Fusion Rules in Navier-Stokes Turbulence: First Experimental Tests

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We present the first experimental tests of the recently derived fusion rules for Navier-Stokes turbulence. The fusion rules address the asymptotic properties of many-point correlation functions as some of the coordinates coalesce, and form an important ingredient of the nonperturbative statistical theory of turbulence. Here we test the fusion rules when the spatial separations lie within the inertial range, and find good agreement between experiment and theory. For inertial-range separations and for velocity increments which are not too large, a simple linear relation appears to exist for the Laplacian of the velocity fluctuation conditioned on velocity increments. [S0031-9007(97)04425-6]

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In this Letter the predictions of the recently proposed fusion rules [1] are tested by analyzing a turbulent velocity signal at high Reynolds number. We start with a short theoretical summary.

The theory focuses on two-point velocity differences

$$\boldsymbol{w}(\boldsymbol{x},\boldsymbol{x}',t) \equiv \boldsymbol{u}(\boldsymbol{x}',t) - \boldsymbol{u}(\boldsymbol{x},t)\,, \qquad (1)$$

where $\boldsymbol{u}(\boldsymbol{x}',t)$ is the Eulerian velocity field accessible to experiment. One attempts to extract predictable and computable results by considering the statistical properties of $\boldsymbol{w}$ [2,3]. The most informative statistical quantities are the equal-time rank-n tensor correlation functions of velocity differences

$$\mathcal{F}_n(\boldsymbol{x}_1,\boldsymbol{x}_1';\boldsymbol{x}_2,\boldsymbol{x}_2';\ldots;\boldsymbol{x}_n,\boldsymbol{x}_n') = \langle \boldsymbol{w}(\boldsymbol{x}_1,\boldsymbol{x}_1')\boldsymbol{w}(\boldsymbol{x}_2,\boldsymbol{x}_2') \cdots \boldsymbol{w}(\boldsymbol{x}_n,\boldsymbol{x}_n')\rangle\,, \qquad (2)$$

where $\langle\cdot\rangle$ denotes averaging, and all coordinates are distinct. We consider Navier-Stokes (N-S) turbulence for which the scaling exponents are presumed to be universal (i.e., they do not depend on the detailed form of forcing), and where the correlations $\mathcal{F}_n$ are homogeneous functions [2], namely

$$\mathcal{F}_n(\lambda\boldsymbol{x}_1,\lambda\boldsymbol{x}_1',\ldots,\lambda\boldsymbol{x}_n') = \lambda^{\zeta_n}\mathcal{F}_n(\boldsymbol{x}_1,\boldsymbol{x}_1',\ldots,\boldsymbol{x}_n')\,, \qquad (3)$$

ζ_n being the homogeneity (or scaling) exponent. This form applies when all distances $|\boldsymbol{x}_i - \boldsymbol{x}_i'|$ are in the "inertial range:" between the outer scale L and the dissipative scale η of the system. Our aim here is to describe the behavior of such functions as pairs of coordinates approach one another, or "fuse." The fusion rules, derived in [1,4], govern the analytical structure of the correlation functions under this coalescence.

The statistical function that has been most commonly studied [2,3,5,6] is the structure function $S_n(R)$

$$S_n(R) = \langle|\boldsymbol{w}(\boldsymbol{x},\boldsymbol{x}')|^n\rangle, \qquad \boldsymbol{R} \equiv \boldsymbol{x}' - \boldsymbol{x}\,. \qquad (4)$$

Clearly, the structure function is obtained from (2) by the fusing of all coordinates $\boldsymbol{x}_i$ into one point $\boldsymbol{x}$, and all coordinates $\boldsymbol{x}_i'$ into $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{R}$. In doing so, one crosses the viscous dissipation length scale. One then expects a change of behavior, reflecting the role of the viscosity in the theory for $S_n(R)$. In developing a N-S based theory in terms of $S_n(R)$, one encounters the notorious closure problem: one must balance terms arising from the convective term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ and the dissipative $\nu\nabla^2\boldsymbol{u}$ term, neither of which can be neglected. Hence determining $S_n(R)$ requires information about $S_{n+1}(R)$. All known closures of this hierarchy of equations are arbitrary. However, according to the theory of Refs. [1,4], the fully unfused $\mathcal{F}_n$ does not suffer from this problem. When all separations are in the inertial range, the viscous term may be neglected, and one obtains [4] homogeneous equations for $\mathcal{F}_n$ in terms of $\mathcal{F}_n$ only. Such homogeneous equations may exhibit new, anomalous scaling solutions for the correlation functions $\mathcal{F}_n$.

