is the principal and outstanding challenge of the subject.

The goal just mentioned is no different from that of statistical thermodynamics. The statistical assumptions made there possess vast applicability and powerful predictive capability. Unfortunately, those made in turbulence have enjoyed far less success, even though much about the behavior of turbulence has been learned in the process of their application. The era in which the statistical approach was the norm—one in which developments in turbulence occurred, on the whole, in the con-

[1]Reynolds number is the dimensionless parameter UL/ν, where U and L are the characteristic velocity and length scales of a turbulent flow and ν is the fluid viscosity. Depending on the purpose, different velocity and length scales become relevant.

[2]The boundary layer is the thin region close to a solid body moving relative to the fluid. Processes in this thin layer are the source, among other things, of fluid dynamical resistance and aerodynamic lift.

text of fluid dynamics—is called here the "classical era." Because of the continuing awareness of the limitations of statistical theories, one has more recently begun to ask whether this basic approach needs to be augmented by a different outlook. This outlook has the common element that it focuses on mechanisms rather than flows, and is influenced by developments in neighboring fields such as bifurcations, chaos, multifractals, and modern field theory. The intent is often to acquire qualitative understanding of fluid turbulence through model nonlinear equations. This era, which we shall loosely call "modern," has benefitted tremendously by the availability of powerful computers and the qualitative theory of differential equations (e.g., the study of space-time singularities).

III. THE CLASSICAL ERA

A. Before Osborne Reynolds

Unlike many other problems in condensed matter physics, the equations governing turbulence—the Navier-Stokes equations—have been known for some 150 years. All available evidence suggests that the phenomenon of turbulence is consistent with these equations, and that the molecular structure makes little difference (except for their role in prescribing gross parameters such as the viscosity coefficient). The Navier-Stokes equations and the use of proper boundary conditions are the result of the cumulative work of heroes such as J. R. d'Alembert, L. Euler, L. M. H. Navier, A. L. Cauchy, S. D. Poisson, J.-C. B. Saint-Venant, and G. G. Stokes. Even as the equations were being refined, controlled experiments were discovering, or rediscovering, that fluid motion occurs in two states—laminar and turbulent—and that a transition from the former to the latter occurs in distinctive ways. It was realized that turbulent flows transport heat, matter, and momentum far better than laminar flows. The concept of "eddy viscosity," attesting to this enhancement of transport, was discussed by Saint-Venant and J. Boussinesq. From observations in water canals, the latter deduced that an apparent analogy exists between gas molecules and turbulent eddies as they carry and exchange momentum.

B. Contributions of Osborne Reynolds

It was Reynolds (1883, 1894) who heralded a new beginning of the study of turbulence: he visualized laminar and turbulent motions in pipe flows; identified the criterion for the onset of turbulence in terms of the nondimensional parameter that now bears his name; showed that the onset is in the form of intensely choatic "flashes" in the midst of otherwise laminar motion; introduced statistical methods by splitting the fluid motion into mean and fluctuating parts ("Reynolds decomposition"); and identified that nonlinear terms in the Navier-Stokes equations yield additional stresses ("Reynolds stresses" or "turbulent stresses") when the equations are recast for the mean part. A *tour de force* indeed! Reynolds' equations for the mean velocity demonstrated the so-called "closure problem" in turbulence: if one generates from the Navier-Stokes equations an auxiliary equation for a low-order moment such as the mean value, that equation contains higher-order moments, so that, at any level in the hierarchy of moments, there is always one unknown more than the available equations. High-order moments are not related to low-order moments as (for example) in a Gaussian process. Thus, even though the Navier-Stokes equations are themselves closed, some additional assumptions are required to close the set of auxiliary equations at any finite level. This feature has defined the framework for much of the turbulence research that has followed.

C. From Reynolds until the 1960s

1. Closure models

Although the closure problem was apparent in Reynolds' work, its fundamentals seem to have been spelled out first by Keller and Friedmann (1924). They derived the general dynamical equations for two-point velocity moments and showed that the equations for each moment also contain high-order moments. Since there is no apparent small parameter in the problem, there is no rational procedure for closing the system of equations at any finite level. The moment equations have been closed by invoking various statistical hypotheses. The simplest of them is Boussinseq's pedagogical analogy—already mentioned—between gas molecules and turbulent eddies. Taylor (1915, 1932), Prandtl (1925), and von Kármán (1930) postulated various relations between turbulent stresses and the gradient of mean velocity (the so-called mixing length models) and closed the equations. Truncated expansions, cumulant discards, infinite partial summations, etc., have all been attempted (see, e.g., Monin and Yaglom, 1975; Narasimha, 1990). Another interesting idea (Malkus, 1956) is that the mean velocity distribution is maintained in a kind of marginally stable state, the turbulence being self-regulated by the transport it produces.

In a paper less known than it deserves, Kolmogorov (1942) augmented the mean velocity equation by two *differential equations* for turbulent energy and (effectively) the energy dissipation, thus anticipating the so-called two-equation models of turbulence; this is a common practice even today in turbulence modeling (although its development was essentially independent of Kolmogorov's original proposal). Other schemes of varying sophistication and complexity have been developed (see, e.g., Reynolds, 1976; Lesieur, 1990).

