Refined Similarity Hypothesis for Transverse Structure Functions in Fluid Turbulence

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We argue on the basis of empirical data that Kolmogorov's refined similarity hypothesis (RSH) needs to be modified for transverse velocity increments, and propose an alternative. In this new form, transverse velocity increments bear the same relation to locally averaged enstrophy (squared vorticity) as longitudinal velocity increments bear in RSH to locally averaged dissipation. We support this hypothesis by analyzing high-resolution numerical simulation data for isotropic turbulence. RSH and its proposed modification for transverse velocity increments appear to represent two independent scaling groups.
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In all small-scale turbulence research driven by expectations of universality, one deals with so-called velocity increments, which are differences of velocities between two spatial positions separated by a fixed distance. Of special interest are velocity increments when the separation distances belong to the inertial range, that is, length scales which are small compared to a typical large-scale motion but large compared to a typical viscous cutoff scale. It is fair to say that much of the phenomenological work on this subject, summarized in Refs. [1–3], is based to some degree or another on two sets of similarity hypotheses proposed by Kolmogorov, abbreviated as K41 [4] and K62 [5]. The latter, also called the refined similarity hypotheses (RSH), have been verified in both real experiments and numerical simulations of turbulence—at least to an extent that certifies them as reasonable. The verification has focused, largely for historical reasons of experimental convenience, on longitudinal velocity increments, that is, velocity increments for which the separation distance is aligned with the velocity component considered. The theme of this Letter is that RSH is inadequate for transverse velocity increments, for which the separation distance is transverse to the direction of the velocity component considered, and that a nontrivial modification (RSHT) is necessary to account for experimental facts. We conclude that RSH and RSHT form two independent scaling groups in small-scale turbulence.

We need a few definitions before proceeding further. The rate of energy dissipation is given by $\varepsilon = \frac{\nu}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)^2$, where ν is the fluid viscosity, and its local average is defined as $\varepsilon_r(\mathbf{x}, t) = \int_{V_r} \varepsilon \, dV$, where the integration volume $V(r)$, with the characteristic linear dimension r, is centered at the spatial position $\mathbf{x}$. The longitudinal velocity increment is defined as $\delta u_r \equiv u(x + r) - u(x)$, where the velocity component u and the separation distance r are both in the same direction, say x. The form of RSH that has been verified

extensively [6–8] is given by the relation

$$\delta u_r = \beta_1 (r \varepsilon_r)^{1/3}, \qquad (1)$$

where β_1 is a stochastic variable independent of r and ε_r. Assume now that the pth order longitudinal structure function $S_p^L(r) \equiv \langle (\delta u_r)^p \rangle \sim r^{\zeta_p^L}$ and $\langle \varepsilon_r^p \rangle \sim r^{\tau_p}$. Here the angular brackets denote suitable averages. It then follows from Eq. (1) that

$$\zeta_p^L = \frac{p}{3} + \tau_{p/3}. \qquad (2)$$

This equation connects the scaling exponents of the longitudinal structure functions with those of the locally averaged dissipation function. If a further assumption about the statistics of ε_r can be made, such as log-normal [5], multifractal [9], or log-Poisson [10], τ_p can be obtained analytically via Eq. (2).

The thinking in K62 is that Eq. (1) holds equally well if the longitudinal velocity increment is replaced by a transverse velocity increment, namely, $\delta v_r = v(x + r) - v(x)$, where the separation distance r is transverse to the velocity component v. If true, this would imply that the scaling exponents ζ_p^T and ζ_p^L, for longitudinal and transverse structure functions, respectively, are equal. A kinematic constraint from isotropy [1,11] assures us that $\zeta_2^T = \zeta_2^L$. Two sets of measurements [12] appear to suggest (or imply) that the equality holds, within experimental uncertainty, for larger p as well. On the other hand, more recent numerical [13,14] and experimental work [15] has revealed that transverse velocity increments are more intermittent than longitudinal ones, and that ζ_p^T are measurably smaller than ζ_p^L, at least for $p > 4$. These results imply that the RSH cannot be equally true for both longitudinal and transverse velocity increments. With this in mind, we first obtain the scaling exponents of longitudinal and transverse
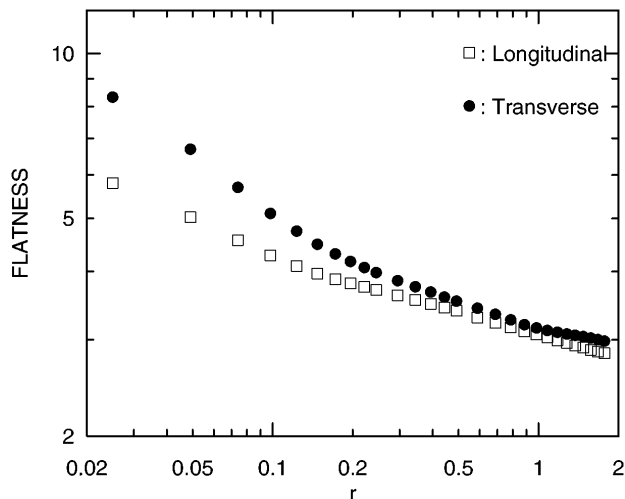
FIG. 1. Flatness of velocity increments, $S_4^L(r)/S_2^L(r)^2$ and $S_4^T(r)/S_2^T(r)^2$, as functions of r.