There are various possible configurations of coalescence. We will test only those in which the coalescing points are those of velocity differences. One can also consider the coalescence of points from different velocity differences, but they are experimentally more difficult to measure; we will "precoalesce" all such points here and comment on the effect of this [4]. The first set of fusion rules that we examine concerns $\mathcal{F}_n$ when p pairs of coordinates $\boldsymbol{x}_1,\boldsymbol{x}_1',\ldots,\boldsymbol{x}_p,\boldsymbol{x}_p'$, $(p < n)$ of p velocity differences coalesce, with typical separations between coordinates $|\boldsymbol{x}_i - \boldsymbol{x}_i'| \sim r$ for $i \leq p$, and all other separations of the order of R, $r \ll R \ll L$. The fusion rules predict

$$\mathcal{F}_n(\boldsymbol{x}_1,\boldsymbol{x}_1';\ldots;\boldsymbol{x}_n,\boldsymbol{x}_n') = \tilde{\mathcal{F}}_p(\boldsymbol{x}_1,\boldsymbol{x}_1';\ldots;\boldsymbol{x}_p,\boldsymbol{x}_p') \times \boldsymbol{\Psi}_{n,p}(\boldsymbol{x}_{p+1},\boldsymbol{x}_{p+1}';\ldots;\boldsymbol{x}_n,\boldsymbol{x}_n')\,, \qquad (5)$$

where $\tilde{\mathcal{F}}_p$ is a tensor of rank p associated with the first p tensor indices of $\mathcal{F}_n$, and it has a homogeneity exponent ζ_p. The $(n-p)$-rank tensor $\boldsymbol{\Psi}_{n,p}(\boldsymbol{x}_{p+1},\boldsymbol{x}_{p+1}';\ldots;\boldsymbol{x}_n,\mathbf{r}_n')$ is a homogeneous function with a scaling exponent $\zeta_n - \zeta_p$, and is associated with the other $n-p$ indices of $\mathcal{F}_n$. In the special case $p = 1$, the leading order evaluation cancels by symmetry. The next-order result, for a randomly oriented set of pairs

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with separation R, is

$$\mathcal{F}_n(\boldsymbol{x}_1,\boldsymbol{x}_1';\ldots;\boldsymbol{x}_n,\boldsymbol{x}_n') \sim (r/R)S_n(R)\,. \quad (6)$$

One can also consider correlation functions under the operation of gradients. One such set of functions, denoted $J_n(R)$, has particular significance. It arises from the dissipative term when one obtains a statistical balance from the N-S equations. For comparison with experiments we define $J_n(R)$ in terms of the longitudinal velocity difference $\delta u_R \equiv \boldsymbol{w}(\boldsymbol{x},\boldsymbol{x}+\boldsymbol{R})\cdot\boldsymbol{R}/R$ as

$$J_n(R) = \langle\tilde{\nabla}^2\boldsymbol{u}(\boldsymbol{x})\,[\delta u_R]^{n-1}\rangle\,. \quad (7)$$

The Laplacian operator in (7) is interpreted as a finite difference of longitudinal components of the velocity

$$\hat{\nabla}^2\boldsymbol{u}(\boldsymbol{x}) = [\boldsymbol{w}(\boldsymbol{x},\boldsymbol{x}+\boldsymbol{\rho}) - \boldsymbol{w}(\boldsymbol{x},\boldsymbol{x}-\boldsymbol{\rho})]\cdot\boldsymbol{\rho}/\rho^3. \quad (8)$$

The predictions of the fusion rules are different for ρ above and below the scale η. This is due to the fact that the scale at which correlation functions reach the dissipative regime depends on the order n of the correlation function, which affects the scaling of gradient correlations. This delicate point is addressed in [7].