2. Similarity arguments

Given that the equations governing turbulence dynamics have been known for so long, the paucity of results that follow from them exactly is astonishing (for an exception under certain conditions, see Kolmogorov 1941a). This situation speaks for the complexity of the equations. Much effort has thus been expended on dimensional and similarity arguments,[3] as well as asymptotics, to arrive at various scaling relations. This type of work continues unabated and with varying degrees of success (see, e.g., Townsend, 1956; Tennekes and Lumley, 1972; Narasimha 1983). For instance, a result from similarity arguments is that the *average* growth of turbulent jets, of the sort shown in Fig. 1, is linear with downstream distance, with the proportionality constant independent of the detailed initial conditions at the jet orifice. Likewise, the energy dissipation on the jet axis away from the orifice depends solely on the ratio U_o^3/D, with the coefficient of proportionality of order unity. Here, D and U_o are the orifice diameter and the velocity at its exit, respectively. These (and similar) scaling results seem to be correct to first order, and so have been used routinely in practice. However, they are working approximations at best: the conditions under which similarity arguments hold are not strictly understood, and the constants of proportionality cannot be extracted from dynamical equations in any case. One should therefore not be too surprised if such relations do not work in every instance (Wygnanski *et al.*, 1986): there is some reason or another to hesitate about the bedrock accuracy of almost every such relation used in the literature. Yet, this should not detract us from appreciating that such results are extremely useful for solving practical problems.

An important relation obtained by asymptotic arguments and supplementary assumptions concerns the distribution of mean velocity in boundary layers, pipes, and flow between parallel plates (e.g., Millikan, 1939). The result is that, in an intermediate region not too close to the surface nor too close to the pipe axis or the boundary layer edge, the mean velocity is proportional to the logarithm of the distance from the wall. This so-called log-law has for a long time enjoyed a preeminent status in turbulence theory (see, however, Sec. III.D). Again, the additive and proportionality constants in the log-law are known only from empirical data.

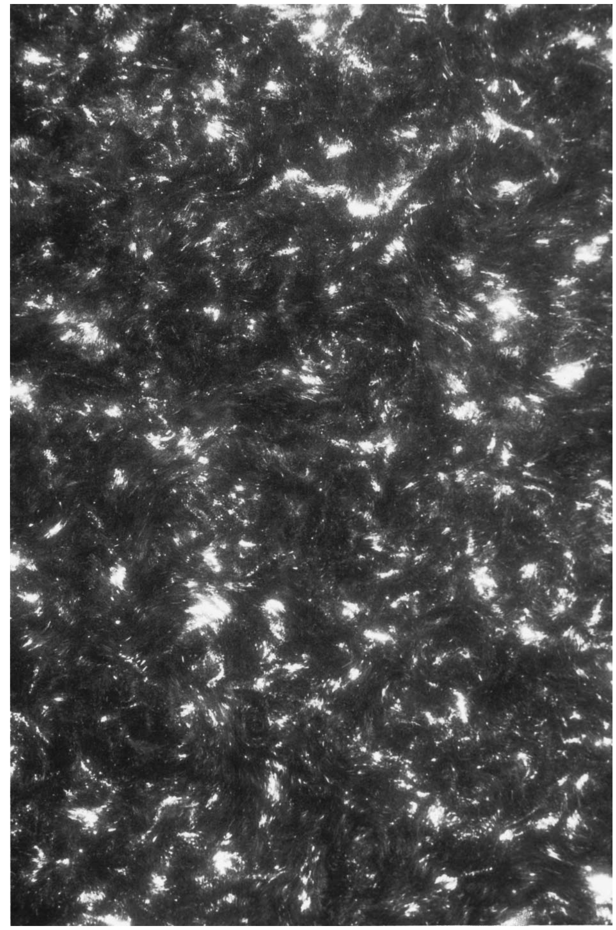

FIG. 2. This picture depicts homogeneous and isotropic turbulence produced by sweeping a grid of bars at a uniform speed through a tank of still water. Unlike the jet turbulence of Fig. 1, turbulence here does not have a preferred direction or orientation. On the average, it does not possess significant spatial inhomogeneities or anisotropies. The strength of the structures, such as they are, is weak in comparison with such structures in Fig. 1. Homogeneous and isotropic turbulence offers considerable theoretical simplifications, and is the object of many studies.

3. Homogeneous and isotropic turbulence

In another important turn of events, a considerable simplification of the general dynamical problem of turbulence was achieved by Taylor (1935) with the introduction of the concept of homogeneous and isotropic turbulence, that is, turbulence that is statistically invariant under translation, rotation and reflection of coordinate axes. Experimentally, nearly homogeneous and isotropic turbulence (see Fig. 2) was developed in the late 1930's using uniform grids of bars in a wind tunnel (e.g., Comte-Bellot and Corrsin 1966). The use of tensors in isotropic turbulence was introduced by von Kármán (1937) who also studied the dynamical consequences of isotropy (von Kármán and Howarth, 1938). Taylor (1938) derived an equation for turbulent vorticity and, almost simultaneously, initiated the use of Fourier transform and spectral representation. Since that time, isotropic turbulence has been the testing ground for most of the analytical theories of turbulence.