FIG. 2. The transverse structure functions $S_p^T(r)$ as functions of r for $p = 2, 4, \ldots, 10$.

structure functions. We employ the numerical data from simulations of isotropic turbulence, carried out using 512^3 mesh points in a periodic box; a statistically steady state was obtained by forcing low Fourier modes. For details, see the paper cited first in [8]. The Taylor microscale Reynolds number $R_\lambda = 216$, which is close to the limit of the current computational capability. Averages over ten large-eddy turnover times were used in the data analysis.

Figure 1 shows the flatnesses of longitudinal and transverse velocity increments as functions of r. For all values of r except perhaps those comparable to the box size, the velocity increment along the transverse direction has a larger flatness, suggesting that the transverse velocity increment is more intermittent [14]. This shows that significantly more averaging is required for transverse quantities than for longitudinal quantities.

In Fig 2, the transverse structure functions $S_p^T(r)$ are plotted as functions of r for $p = 2, 4, 6, 8, 10$. An inertial-range power-law scaling can be identified for $0.2 \leq r \leq 0.6$ (the whole box size being 2π). This is also the scaling range determined [16] from Kolmogorov's 4/5 law for the third-order structure function [17].

The exponents ζ_p^T, determined by least-squares fits within the inertial range just mentioned, are plotted in Fig. 3. The use of the extended self-similarity (ESS) technique [18] yields very similar results [19]. The results for ζ_p^L, obtained earlier [20], have also been plotted for comparison. These results demonstrate that while both exponents show considerable anomaly due to inter-mittency effects (as evidenced by the deviation from the dotted line given by K41), the transverse exponents are systematically smaller than the longitudinal ones for $p > 3$. Typical scaling exponents for S_p^T are $\zeta_2^T = 0.71 \pm 0.04$, $\zeta_4^T = 1.25 \pm 0.067$, $\zeta_6^T = 1.63 \pm 0.079$, and $\zeta_8^T = 1.87 \pm 0.078$. While the longitudinal exponents agree quite well with existing scaling models [9,10,21], the difference between the transverse exponents and the models
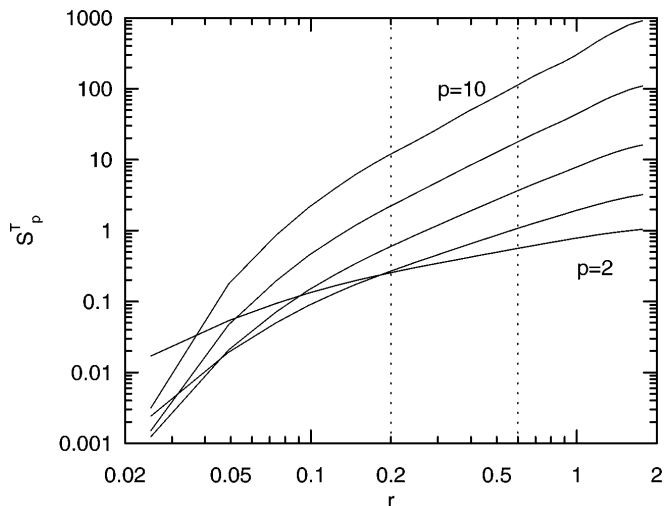
increases with p. For brevity, we show only a comparison with the log-Poisson model [10], which is typical.

