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Fractal geometry and multifractal measures in fluid mechanics

Katepalli R. Sreenivasan
Mason Laboratory, Yale University, New Haven, CT 06520-2159

> *Philosophy is written in this vast book – I mean the universe – ... in the language of mathematics, and its characters are triangles, circles and other geometric figures, without which it is humanly impossible to understand a single word of it; without which, one wanders in vain in a dark labyrinth...*
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> *Galileo Galilei [1]*

1. INTRODUCTION

Galileo Galilei was merely expressing the sentiment that had reigned supreme in the days of the ancient Greeks, in whose world truth and beauty were sought in terms of composites of perfect Euclidean solids. It is a ceaseless wonder that the scientific enterprise, which is based on such perfect and abstract forms, has the impact it does on human civilization. In reality, Nature is quite irregular, even at low levels of energy: as Mandelbrot [2] remarks, 'Clouds are not spheres, mountains are not cones, coastlines are not circles, and the bark is not smooth, nor does the lightning travel in a straight line....Nature exhibits not simply a higher degree but an altogether different level of complexity.' Modern science has embraced the view that irregularity is a legitimate subject for study; and that, with this study, comes a deeper level of understanding of Nature; with this understanding (and some ingenuity) come applications and, perhaps, also the betterment of human society.

Many seemingly irregular objects possess some degree of order. While the intrinsic reason for this order is not always clear, the claim [2] is that notions of fractal geometry and multifractal measures are central to discerning the order in a large class of systems. Nature is richer in its many manifestations than can be gift-wrapped conveniently in the garb of fractals and multifractals, but there is little doubt that they provide powerful tools for describing, and often understanding, aspects of human experience that were not quantifiable before.

The concepts basic to fractal geometry are relatively simple; in their origin, they are not all that new either. But the recognition that these simple notions form a unified language for a variety of disciplines in natural science is due to Mandelbrot [2].

Research into fractals and multifractals has proceeded roughly in three related directions: (i) the characterization of fractal-like objects by experiment and numerical simulation; (ii) the search for a theoretical understanding of the physical mechanisms governing the observed fractal features; (iii) the evaluation of various physical properties of interest on the basis of fractal geometry.

Our objective is to assess briefly the role of fractals and multifractals in a broad class of fluid flows including turbulence and combustion. A projection for the near future is an intended purpose of this report. However, since such an attempt would have no substance to it without some discussion of the current status, the latter will form the bulk of the report. As applications of fractals in fluid dynamics have yet to mature, the report captures a snap-shot of the changing scene. We focus on activities that are common to both fluid dynamics and fractals. We ignore some isolated aspects as well as those that are likely to be covered elsewhere (such as fractal structure in chaotic mixing). We restrict emphasis to how fractals enter the physical problems, rather than describe classical results.

Since fractals (especially multifractals) are not likely to be familiar to all potential readers, an informal review is included (sections 2 and 3). Much of the material to be covered below can be found in [2-8]. Other references cited are not meant to be exhaustive.

2. BASIC NOTIONS CONCERNING FRACTALS

2.1 Self-similarity and self-affinity

Two basic notions are central to fractals. The first one concerns *self-similarity* and *self-affinity*. In rough terms, an object is *self-similar* if it is composed of smaller pieces, each of which is a replica of the whole. Each small piece can be obtained from the original whole by a *similarity* transformation, or a contraction which reduces the original object by *the same scale factor in all coordinates.* An *affine* transformation is one in which different coordinates are contracted by different factors. Objects with parts that are affine copies of the whole are called *self-affine*. Note that both similarity and affinity are linear transformations.

2.2 Non-integer dimensions

The second important notion is that the dimension of an object need not be an integer. We intuitively understand that a point has the dimension zero, a straight line the dimension one, a square the dimension two, a cube the dimension three, and so forth. A fractal curve, because of its infinite detail, is more space-filling than an Euclidean curve. It is therefore natural to associate with a fractal curve a dimension greater than unity. In general, fractal objects have dimensions which exceed their topological dimensions [2].