4. Local isotropy and universality of small scales: The Kolmogorov turbulence

The reality is that no turbulent flow is homogeneous and isotropic. Further, there are many types of turbulence—depending on boundary conditions, body forces, and other auxiliary parameters: incompressible,

[3]A similarity transformation is an affine transformation that reduces a set of partial differential equations to an ordinary differential equation. For a turbulent jet far away from the orifice, it takes the form that mean velocity distribution preserves its shape when scaled on *local* velocity and length scales. By demanding that the coefficients of the resulting ordinary differential equation be constants, one obtains power-laws for the variation of these velocity and length scales along the jet axis. However, only the power-law exponent can be determined in this way, not prefactors.

compressible, homogeneously sheared, inhomogeneous, stratified, magnetohydrodynamic, superfluid turbulence, and so forth. They are all similar in some respects (e.g., they are highly dissipative), but also different in some respects (e.g., the topology of the large structure is different). This situation is somewhat similar to that in chemistry: while all compounds have the same essential elements, they are also different from each other. It is therefore useful to ask whether "turbulence"—when divorced from a specific context—has a meaningful existence at all.

Enter Kolmogorov (1941b) and his revolutionary postulate that small scales of turbulence are *statistically* isotropic—no matter how the turbulence is produced. This postulate,[4] coupled with Kolmogorov's other hypothesis—known for short in the jargon as K41—has allowed several detailed predictions to be made with regard to the scaling properties of "small-scale" turbulence. The spirit of K41, a major fore-runner for which are Richardson's (1922) qualitative ideas of self-similar distribution of turbulent eddies, is to assume that the "small" scales of turbulence are universal, even though the "large" scales are specific to a given flow—or class of flows with the same boundary conditions. While a full understanding of a turbulent flow requires attention to large as well as small scales (whose mix varies from flow to flow), K41 presupposes that the small scales can be understoood independent of the specifics that determine the large scales. In particular, towards the upper end of the small-scale range (the so-called inertial subrange), K41 shows that the energy spectral density $\phi(\kappa)$ varies with the wave number κ according to $\phi(\kappa)=c_\kappa \varepsilon^{2/3}\kappa^{-5/3}$. Here ε is the rate at which energy is dissipated by the low end of the small scales, and c_κ is an unknown but universal constant. Embedded in K41 is the notion that the large scales—at which the energy is injected—transfer it to the small scales—where it is dissipated—through a series of steps, each of which is dissipationless and involves the interaction of only neighboring scales (instead of all possible triads of wave numbers allowed by the Navier-Stokes equations). The transfer is supposed to occur with ever-increasing rapidity as one approaches increasingly smaller scales. This process of energy transfer, which is at best a good abstraction of a more complex reality, is picturesquely known as energy cascade (Onsager, 1945). Besides Onsager, the other early workers who independently contributed to the understanding of the inertial subrange are von Weizsäcker (1948) and Heisenberg (1948). It is worth stressing that K41 makes no direct connection to the Navier-Stokes equations.

[4]A second important postulate, already mentioned in the specific context of the turbulent jet, is that the rate of energy dissipation at high Reynolds numbers far away from solid boundaries—although mediated by fluid viscosity—is independent of it. Experiments support the postulate on balance, but the evidence leaves much to be desired.

We shall not discuss here Kolmogorov's form for the dissipative scales, but refer to Monin and Yaglom (1975) and Frisch (1995). We shall also say nothing about the consequences of K41 for turbulent diffusion except to note that Richardson's (1926) law for the diffusion of particle pairs can be recovered from its application.

A first-order verification of K41 in a tidal channel at very high Reynolds numbers (Grant *et al.* 1962) is a milestone in the history of turbulence. This rough experimental confirmation and its alluring simplicity have made K41 a staple of turbulence research. However, we shall presently see that K41 is not correct in detail.

5. Experimental tools

Until the late 1920s, the types of turbulence measurements that could be made were limited to time-average properties such as mean velocity and pressures differences. It was not possible to measure fluctuations faithfully because of the demands of spatial and temporal resolution: the spatial resolution required is $O(Re^{-9/4})$ and the temporal resolution $O(Re^{-1/2})$. The technique commonly used for the study of turbulent characteristics was the visualization of flow by injecting a dye or a tracer. This type of work led to valuable insights in the hands of stalwarts such as Prandtl (see Prandtl and Tietjens, 1934). Since the 1950's, which is when thermal anemometry came into being in a robust form, the technique has been the workhorse of turbulence research. Briefly, a fine wire of low thermal capacity is heated to a certain temperature above the ambient, and the change in resistance encountered by it, as a fluid with fluctuating velocity flows around it, is measured. This change is related to the flow velocity through a calibration. Late in the period being considered here, optical techniques such as laser Doppler velocimetry began to make inroads, but hotwires are still the probes of choice in a number of situations.

D. A brief assessment of the classical era

As already mentioned, the statistical principles used for closing the moment equations have enjoyed only transient success. The eddy viscosity and mixing length principles, despite the initial triumphs (see Schlichting, 1956), proved to be flawed (though this has not prevented their use—with varying levels of discernment). Similarly, some closure models (e.g., the so-called quasinormal approximation) often violate the *realizability* condition, namely the positivity of probabilities (or other related results that follow). Kraichnan (1959, and later) has emphasized the need for dealing with this issue directly, and devised models that ensure realizability. These models have certain consistency properties that conventional closure schemes may not. Realizability constraints have now become a standard test in turbulence modeling (Speziale, 1991), especially in modern computing efforts. Further, the general scaling results such as for the overall growth of turbulent flows and energy dissipation (see Sec. III.C.2) seem to drive rough experimental suuport, but reveal many open problems upon close scrutiny. Even the log-law, long regarded as a crowning achievement in turbulence, has been questioned vigorously in recent years (Barenblatt *et al.*, 1997). The issue on hand is not simply whether the log-law or an alternative power-law fits the data better. At stake is the validity of the underlying principles of similarity that each argument employs.