If the scaling exponents for longitudinal and transverse structure functions are different, it is clear that Eq. (1), now known to be valid for the former, cannot be valid for the latter; some modification will therefore be required. One possibility suggests itself from kinematic considerations. Recall from the second reference of [6] that RSH may be expected because the longitudinal velocity increment δu_r is an integral over the separation distance r of the velocity derivative $\partial u/\partial x$, whose square is one of the components of the energy dissipation. In a similar way, the transverse structure function δv_r is the integral of $\partial v/\partial x$, which is one of the two terms of the vorticity component $\omega_z = \partial v/\partial x - \partial u/\partial y$. It is reasonable to
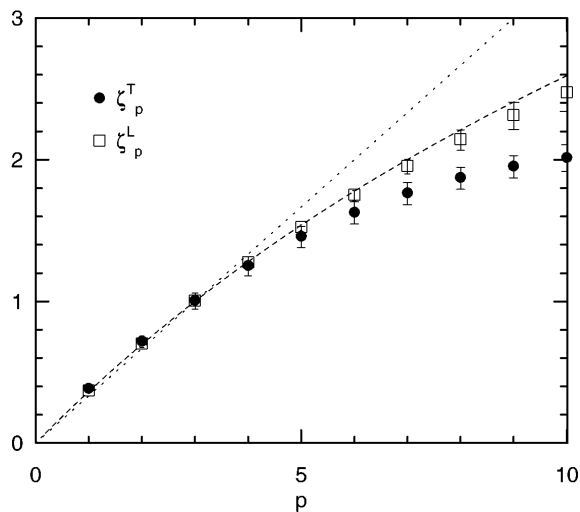

FIG. 3. Numerical results for the transverse scaling exponents, ζ_p^T and ζ_p^L, as functions of p. The dotted line is for the normal scaling relation (K41) and the dashed line is for the log-Poisson model [10].

guess, then, that a suitable modification of Eq. (1) would be

$$\delta v_r = \beta_2 (r \Omega_r)^{1/3}, \qquad (3)$$

where $\Omega = \nu \omega^2$, $\omega = \nabla \times \mathbf{u}$ is the vorticity, and Ω_r is the local average of Ω in the same way as ε_r is the local average of ε in Eq. (1). A necessary condition for the above equation to be plausible is that ε_r should scale differently from Ω_r and that the difference must be compatible in sign with that of the difference between ζ_p^L and ζ_p^T. That this is indeed so was demonstrated recently [22]; that is, if $\langle \Omega_r^p \rangle \sim r^{o_p}$, $o_p < \tau_p$. Thus, Eq. (3) is *a priori* worth exploring as a candidate for the refined similarity hypothesis for transverse velocity increments (RSHT).

We now test the validity of Eq. (3). In particular, we test whether the relation

$$\zeta_p^T = \frac{p}{3} + o_{p/3}, \qquad (4)$$

which follows from Eq. (3), is true. We should note that the equations

$$\delta v_r = \beta_3 (r \varepsilon_r)^{1/3}, \qquad (5)$$

$$\delta u_r = \beta_4 (r \Omega_r)^{1/3} \qquad (6)$$

are both dimensionally plausible, and would yield

$$\zeta_p^L = \frac{p}{3} + o_{p/3}, \qquad (7)$$

$$\zeta_p^T = \frac{p}{3} + \tau_{p/3}. \qquad (8)$$

Equation (4) would derive greater support if the alternative relations (7) and (8) can be shown to be unsatisfactory.

In Figs. 4(a) and 4(b), we show comparisons of numerical results for p up to 10 with predictions of (4) and (8) for the transverse case, and with predictions of (2) and (7) for the longitudinal case. (For odd order, we have used absolute values of the appropriate velocity differences.) The exponents $\tau_{p/3}$ and $o_{p/3}$ used in the above relations were taken from previously published results for the same data [20,22]. For $p < 3$, the differences between the implied relations (2), (4), and (7), (8) and numerical results are relatively small, and cannot be distinguished easily. On the other hand, with increasing p, only $\zeta_p^L - \tau_{p/3}$ and $\zeta_p^T - o_{p/3}$ coincide with $p/3$, showing that Eqs. (2) and (4) are valid to a good approximation. The numerical evidence that $\zeta_p^L - o_{p/3}$ and $\zeta_p^T - \tau_{p/3}$ depart from $p/3$ for large p effectively disqualifies (5) and (6). Note that the error bars in Fig. 4 are quite small and do not affect this conclusion.