The predictions read:

$$J_n(R) = nC_nJ_2S_n(R)/2S_2(R), \qquad \rho \gg \eta\,, \quad (9)$$

$$J_n(R) = n\tilde{C}_nJ_2S_{n+1}(R)/S_3(R), \qquad \rho \ll \eta\,, \quad (10)$$

where C_n and $\tilde{C}_n$ are R-independent dimensionless constants. The fusion rules do not rule out an n dependence of these coefficients. J_2 is equal to the mean dissipation $\langle|\nabla u(x)|^2\rangle$, thus is expected to be R independent; its dependence on ρ is easily estimated as $J_2(\rho) \sim S_2(\rho)/\rho^2$. All higher order J_ns depend on ρ in the same way.

Here we test these predictions using atmospheric turbulence data obtained using a single hot-wire probe mounted at a height of 35 m on the meteorological tower at the Brookhaven National Laboratory. The hot-wire was about 0.7 mm in length and 6 μm in diameter. It was operated on a DISA 55M01 anemometer in constant temperature mode. The wind direction, measured independently by a vane anemometer, was approximately constant. The frequency response of the hot-wire was good up to 20 kHz. The voltage from the anemometer was low-pass filtered at 2 kHz and sampled at 5 kHz, and later converted to velocity through an *in situ* calibration. The mean wind speed was 7.6 ms^{-1} and the root-mean-square (rms) velocity was 1.3 ms^{-1}. Surrogating time for space (by "Taylor's hypothesis"), we obtain the Taylor microscale Reynolds number to be 9540 and the Kolmogorov microscale η to be 0.57 mm.

In Fig. 1 we present the structure functions $S_n(R)$ as a function of R. They were computed using up to 40 million data samples. Here and in all figures, spatial separations have units of sampling times, and the velocity is normalized by the rms velocity. In these units, η is less than 1 sampling time. This figure shows that we have almost three decades of inertial range (between, say, 10 and 10^4 sampling units). The scales below 10 units appear to suffer from an unexplained bump. For now, therefore, we have not considered velocity increments with smaller separation units and cannot test the theory for subdissipation scales. Structure functions of orders $n > 8$ are less reliable and will not be considered.

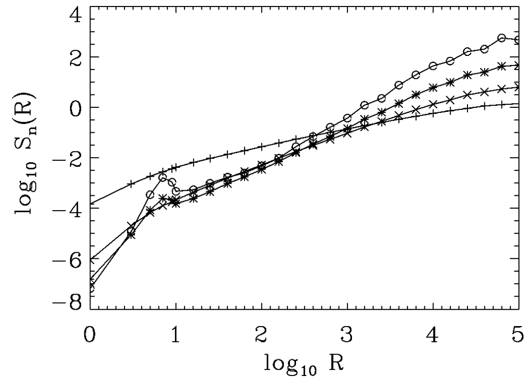

FIG. 1. Log-log plot of the structure functions $S_n(R)$ as a function of R for $n = 2, 4, 6, 8$ denoted by $+$, $\times$, $*$, and $\circ$, respectively.