2.3 Dimensions of self-similar objects

For fractal curves, the question of what constitutes their length – and likewise, in other cases, of area or volume – is ill-posed. Length estimates for a classical curve, made for instance by walking along its length with increasing resolution, will converge to a finite value – which we may call its true length. This is not so for fractal curves. In general, the estimates for the length, area or volume of fractal objects increase as one measures them with respect to smaller and smaller scales. Further, the two sets of quantities – length, area, and volume on the one hand, and the resolution on the other – do not vary arbitrarily but are related by power laws. For example, the length estimate $L(r)$ of a fractal curve varies with the resolution r as $L(r) \sim r^{1-D}$, where D is the dimension of the curve; D is clearly equal to unity for classical curves (in the limit of small r), and is larger than unity for fractal curves.

In practice, for any physical phenomena, there are cut-offs at some scales on either end, called the inner and outer cut-offs, beyond which power-law relations do not hold. The cut-offs play an important role in the interpretation of fractal scaling.

There are many definitions of non-integer dimensions (Hausdorff dimension, similarity dimension, box-dimension, compass dimension, capacity dimension, information dimension, correlation dimension, and so forth). This situation can be quite confusing for a beginner. Some useful references are [2,4,5,9]. The various dimensions are related by equalities in some cases and inequalities in others. There is no simple guide as to which dimension is the most suitable in a given context. Sometimes one dimension is easier to measure, and even more appropriate, than the others.

For later convenience, we briefly define the box-dimension here. The box-counting method has the advantage that is easily programmed on a computer, and can be used on objects with or without visible evidence of perfect self-similarity. Consider a fractal object residing in a d-dimensional Euclidean space. Divide the space into d-dimensional boxes of a certain size and count the number of boxes containing the object. Repeat the procedure by successively reducing the box size. Let N(r) be the number of boxes of size r which contain the object. The box-dimension is defined as $D_b = \log N(r)/\log(1/r)$. Although the dimension is a limiting concept, in practice, there is a range of scales for which D_b is a constant and obtained from the (constant) slope in the log-log plot. A straight line in such log-log plots is often a sign of statistical (rather than exact) self-similarity.

2.3 Time records

Consider a trace of turbulent velocity, say, measured as a function of time. Since the two quantities (velocity and time) are independent of each other, the coordinate axes can be stretched (or contracted) independently. As we shall see presently, such objects are not candidates for self-similarity, but self-affinity instead.

For illustrative purposes, consider the so-called fractional Brownian motion (fBm), which is a generalization [10] of the standard Brownian motion. The fractional Brownian motion X(t) is a single valued stationary function of the time variable t, such that its increments $\Delta X(\tau) = X(t+\tau) - X(t)$ have a Gaussian distribution with variance $<[\Delta X(\tau)]^2> \sim \tau^{2H}$. The case $H = 1/2$ yields the classical Brownian motion.

If the time axis of an fBm trace is stretched excessively, the trace looks smooth on all scales and intuition suggests (and detailed calculations confirm) that the dimension of this highly stretched trace is unity. This is the so-called global dimension of the trace. On the other hand, if time axis is compressed excessively, the signal will look rough on all scales and possesses a dimension greater than unity. This dimension, called the latent dimension, can be shown to be either 2 or 1/H, whichever is smaller. For a range of the stretching factor such that the vertical and horizontal scales are of the same order, one can associate a non-trivial (local) dimension D with the trace. It can be shown that $D = 2-H$. (Since D for a graph can lie only between 1 and 2, it follows that $0<H<1$.)

The notion of the dimension for a self-affine graph is not as straight-forward as that for a self-similar object, and its measurement requires a good understanding of the various pitfalls. The lack of appreciation of this elementary fact has occasionally resulted in false claims.

The exponent H appears also in the analysis of time records with memory. The auto-correlation function of the increments of the classical Brownian motion is a delta-function centered at zero (that is, it has no memory). On the other hand, processes for which $H>1/2$, possess a positive correlation that extends indefinitely (in practice, up to some cut-off scale); such processes are persistent in that an increasing trend in the present is followed, on the average, by a tendency to increase also in the future. Hurst (see [2]) showed that such phenomena are preponderant (floods, rainfall, sunspot numbers, and other phenomena unrelated to fluid dynamics). When $H<1/2$, the tendency is for the trace to show a decreasing tendency in the future when the present trend is to increase (anti-persistence).