The lack of successful closure models on the one hand, and the apparent success of K41 in describing low-order statistics of the small-scale on the other, have led to an excessive tendency to regard turbulence as a single, unified phenomenon. This development has not always been healthy.

One cannot escape the feeling that much of the work has a tentative character to it. This is not the norm in mechanics or other branches of classical physics.

IV. THE MODERN ERA

A. Large-scale coherent structures

Even casual observations of turbulent flows reveal well-organized motions on scales comparable to the flow width (see the splendid collection of pictures by Van Dyke, 1982); indeed, experimentally measured correlation functions had occasionally pointed to the existence of organized large scales (Liepmann, 1952; Favre *et al.*, 1962). Yet, this aspect was not the central theme of turbulence research in the classical era. On hindsight, many aspects contributed to this neglect: the realization that statistical description was inevitable, preoccupation with isotropic turbulence where the spatial organization is minimal, the absence of historical precedents of physical systems in which order and chaos coexist, and so forth. The important role of large-scale organized motions for transport processes has since been emphasized (Kline *et al.*, 1967; Brown and Roshko, 1974; Head and Bandyopadhyay, 1981), leading to a resurgence of interest in them.

It is a nontrivial matter that the large scales can maintain their coherence in the presence of a superimposed incoherent activity. The origin of the large structure has often been sought in terms of the instability of the (hypothetical) mean velocity distribution, or something even simpler, but there are conspicuous gaps in the arguments employed. It is worth recalling that complicated, nonlinear, systems with many degrees of freedom do sometimes develop organized structures such as solitons (Zabusky and Kruskal, 1965). If solitons have anything to do with coherent structures in turbulence, that connection remains obscure.

Taking for granted the importance of the large scales, the question is how to identify them objectively. An experimentally useful tool is the so-called conditional averaging (e.g., Kovasznay *et al.*, 1970), in which one averages over preselected members of an ensemble. Suitable wavelets have sometime been used as templates for the large scale. The difficult question is how to describe them analytically and construct usefully approximate dynamical systems, preferably of low dimensions. This is not a simple task, but some success has been attained in special cases via the so-called Karhunen-Lóeve procedure (e.g., Sirovich, 1987; Holmes *et al.*, 1998).

A hope in the work on coherent structures has been that they could lead to efficient methods for predicting overall features of turbulent flows. The verdict on this effort is still unclear (e.g., Hussain, 1983). Another quest has been to control, or manage, turbulent flows via large-scale coherent structures. The verdict on this line of inquiry is mixed (e.g., Gad-el-Hak *et al.*, 1998).

B. Small-scale turbulence: Repercussions of Kolmogorov's "refinement"

It has been hinted already that Kolmogorov's arguments of local isotropy and small-scale universality have pervaded all aspects of turbulence research (e.g., Monin and Yaglom, 1975; Frisch, 1995). Deeper exploration has revealed that strong departures from the K41 universality exist, and that they are due to less benign interactions between large and small scales than was visualized in K41. Following a remark of Landau (see Frisch, 1995), Kolmogorov (1962) himself provided a "refinement" of his earlier hypotheses. In reality, this refinement is a vital revision (Kraichnan, 1974), and its repercussions are being felt even today (e.g., Chorin, 1994; Stolovitzky and Sreenivasan, 1994). One of its manifestations is that the various scaling exponents characterizing small-scale statistics are anomalous (that is, the exponent for each order of the moment has to be determined individually in a nontrivial manner, and cannot be guessed from dimensional arguments). Although the anomaly is still *essentially* an empirical fact, and its existence has yet to be established beyond blemish due to various experimental ambiguities,[5] it seems unlikely that we will return to K41 universality. Even the nature of anomaly seems to depend on the particular class of flows. However, these subtle differences might arise from finite Reynolds number effects, large-scale anisotropies, and so forth; without quantitative ability to calculate these effects, one will always have lingering doubts about the true nature of anomaly and of scaling itself (e.g., Barenblatt

[5]There are several of them. First, measured time traces of turbulent quantities are interpreted as spatial cuts by assuming that turbulence gets convected by the mean velocity without distortion. This is the so-called Taylor's hypothesis. Second, one cannot often measure the quantity of theoretical interest in its entirety, but only a part of it. The practice of replacing one quantity by a similar one is called surrogacy. Surrogacy is often a necessary evil in turbulence work, and makes the interpretation of measurements ambiguous (e.g., Chen *et al.*, 1993). Finally, the scaling region depends on some power of the Reynolds number, and also on the nature of large-scale forcing. The scaling range available in most accessible flows—especially in numerical simulations where the first two issues are not relevant—is small because the Reynolds numbers are not large enough.