While the fusion rules are formulated for differences in d-dimensional space, the surrogated data represent a one-dimensional cut. This has implications for the choice of positioning of the coordinates. In d-dimensional space we can choose separations to fall within balls of size R and r, respectively. In our case this ball collapses onto a line, and best results are obtained when the pairs of coordinates in the two groups coincide. As a simple demonstration one may calculate explicitly the second-order quantity $\mathcal{F}_2(\boldsymbol{x}_1,\boldsymbol{x}_1',\boldsymbol{x}_2,\boldsymbol{x}_2') = \langle(x_1 - x_1')(x_2 - x_2')\rangle$ with the two pairs of coordinates chosen, respectively, to coincide; to be displaced by r; and to be displaced by $2r$. That is, we take (i) $x_1 = x_1' = x$, $x_2 = x_2' = y$, $r = |y - x|$, (ii) $|x_1 - x_1'| = |x_2 - x_2'| = r$, $x_1' = x_2$, and (iii) $|x_1 - x_1'| = |x_2 - x_2'| = r$, where x_1' and x_2 are also separated by r. Computing the correlation functions one finds that all have the same power-law dependence on r but with a reduction factor in cases (ii) and (iii) with respect to case (i) of $2^{\zeta_2-1} - 1 \approx -0.2$ and $(-2^{\zeta_2+1} + 1 + 3^{\zeta_2})/2 \approx -0.05$. We thus see that there is a rapid decrease in amplitude when the distances are not overlapping, and so all averaging is done using maximally overlapped configurations [i.e., case (i)]. As remarked above, in doing so, all the "unprimed" points *not across velocity differences* are already fused. This procedure does not affect predictions (5) and (6); see [4].

Explicitly, therefore, we examine the behavior of the correlation function

$$\mathcal{F}_{p+q}(r,R) \equiv \langle[u(x+r) - u(x)]^p[u(x+R) - u(x)]^q\rangle \quad (11)$$

as a function of both r and R for several values of the powers p and q. From Eq. (5) one expects

$$\mathcal{F}_{p+q}(r,R) \sim S_p(r)S_{q+p}(R)/S_p(R)\,. \quad (12)$$

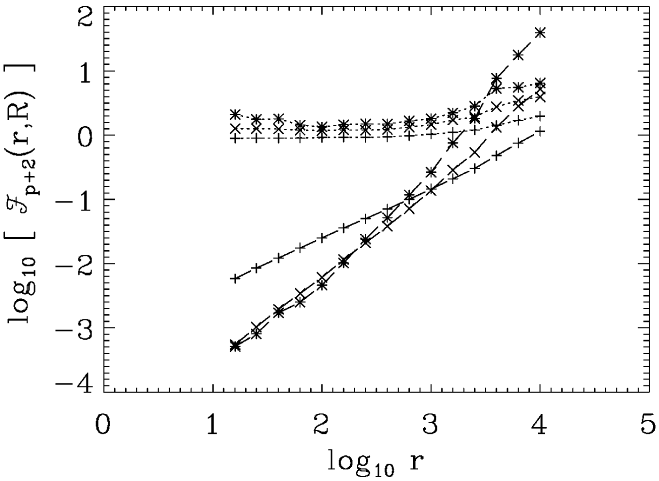

FIG. 2. Log-log plot of $\mathcal{F}_{p+q}(r,R)$ as a function of r at fixed R for $q = 2$ and $p = 2, 4, 6$ denoted by $+$, $\times$, and $*$, respectively, with dashed lines. Shown with dotted lines are the same quantities divided by $S_p(r)$.

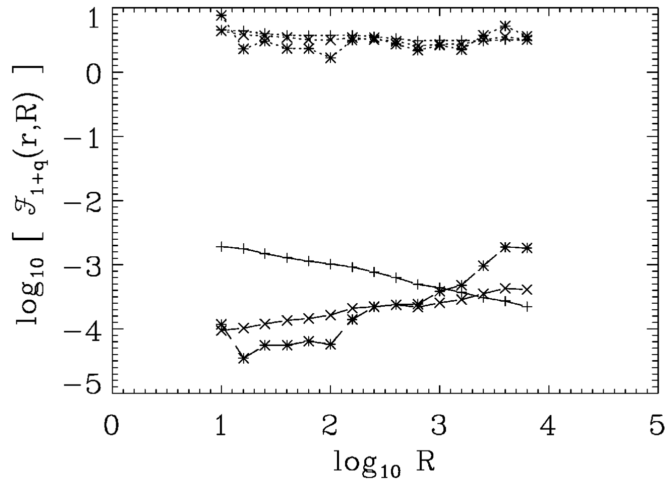

FIG. 4. Log-log plot of $\mathcal{F}_{1+q}(r,R)$ as a function of R for $q = 1, 3$, and 5 denoted by $+$, $\times$, and $*$, respectively. Shown with dotted lines are the same quantities divided by $S_{1+q}(R)/R$.