3. MULTIFRACTALS

The notion central to multifractals is a singular measure [11]. It is enough here to think of a measure as a positive definite, additive quantity distributed on an interval (or set). By singular measure, we mean that the measure density is not everywhere defined. For the formalism of multifractals to be useful, however, it must be applicable to singularities smoothed out by some physical mechanism (such as viscosity). This is indeed the case.

As an example of near-singular measures, consider the turbulent energy dissipation rate distributed on a line at a high Reynolds number. Figure 1 shows a typical trace. The signal is highly intermittent; a few peaks are several hundred times larger than the mean of the signal. The higher the Reynolds number, the stronger the peaks; in the limit of infinite Reynolds number, the peaks will be infinitely large and truly singular. At all large but finite Reynolds numbers, the peaks (if resolved adequately) can be fitted by algebraic singularities whose middle is smoothed by viscosity. It is clear from inspection that the first few moments (such as the mean, variance,...) will not provide adequate information about such near-singular distributions; many high-order moments, or the entire probability density (PDF) itself, will be needed. Unfortunately, the PDFs depend strongly on the Reynolds number. The point, however, is that the mechanism that produces these intermittent distributions (namely the energy cascade, which itself is a partial abstraction of the vortex stretching process producing small scales of turbulence) is thought to be Reynolds-number-independent (as long as the Reynolds number is sufficiently high). If this is true, one needs a statistical quantity which sheds light on this Reynolds-number-independent process. Multifractals provide tools for characterizing such singular measures in Reynolds-number-independent ways.

For another intuitive means of introducing multifractals consider the following questions about figure 1. Over what fraction of the line interval does dissipation occur? What is the dimension of the set supporting the dissipation? Since some dissipation occurs everywhere, the trivial answers are, respectively, 'everywhere' and 'unity'. These are not useful answers, however, and researchers have traditionally proceeded by setting a threshold to distinguish significant levels of dissipation from insignificant levels. Unfortunately, the choice of this threshold is very subjective and influences the answers strongly [12]. In any case, it is not enough to seek information about the oversimplified binary picture of whether or not there is dissipation, but it is essential to know something about its different magnitudes. Multifractals provide proper tools for handling these questions. Roughly, a given level of activity is characterized by a fixed value of a well-defined Holder exponent, and information is sought about the dimension of the set supporting this given level of activity.

Formally, if the measure (dissipation) in a box of size r centered at a position x on the interval varies with r as $r^{\alpha(x)}$, the measure density can be written as $\mu' \sim r^{\alpha-1}$. This is singular in the sense that, as r approaches zero, the measure density μ' diverges for all $\alpha < 1$. The smaller the α, the larger the strength of this singularity. Roughly, one can associate [13,14] with every iso-α region a fractal dimension $f(\alpha)$. Since α can vary continuously between α_{min} and α_{max}, so does $f(\alpha)$. The $f(\alpha)$ curve, also called the multifractal spectrum, thus represents an infinity of dimensions, each corresponding to the set supporting singularities of a given strength α.[1]

While this description is intuitively helpful, it is not convenient for obtaining a converged $f(\alpha)$ curve. For this purpose, an equivalent description [15] in terms of the so-called generalized dimensions, D_q, is more useful. It has been shown [11,13,14] that the pair of variables (τ,q), where $\tau = (q-1)D_q$ and the real number q lies between $-\infty$ and $+\infty$, are Legendre transforms of the pair (f,α). By evaluating generalized dimensions and Legendre

[1] The multifractal formalism of Ref. [13] does not deal with measures but with velocity increments which have the property that a velocity increment over two adjacent subintervals equals that over the whole interval.

transforming them, one can obtain the f(α) curve.

One can also measure the f(α) curve directly [16] by noting a formal connection that exists between multifractals and thermodynamics of statistical mechanical systems [17].[2]

More basic to multifractals than the f(α) curve is the self-similarity of the underlying multiplicative process that generates the multifractal [11,18]. To understand this, let us start with a uniform measure distributed on unit interval, divide it into a specified number of subintervals and redistribute the measure unequally on them. The redistribution is obtained by multiplying the parent measure by 'multipliers' taken from a certain PDF ('multiplier distribution') defined on the interval. Imagine that this process continues indefinitely. If the multiplier distribution is the same at each stage of refinement, one obtains a multifractal with scaling properties.