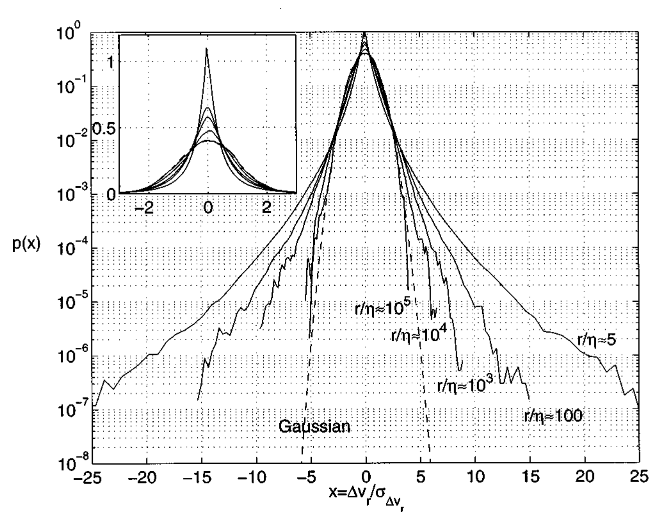

FIG. 3. The probability density functions, of differences of velocity fluctuations, obtained in atmospheric turbulence about 30 m above the ground. The ordinate is logarithmic in the main figure and linear in the inset. Each curve is for a different separation distance (using Taylor's hypothesis). The separation distance is transverse to the direction of the velocity component. The smallest separation distance (about 2.5 mm) is only five times the Kolmogorov scale η, denoting the smallest scale of fluctuations, while the largest (about 50 m) is comparable to the height of the measurement point. For small separation distances, very large excursions (even as large as 25 standard deviations) occur with nontrivial frequency; they are far more frequent than is given by a Gaussian distribution (shown by the full line), which is approached only for large separation distances. Extended tails over a wide range of scales is related to the phenomenon of small-scale intermittency (that is, uneven distribution in space of the small scales). These probability density functions are nonskewed. If the separation distance is in the direction of the velocity component measured, the probability density functions possess a definite skewness, as shown by Kolmogorov (1941a). This skewness is related to the energy transfer from large to small scales. In contrast to velocity increments, velocity fluctuations themselves have a nearly Gaussian character at this height above the ground. The shape of the probability density function depends on the flow and the spatial position in an inhomogeneous flow. For isotropic and homogeneous turbulence, it is marginally sub-Gaussian for high fluctuation amplitudes.

and Goldenfeld, 1995). These issues are being constantly investigated with increasing precision (e.g., Anselmet *et al.*, 1983; Benzi *et al.*, 1993; L'vov and Procaccia, 1995; Arneodo *et al.*, 1996; Cao *et al.*, 1996; Tabeling *et al.*, 1996; Sreenivasan and Dhruva, 1998).

The anomaly of scaling exponents is related to small-scale intermittency. Roughly speaking, intermittency means that extreme events are far more probable than can be expected from Gaussian statistics and that the probability density functions of increasingly smaller scales are increasingly non-Gaussian (Fig. 3). This is a statistical consequence of uneven spatial distribution of the small-scale (Fig. 4), and can be modeled by multifractals (Mandelbrot, 1974; Parisi and Frisch, 1985; Meneveau and Sreenivasan, 1991). Most nonlinear systems are intermittent in time, space, or both, and the study of intermittency in turbulence is useful in a broad range of circumstances (e.g., Halsey *et al.*, 1986).

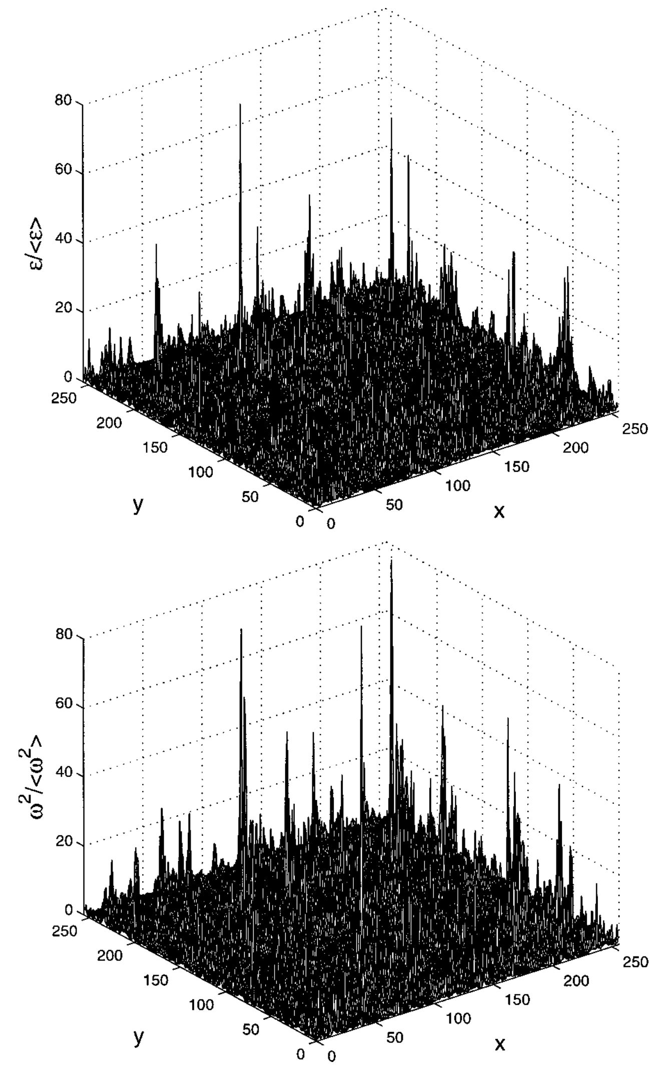

FIG. 4. Planar cuts of the three-dimensional fields of (a) energy dissipation and (b) squared vorticity in a box of homogeneous and isotropic turbulence. The data are obtained by solving the Navier-Stokes equations on a computer. Not uncommon are amplitudes much larger than the mean; these large events become stronger with increasing Reynolds number. Such quantities are not governed by the central limit theorem. The statistics of large deviations are relevant here, as in many other broad contexts of modern interest. Kolmogorov (1962) proposed log-normal distribution to model energy dissipation (and, by inference, squared vorticity), but there seems to be a general agreement that lognormality is in principle incorrect (e.g., Mandelbrot, 1974; Narasimha, 1990; Novikov, 1990; Frisch 1995). Both these quantities have been modeled successfully by multifractals (Meneveau and Sreenivasan, 1991). A promising alternative is the log-Poisson model (She and Leveque, 1994).