In Fig. 2 we display the results for $q = 2$ with even values of p as a function of r for r in the inertial range. Only even values are displayed as the odd correlations fluctuate in sign. The large scale R was fixed at the upper end of the inertial range, $R = 15\,849$ in the sampling units of Fig. 1. The data show clean scaling in the inertial range. Overlaid are the averages corrected by the prediction of the fusion rule (12). Here and in all other figures the averages themselves are connected with dashed lines, whereas compensated results are shown dotted. One observes a change of behavior as r approaches R at the upper limits and the average increases in size towards the "fully fused" quantity $S_{p+q}(R)$. Similarly convincing results were obtained for other values of p and q.

In Figure 3 we show $\mathcal{F}_{p+q}(r,R)$ as a function of the large scale R with the small separation r fixed at $r = 16$, together with the values corrected by (12). There is a clear trend towards zero slope in the corrected quantity in the upper inertial range.

We consider now the special case that a single pair of points in a velocity difference approach one another. The prediction of Eq. (6) is tested for $r = 16$ and the expected dependence on R is well verified in Fig. 4. The results found by varying r are not shown here: the linear configuration of the measurement points leads to a competition between the leading and next-order scaling.

The function J_2 was computed and found to be independent of R in the inertial range and to scale as $S_2(\rho)/\rho^2$ (the ρ dependence of all J_ns was verified [8] to scale as J_2). In Fig. 5, $J_n(R)$ is shown as a function of $nJ_2S_n(R)/2S_2(R)$ for $n = 2, 4, 6, 8$ and inertial range R. The finite difference Laplacian Eq. (8) was computed with $\rho = 10$. The straight line $y = x$ passing through the data is not a fit.

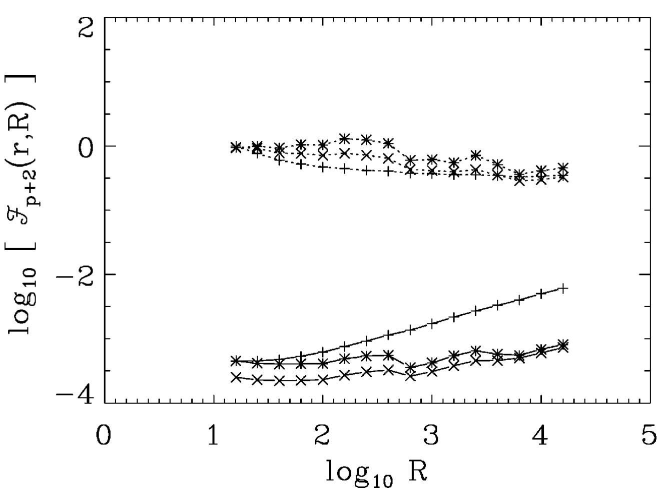

FIG. 3. As in Fig. 2 as a function of R at fixed r, with the dotted lines representing the quantities divided by $S_{p+q}(R)/S_p(R)$.

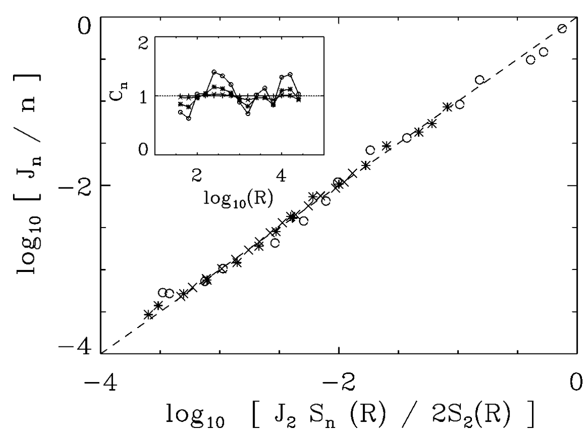

FIG. 5. $\mathrm{Log}_{10}[J_n(R)]$ as a function of the fusion rule prediction $\log_{10}[J_2S_n(R)/S_2(R)]$ for $n = 2, 4, 6$, and 8 denoted by $+$, $\times$, $*$, and $\circ$, respectively. Inset: The coefficient C_n with the same notation.