4. AGGREGATION IN PARTICLE-LADEN FLOWS

Fluid dynamics of particle-laden flows is replete with applications. If interparticle interactions are ignored, the essential problem is one of understanding how various physical properties of the flow (such as effective viscosity) are altered by particle loading. Alternatively, the effect of the fluid flow on the particle motion is also of interest. Under certain circumstances dictated by hydrodynamics, inter-particle interactions may become important and lead to the formation of aggregates (that is, structures in which particles stick together irreversibly).

There are two basic aspects to the study of aggregation: kinetics and geometry. Kinetics involve the quantitative description of the time evolution of the aggregates and their size distribution, whereas geometry is concerned with a quantitative description of the structure of the aggregates. The first aspect has been studied for a long time, whereas the second aspect, which used to be the backwaters of aggregation studies until recently, has taken on a life of its own since the advent of fractals.

Aggregation can occur in a variety of ways: electrolysis (induced by electric field), sedimentation (induced by gravity), filtration (caused by the particle motion stopped by small pores), and so forth, and can be either of the particle-cluster type or cluster-cluster type. A simple kinetic model [19] called the diffusion-limited-aggregation (DLA) has been studied extensively and thought of (with some minor modifications) as a paradigm for a number of processes such as protein aggregation, colloid clusters of gold and silica, soot formation, viscous fingering in porous and non-porous media, dielectric breakdown, dendritic solidification, and so forth. The common feature among many of these phenomena is that a suitably defined potential governed by the Laplace equation can be defined.[3] It is therefore worth examining DLA briefly.

In the two-dimensional version of DLA, one considers a square lattice on a plane and first chooses the origin for the cluster. A particle, the seed for the aggregate, is placed at the

[2] The quantities f, α, τ and q are anologous, respectively, to entropy, internal energy, free energy and inverse temperature. This means that, if small scale turbulence is treated in some rough sense as a statistical mechanical system, the measurement of these multifractal quantities is equivalent to measuring the thermodynamic properties. It should be noted that this thermodynamic analogy breaks down when f(α) becomes negative [11], as indeed happens in stochastic multiplicative processes [18].

[3] Just as the Ising model describes the essential physics of a wide variety of materials near the critical point, the hope has been expressed that DLA would describe a variety of growth processes. DLA and turbulence (see section 7) are often thought to be the paradigm problems for serious multifractal applications.

origin. One then considers a large circle of radius R, centered at the origin, and chooses a point at random on this circle. A particle is released at a site nearest to this point and is allowed to execute a random walk on the lattice. If the random walker reaches a site nearest to the origin, it stops and stays stuck to the seed. (If the particle reaches the circle without getting close to the origin, it is abandoned.) Another particle is released from a lattice point close to another randomly chosen point on the circle, and allowed to stick to the seed or the two-particle cluster (as the case may be). The process is continued until a cluster of the desired size is reached. The algorithm can be extended to any higher dimension.

The DLA aggregates are fractal structures with a dimension of about 1.7 in two dimensions and 2.5 in three dimensions [3,20]. The growth of the cluster is governed by the so-called harmonic measure which is the probability that a random walker approaching the cluster from the far-away circle hits the cluster in a certain infinitesimal interval along the boundary. The harmonic measure is a solution of the Laplace's equation for the electrostatic potential when the cluster boundary is taken to be at zero potential and the circle at a potential of unity. It is intermittent in appearance and amenable to multifractal analysis [21,22]. As remarked in section 3, the basic evidence for the self-similarity in the DLA structure comes from the self-similarity of the multiplicative process [11]; it has also been pointed out that $\tau(q)$ does not exist for negative values of q.

Most fractal and multifractal characteristics of DLA have been extracted from computer simulations [3,22]; to our knowledge, there are no exact analytical results.

5. APPLICATIONS IN POROUS MEDIA AND VISCOUS FINGERING

5.1 Porous media

The Stokes equation governing the fluid motion in porous media is linear. It can be reduced to the Darcy law (according to which the velocity is linearly proportional to the pressure drop) by using appropriate assumptions of homogeneity. Equivalently, this can be written as the Laplace equation for the pressure. This suggests analogies with DLA, except that the boundary conditions in the porous media are more difficult to handle.