Taken together, a major thrust of theoretical efforts has been the understanding of intermittency, multifractality, and the anomaly of scaling exponents. Many pedagogically illuminating models have been invented (see Sreenivasan and Antonia, 1997, for a summary), and a few rigorous inequalities are known (e.g., Constantin and Fefferman, 1994).

C. Some recent efforts

1. Theoretical issues

As a guide to further discussion, it is helpful to recall the mathematical problems associated with the Navier-Stokes equations. First, as already emphasized, there is no obvious small parameter on which to base a systematic perturbation theory. Second, the equations are nonlinear. The effects include energy redistribution among the constituent scales, as well as the so-called sweeping effect, which represents the manner in which the small scales are swept by the large. Third, the equations are dissipative even when the fluid viscosity is infinitesimally small ($Re \to \infty$). Fourth, there are dominant nonlocal effects arising from pressure.

The desire to understand qualitative aspects of each of these effects has led to different approaches. For instance, the inadequacy of perturbation methods have led to the exploration of nonperturbative alternatives. (An incomplete list of references in this regard, not necessarily alike in philosophy or detail, are Kraichnan, 1959; Martin *et al.*, 1973; Forster *et al.*, 1977; Yakhot and Orszag, 1986; McComb, 1990; Avellaneda and Majda, 1994; Eyink, 1994; Mou and Weichman, 1995; L'vov and Procaccia, 1996.) To understand nonlinear effects in forced systems, researchers have explored various alternatives such as Burgers equation with stochastic forcing (e.g., Cheklov and Yakhot, 1995; Polyakov, 1995), and shell models (e.g., Jensen *et al.*, 1992) or their variants (Grossmann and Lohse, 1994).[6] Some attention has been paid to possible depletion of nonlinearity in parts of the real space (e.g., Frisch and Orszag, 1990). For passive scalars, the anomaly of scaling exponents is being explored via the rapidly-varying-velocity model for passive scalars (e.g., Kraichnan, 1994; Frisch *et al.*, 1998). The interest in the small viscosity limit in the problem has led to serious studies of the singularities of the governing equations (Caferelli *et al.*, 1982), especially of the inviscid counterpart—namely, the Euler equations (see, e.g., Beale *et al.*, 1989). The multifractal analysis of dissipation fits in this broad picture. There is substantial interest in the physics of vortex dynamics (e.g., Saffman, 1992), particularly vortex reconnections (e.g., Kida and Takaoka, 1994). It is not always clear how centrally these studies bear on developed turbulence.

2. Advanced experimental methods

Traditional turbulence measurements are made at a single spatial position or at a few positions as functions of time, and yield time traces of velocity, temperature, or other quantities. These are treated as spatial cuts through the flow by invoking Taylor's hypothesis, whose limitations are not fully understood (Lumley, 1965). A major accomplishment in recent years is the direct measurement of spatio-temporal fields of turbulence, obviating the need for this plausible but uncertain assumption. The techniques are typically the laser-induced fluorescence for passive scalars (e.g., Dahm *et al.*, 1991) and particle image velocimetry for flow velocity (e.g., Adrian, 1991). Unfortunately, available technology restricts true spatio-temporal measurements to low Reynolds numbers.

An experimental goal is to produce high Reynolds number turbulence and measure all the desired properties with adequate resolution in space and time. To obtain high Re, one may use the high speeds of fluid (but one is then limited by compressibility effects for gases and cavitation problems for liquids), a large-scale apparatus (which is limited by cost and available space), or use fluids of low viscosity (such as air at very high pressures or cryogenic fluids such as He I). For He I, the exquisite control on viscosity allows one to obtain, in an apparatus of a fixed size, a large *range* of Reynolds numbers than is possible by varying flow speed alone. This advantage has been exploited adroitly in a few instances (e.g., Castaing *et al.*, 1989; Tabeling *et al.*, 1996). In these instances, one has been forced to limit oneself to single-point data; the challenge is to develop instrumentation for obtaining spatial data, especially resolving small scales (for an account of some progress, see, e.g., Donnelly, 1991).[7]

3. Computational efforts

Another major advance is the use of powerful computers to solve Navier-Stokes equations exactly to produce turbulent solutions (e.g., Chorin, 1967; Orszag and Patterson, 1972). These are called direct numerical simulations (DNS). The DNS data are in some respects superior to experimental data because one can study experimentally inaccessible quantities such as tensorial invariants or pressure fluctuations at an interior point in the flow. The DNS data have allowed us to visualize details of small-scale vorticity and other similar features. For instance, they show that intense vorticity is often concentrated in tubes[8] (She *et al.*, 1990; Jimenez *et al.*, 1993); see Fig. 5. Yet, available computer memory and

[6]Burger's equation is the one-dimensional version of the Navier-Stokes equation, but without the pressure term; it possesses no chaotic solutions without forcing. Shell models are severe truncations of the Navier-Stokes equations, retaining only a few representative Fourier modes in any wave-number band. Only nearest, or the next nearest, couplings are allowed. The models retain several symmetry properties of the Navier-Stokes equations.