A more detailed test of Eq. (3) is given in Fig. 5, where we plot the normalized probability density function (PDF) of $\beta_2 \equiv \Delta v_r / (r \Omega_r)^{1/3}$ with $r = 0.2$, 0.34, and 0.44, all
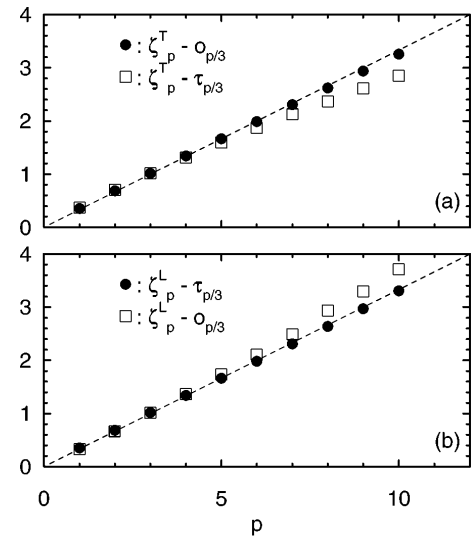

FIG. 4. Verification of the scaling relations in RSHT. (a) shows the results for the transverse structure functions and (b) for the longitudinal case. The dashed lines correspond to $p/3$, which would be followed by the numerical data if the similarity hypothesis used is correct. See the equation pairs (2) and (4), and (7) and (8).

of which lie in the inertial range. The three PDFs collapse quite well. It is seen that in the region $\beta_2 / \langle \beta_2^2 \rangle^{1/2} < 3$, the PDFs agree with the standard Gaussian distribution. For larger values of $\beta_2 / \langle (\beta_2)^2 \rangle^{1/2}$, the PDFs seem to depart slightly from Gaussian, with a tendency perhaps towards exponential. This trend has also been seen for the longitudinal counterpart, β_1, investigated by Wang *et al.* [8]. One difference between the PDFs of β_1 and β_2, which follows from Kolmogorov's 4/5ths law [17], is that β_2 is not expected to be skewed, whereas β_1 should have a third moment equal to $-4/5$. The PDFs of β_3 and β_4 have also been obtained but are not shown here. They do not collapse for inertial range separations, nor do
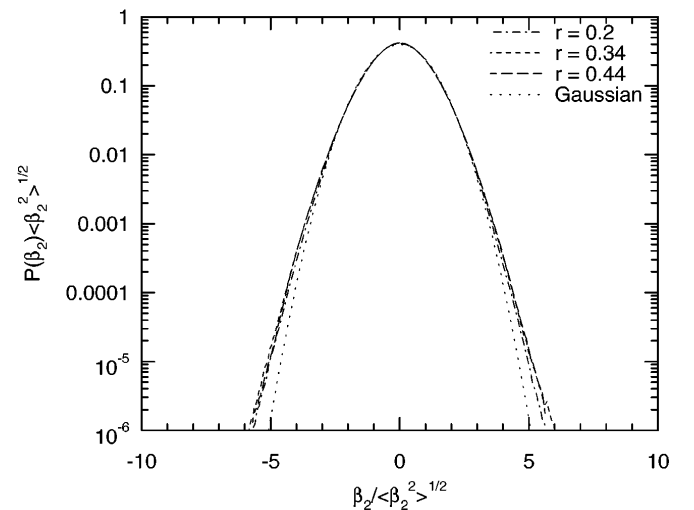

FIG. 5. Normalized PDFs of $\beta_2 / \langle (\beta_2)^2 \rangle^{1/2}$ for $r = 0.2$, 0.34, and 0.44.

they agree with the Gaussian distribution even in the core region, and they possess larger departures from Gaussian near the tails.

The results in Figs. 4 and 5 suggest that RSHT proposed in (3) is a good working approximation connecting the transverse velocity increment with the enstrophy field. In addition, the differences of the intermittency properties between δu_r and δv_r as well as ε_r and Ω_r strongly suggest the possibility of two independent scaling groups. Note that ε_r and Ω_r are the symmetric and antisymmetric parts of the strain-rate tensor, respectively; so we expect that this difference, reflecting the difference in the inertial-range physics of the dissipation and vortex dynamics, may quite plausibly carry over to the high Reynolds number limit as well.

It should be pointed out that a more general refined similarity hypothesis, which encompasses results for both velocity components, cannot be ruled out. One possible scenario could be

$$\Delta v_r = \beta_5 (r \Omega_r^{\alpha} \varepsilon_r^{1-\alpha})^{1/3}, \tag{9}$$

where β_5 is again a universal stochastic variable and α could, in general, be a function of the order index p [23]. Equation (9) is more complex than (1) and (3), since the scalings of the cross-correlation functions $\langle \Omega_r^p \epsilon_r^q \rangle$ have to be taken into account.