These results show that (9) is obeyed well with $C_n = 1$. The R independence of C_n is a direct confirmation of the fusion rules for the fusion of two points. On the other hand, that C_n is independent of n is a surprise that does not follow from fusion rules, and has interesting implications for the statistical theory of turbulence. A more sensitive check of the value of C_n is obtained by dividing $nJ_2S_n(R)/S_2(R)$ by $J_n(R)$ for individual values of n and R. This is displayed in the inset in Fig. 5. Clearly, there are statistical fluctuations that increase with increasing n, but the data show that C_n is approximately constant in R and n, with a value of about unity.

An n-independent C_n has surprising consequences for the conditional statistics of our field. Rewrite $J_n(R)$ as

$$J_n(R) = \int d\delta u_R P[\delta u_R] \langle \hat{\nabla}^2 \boldsymbol{u}(\boldsymbol{x}) | \delta u_R \rangle \delta u_R^{n-1}. \quad (13)$$

Here $\langle \hat{\nabla}^2 \boldsymbol{u}(\boldsymbol{x}) | \delta u_R \rangle$ is the average of the finite difference Laplacian conditioned on a value of δu_R. The simplest way to satisfy both (13) and (9) with C_n that is independent of n and R is to assert that the conditional average, which is in general a function of δu_R and R, can be factored into a function of R and a linear function of δu_R:

$$\langle \hat{\nabla}^2 \boldsymbol{u}(\boldsymbol{x}) | \delta u_R \rangle = \frac{J_2}{2S_2(R)}\, \delta u_R\,. \quad (14)$$

Such linear laws have been discussed in the context of conditional statistics of passive scalar advection [9], and were thought to be reasonable because of the linear nature of the advection-diffusion equation for the scalar. However, linear laws for N-S turbulence are unexpected. In Fig. 6 we display a direct calculation of the conditional average of the surrogate Laplacian with $\rho = 10$, multiplied by $2S_2(R)/J_2$ as a function of δu_R for four values of R in the inertial range. For velocity increments that are not too large, the data appear to collapse on a straight line with slope unity.

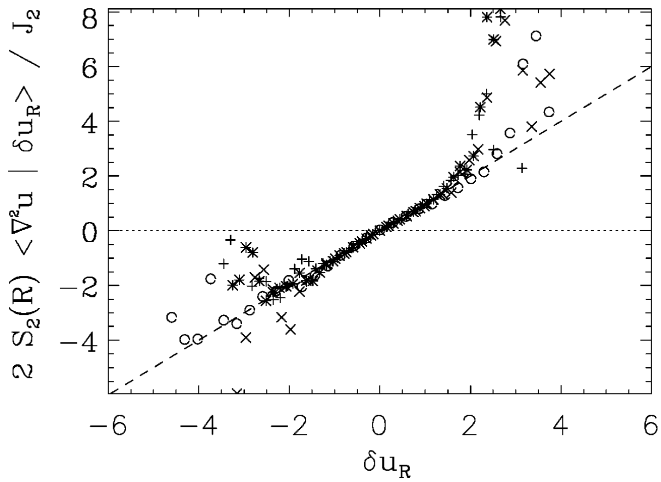

FIG. 6. The conditional averages of the Laplacian, for $R = 50$, 100, 200, and 1000, respectively, marked by $+$, $\times$, $*$, and $\circ$. The large and rare velocity differences are insufficiently converged to determine whether linearity holds for the outer values.

Unfortunately, we cannot test (10) with this data because subdissipation scales are unresolved. The prediction implies that the nature of the conditional average changes qualitatively when ρ decreases below the dissipative scale. Such changes have important consequences for the ultraviolet properties of the statistical theory of turbulence, and a rich variety of predictions are already available [7]. Efforts to acquire subdissipation-scale data are under way.

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