[7]For a fixed Re, a far smaller apparatus suffices when He is used instead of say, air, which makes the smallest scale that much smaller: recall that the ratio of the smallest scale to the flow apparatus is $O(Re^{-3/4})$.

[8]Experimental demonstration that vortex tubes can often be as long as the large-scale of turbulence can be found in Bonn *et al.* (1993).

FIG. 5. (Color) Demonstration that vorticity at large amplitudes, say greater than 3 standard deviations, organizes itself in the form of tubes (shown in yellow), even though the turbulence is globally homogeneous and isotropic. Large-amplitude dissipation (shown in red) is not as organized, and seems to surround regions of high vorticity. Smaller amplitudes do not possess such structure even for vorticity. In principle, the multifractal description of the spiky signals of Fig. 4 is capable of discerning geometric structures such as sheets and tubes, but no particular shape plays a central role in that description. The dynamical reason for this organization of large-amplitude vorticity is unclear. The ubiquitous presence of vortex tubes raises a number of interesting questions, some of which are mentioned in the text. At present, elementary properties of these tubes, such as their mean length and scaling of their thickness with Reynolds number, have not been quantified satisfactorily; nor has their dynamical significance.

speed limit calculations to Reynolds numbers of the order of a few thousand. This limit is at present slightly better than the experimental range (see previous subsection).

It thus becomes necessary to adopt different strategies for computing high-Reynolds-number flows (e.g., Leonard, 1985; Lesieur and Metais, 1996; Moin, 1996; Moin and Kim, 1997). A fruitful avenue is the so-called Large Eddy Simulation method, in which one resolves what is possible, and suitably models the unresolved part. The modeling schemes vary in nature from an *a priori* prescription of the properties of the unresolved scales to computing their effect as part of the calculation scheme itself; the latter makes use of the known scaling properties of small-scale motion such as the locality of wavenumber interaction or spectral-scale similarity. As a computational tool for *practical* applications, the Large Eddy Simulation method has much promise. Increasing its versatility and adaptability near a solid surface is a major area of current research.

We shall not remark here at length on engineering models of turbulence. They range from modified mixing length theories to those based on supplementary differential equations (e.g., Reynolds, 1976; Lumley, 1990; Lesieur, 1990) to the adaptation of the renormalization group methods (e.g., Yakhot and Orszag, 1986). These models cleverly exploit symmetries, conservation properties, realizability constraints, and other general principles to make headway in practical problem solving. Their short-term importance cannot be exaggerated.

V. PROSPECTS FOR THE NEAR FUTURE

It is useful to reiterate that turbulence research spans a wide spectrum from practical applications to fundamental physics. At one end of this spectrum are problems such as the prediction of fluctuating pressure field on the skin of an aircraft wing, or the hydrodynamic noise emitted by a submarine. Interactions with complexities such as combustion, rotation, and stratification pose a plethora of further questions: for instance, what is the amount of heat transported by the outer convective motion in the Sun? These problems involve nonlinearly complex interactions of the many parts of which they are composed, and so will necessarily remain too specific to expect general solutions. A sensible goal in such instances will always be to obtain reliable working approximations.

At the other end of the spectrum are deep physics issues arising from the nonperturbative nature of the turbulence problem. How may one understand precisely this many-scale problem with strong coupling among its constituent scales? It is natural to seek clues to this question in the analytic structure of the Navier-Stokes equations, but this task has so far proved hopelessly difficult. Therefore, one often seeks guidance via simpler problems of the same class, even if some essential elements are lost along the way.

Between the two ends is a wide middle, consisting of a study of carefully chosen idealized configurations. Typical problems follow: How much mixing occurs between two parallel streams in a well-conditioned flow apparatus? What is the net force exerted on a flat plate parallel to a smooth stream? What is the best way to parametrize the flow near smooth boundaries where viscosity affects all scales of turbulence? Such problems are approached by several complementary methods, but their broad content is the splitting of the overall motion into large and small scales—the former may well be the mean motion—and mastering the latter by combining phenomenology with aspects of universality. A sensible goal here is to put this practice on firmer physical principles.

Since these physical principles are still unclear, the task has an iterative character to it; thus, each generation of students of the subject has lived through them in different forms and made incremental progress. Progress has demanded that this grand problem (often hailed as the last such problem in classical physics) be split into various sub-problems—some closer to basic physics and some to working practice. Some in either variety may ultimately prove inessential to the overall purpose, but there can be no room for impatience or prejudice.