In summary, we have proposed a refined similarity hypothesis for the transverse direction (RSHT); it connects the statistics of the transverse structure function in the inertial range with the locally averaged enstrophy. Numerical simulations data for the Navier-Stokes equations at moderate Reynolds numbers demonstrate that RSHT is valid for the transverse structure functions in the inertial range, while Kolmogorov's RSH has been shown previously to be essentially correct for the longitudinal case [6–8]; the latter result is confirmed again in this paper. The important implication of RSHT is the possibility of the existence of two independent scaling groups which may correspond to different intermittent physics in fluid turbulence: one is related to the symmetric part of the strain rate, or the dissipation physics, and the other is related to the antisymmetric part of the strain rate, or the vortex dynamics. This view also suggests that no more than two sets of independent exponents are required for describing the scaling of all small-scale features as longitudinal and transverse velocity increments, dissipation, enstrophy, and circulation [24]. Its conclusions are consistent with all previous data.

[1] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975), Vol. 2.

[2] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).

[3] K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid Mech. **29**, 435 (1997).

[4] A. N. Kolmogorov, C. R. Acad. Sci. URSS **32**, 19 (1941).

[5] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).

[6] G. Stolovitzky, P. Kailasnath, and K. R. Sreenivasan, Phys. Rev. Lett. **69**, 1178 (1992); G. Stolovitzky and K. R. Sreenivasan, Rev. Mod. Phys. **66**, 229 (1994).

[7] A. A. Praskovsky, Phys. Fluids **4**, 2589 (1992); S. T. Thoroddsen and C. W. Van Atta, Phys. Fluids A **4**, 2592 (1992).

[8] S. Chen, G. D. Doolen, R. H. Kraichnan, and Z.-S. She, Phys. Fluids A **5**, 458 (1993); S. Chen, G. D. Doolen, R. H. Kraichnan, and L.-P. Wang, Phys. Rev. Lett. **74**, 1755 (1995); L.-P. Wang, S. Chen, J. Brasseur, and J. Wyngaard, J. Fluid Mech. **309**, 113 (1996); V. Borue and S. A. Orszag, Phys. Rev. E **53**, 21 (1996).

[9] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).

[10] Z.-S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994).

[11] M. Nelkin, Adv. Phys. **43**, 143 (1994).

[12] R. Camussi, D. Barbagallo, G. Guj, and F. Stella, Phys. Fluids **8**, 1181 (1996); A. Noullez, G. Wallace, W. Lempert, R. B. Miles, and U. Frisch, J. Fluid Mech. **339**, 287 (1997).

[13] O. N. Boratav and R. B. Pelz, Phys. Fluids **9**, 1400 (1997).

[14] S. Grossmann, D. Lohse, and A. Reeh, "Different Intermittency for Longitudinal and Transverse Turbulence Fluctuations," 1997 (to be published).

[15] R. Camussi and R. Benzi, Phys. Fluids **9**, 257 (1997).

[16] K. R. Sreenivasan, S. I. Vainshtein, R. Bhiladvala, I. San Gil, S. Chen, and N. Cao, Phys. Rev. Lett. **77**, 1488 (1996).

[17] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **32**, 16 (1941) [reprinted in Proc. R. Soc. London A **434**, 9 (1991)].

[18] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, E. Massaioli, and S. Succi, Phys. Rev. E **48**, 29 (1993).

[19] In applying ESS, we have plotted various orders of transverse structure functions against the third-order longitudinal structure function. Hyperviscous calculations also reveal differences between longitudinal and transverse exponents, but it would be difficult to use the hyperviscous data for modifying RSH.

[20] N. Cao, S. Chen, and Z. She, Phys. Rev. Lett. **76**, 3711 (1996); N. Cao, S. Chen, and K. R. Sreenivasan, Phys. Rev. Lett. **77**, 3799 (1996).

[21] C. W. Van Atta, Phys. Fluids **14**, 1803 (1971); E. A. Novikov, Phys. Rev. E **50**, R3303 (1994); M. Nelkin, Phys. Rev. E **52**, R4610 (1995); S. Chen and N. Cao, Phys. Rev. E **52**, R5757 (1995).

[22] S. Chen, K. S. Sreenivasan, and M. Nelkin, Phys. Rev. Lett. **79**, 1253 (1997).

[23] This relation was also suggested by E. Titi (private communication).

[24] K. R. Sreenivasan, A. Juneja, and A. K. Suri, Phys. Rev. Lett. **75**, 433 (1995); N. Cao, S. Chen, and K. R. Sreenivasan, Phys. Rev. Lett. **76**, 616 (1996); R. Benzi, L. Biferale, M. V. Struglia, and R. Tripiccione, Phys. Rev. E **55**, 3739 (1997).