Listing all useful sub-problems without trivializing them is itself a challenge. We will unfortunately not rise to the occasion here, but list a few illustrative ones—making no effort to describe the progress being made. With respect to small scales, one interesting question is the dynamical importance of the highly anisotropic vortex tubes, and whether their existence is consistent with the universal (albeit anomalous) scaling presumed to exist in high-Reynolds-number turbulence (Moffatt, 1994; Moffatt *et al.*, 1994): What is the connection between scaling (which emphasizes the sameness at various scales) and structure (which becomes better defined and topologically more anisotropic at larger fluctuation amplitudes)? In some problems of condensed matter physics—for example, anisotropic ferromagnets near the critical point—the critical indices are oblivious to the magnitude of anisotropy. However, this is not always the case. As Mandelbrot (1982) has emphasized in several contexts, this type of question necessarily forces the marriage of geometry with analysis (for some progress, see, Constantin, 1994); a particular case of this bigger picture is the stochastic geometry of turbulent/nonturbulent interfaces and of isoscalar surfaces (e.g., Constantin *et al.*, 1991). A second question is the understanding of the effect of finite Reynolds number and of finite shear and anisotropy, comparable in scope, say, to that of finite-size effects in typical scaling problems in critical phenomena. This is a crucial undertaking for all issues related to scaling. Third, shifting focus from scaling exponents to scaling functions, and from the tails of probability density functions to the entire distribution, would be a useful relief. Fourth, a study of objects more complex than two-point structure functions would be highly informative (e.g., L'vov and Procaccia, 1996; Chertkov *et al.*, 1998). Fifth, while the overall flux of energy from the large to the small scale is unidirectional on the average, the instantaneous flux is in both directions; is the overall average flux a small difference between the forward and reverse fluxes, or only a small fraction of the average? The answer to this question changes our perception of the degree of non-equilibrium present in the energy cascade, and influences the development of sound Large Eddy Simulation models. Sixth, one may usefully focus attention on other problems where violations of the K41 universality are first-order in importance—e.g., the problem of passive admixtures (Sreenivasan, 1991; Shraiman and Siggia, 1995), of pressure, and of acceleration statistics (e.g., Nelkin, 1994). As far as the large structures are concerned, the outstanding question is the determination of their origin, topology, frequency, and relation to small scales (e.g., Roshko, 1976; Hussain, 1983). Finally, an overarching issue is the abstraction of the small-scale influence on the small scales.

Some degree of progress has occurred on all these fronts, and has accelerated in recent years. Much of it is due to a powerful combination of experimental methods, computer simulations, and analytical advances in neighboring fields. Our hope lies in this synergism, whose importance cannot be exaggerated. It is trite but true to say that advancing experimental methods will imporve our understanding of turbulence significantly. (Recall the motto of Kamerlingh Onnes, the father of low temperature physics: "through measurement to knowledge".) In this regard, the key lies in measurements at high Reynolds numbers. How high a Reynolds number is "high enough" depends on the context and purpose. Yet, without a proper knowledge of Reynolds-number-scaling, one can be lured into false certainty by focusing exclusively on low Reynolds numbers. Presently, one obtains high-Reynolds-number small-scale data either in atmospheric flows or specialized facilities. Among the latter are facilities meant for testing large-scale aeronautical and navy vehicles, or those that use helium (e.g., Castaing *et al.*, 1989; Tabeling *et al.*, 1996), or use compressed air at very high pressures (Zagarola and Smits, 1996). Atmospheric flows are not controlled and stationary over long intervals of time, and only a few probes can be used at a given time. Among the specialized facilities, the large ones are very expensive to operate and, to a first approximation, unavailable for basic research. The smaller specialized flows allow, because of instrumentation limitations, only a small number of quantities to be measured with limited resolution. These shortcomings have been alleviated to some degree by computer simulation of the equations of motion, and a great deal can indeed be learned by combining such simulations at moderate Reynolds numbers with experiments at high Reynolds numbers. It is clear that the next generation of simulations, now already in progress, will produce data at high enough Reynolds numbers to begin to close the existing gap.

VI. CONCLUDING REMARKS

From Osborne Reynolds at the turn of the last century to the present day, much qualitative understanding has been acquired about various aspects of turbulence. This progress has been undoubtedly useful in practice, despite large gaps that exist in our understanding. As a problem in physics or mechanics—contrasted, for example, against the rigor with which potential theory is understood—the problem is still in its infancy.

It has already been remarked that viewing turbulence as one grand problem may be debilitating. The large and diverse clientele it enjoys—such as astrophysicists, atmospheric physicists, aeronautical, mechanical, and chemical engineers—has different needs and approaches the problem with correspondingly different emphases. This makes it difficult to mount a focused frontal attack on a single aspect of the problem. It is therefore intriguing to ask: how may one recognize that the "turbulence problem" has been solved? It would be a great advance for an engineer to determine from fluid equations the pressure needed to push a certain volume of fluid through a circular tube. Even if this particular problem, or another like it, were to be solved, might it be deemed too special unless the effort paved the way for attacking similar problems?

There are two possible scenarios. Our computing abilities may improve so much that any conceivable turbulent problem can be "computed away" with adequate accuracy, so the problem disappears in the face of this formidable weaponry. One may still fret that computing is not understanding, but the issue assumes a more benign complexion. The other scenario—which is common in physics—is that a particular special problem that is sufficiently realistic and close enough to turbulence, will be solved in detail and understood fully. After all, no one can compute the detailed structure of the nitrogen atom from quantum mechanics, yet there is full confidence in the fundamentals of that subject. Unfortunately, the appropriate "hydrogen atom" or the "Ising model" for turbulence remains elusive.

In summary, there is a well-developed body of knowledge in turbulence that is generally self-consistent and useful for problem solving. However, there are lingering uncertainties at almost all levels. Extrapolating from experience so far, future progress will take a zigzag path, and further order will be slow to emerge. What is clear is that progress will depend on controlled measurements and computer simulations at high Reynolds numbers, and the ability to see in them the answers to the right theoretical questions. There is ground for optimism, and a meaningful interaction among theory, experiment, and computations must be able to take us far. It is a matter of time and persistence.

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