The difficulty is in describing the boundary between fluid-filled regions and solid-like regions. Two types of simplifications have been made in the past. The first is to model porous media as a network of capillary tubes. This model conceives of the solid phase as being continuous with interconnected fluid-filled pores running through it. A major simplification is achieved in this way because the Poiseuille law valid for laminar flow through pipes holds for each capillary. The second variety of simplification assumes the fluid phase as continuous and the solid particles as obstacles for the flow.

Fractals appear in studies of flow through porous media because of the random character and the extreme variety of shapes encountered. The pore space, the solid phase and the solid-pore interface all exhibit fractal scaling. Various estimates for the fractal dimensions of these three aspects have been made [4,5].

In practical applications, the one unknown is the permeability of the medium – which, in general, is a tensorial quantity. The permeability in most cases is measured by pressure drop experiments or estimated empirically. A useful goal would be to relate the permeability of a medium to its characteristic fractal dimensions. This has not yet been accomplished.

5.2. Viscous fingering

When a low viscosity fluid is pushed into a high viscosity immiscible fluid, Saffman-

Taylor fingers develop [23]; these fingers occur singly and are broad and smooth in shape. The equation governing viscous fingering in Hele-Shaw cell is the same as that for flow through porous media, except that the permeability in the former is not real but related to the gap between the plates of the Hele-Shaw cell. The Saffman-Taylor fingers correspond to the wavenumber with maximum instability, and their width varies as the square root of the surface tension between the two fluids (if all other conditions are held fixed). As the surface tension is lowered, the fingers split more and more; but there is a practical limit to how low the surface tension can get. If the high-viscosity fluid is a miscible colloidal solution with shear-dependent viscosity, the interface grows to be fractal-like in appearance [24] even when the capillary number is moderately high; the more non-Newtonian the solution, the more the tendency to fractal fingering. In spite of some eminent studies to model this behavior [25], the basic physics of fingering in non-Newtonian fluids is not well-understood.

It has been argued [26] that, if interfacial tension is ignored[4], the viscous fingering problem is analogous to DLA; indeed, the viscous fingering patterns in radial Hele-Shaw cells (which do not have the anisotropic constraining effects of rectangular cells) are similar to the DLA structure and possess roughly the same fractal dimension. The growth of viscous fingers depends on the pressure gradient, which therefore plays a role analogous to the harmonic measure for DLA. Since the pressure field is not easily measured, related growth measures have been defined and characterized by multifractals [4].

If the medium in the Hele-Shaw cell is porous, one obtains fractal structure even if one uses Newtonian fluids and the capillary number is moderately high [4].

5.3 Percolation and diffusion

Percolation involves the spreading of a fluid in a random medium, where the words 'fluid' and 'medium' are used in a certain general sense. The role of randomness in percolation is quite different from that in diffusion. In the latter, the Brownian particle executes a random walk, whereas percolation deals with randomness that is frozen into the medium. While any position in the medium can be reached by diffusion, the spreading in percolation is confined to finite regions except when the so-called 'percolation threshold' is exceeded. In model studies, one considers a square lattice whose sites are either randomly occupied (with probability p) or empty (with probability 1−p). Connected sites form clusters. On an infinite sample, all clusters remain finite below the percolation threshold, $p=p_c$. For $p>p_c$, infinite clusters with a finite probability per site appear, and the probability of this occurrence varies with p as $(p-p_c)^\beta$. Percolation clusters are self-similar and possess [27], in the limit of large clusters, a fractal dimension of 1.89. The 'hull' of the diffusion front is also a fractal with a dimension of 1.75, whereas its external perimeter has a dimension of about 1.37 [28].

One should also mention here the 'invasion percolation' where the water displacing oil in porous rocks may trap regions of oil [29]. Randomness encountered by the invading fluid would now also depend on the trapped regions.

Branched polymers have size distributions that are self-similar [30], and are therefore candidates for fractal description. Some useful analogy exists between polymers and percolation studies [32]. As in percolation, the fractal dimension of branched polymers can be related to other indices characteristic of the polymer size above a 'percolation threshold'.

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[4] In practice, this is far from being correct. The arguments postulating similarity between viscous fingering and DLA must be examined critically in spite of the resemblence of the observed fractal patterns.

6. ONSET OF CHAOS IN NEWTONIAN FLUID FLOWS

Multifractals have played a powerful role in characterizing universality at the onset of chaos in low-dimensional systems. The renormalization theory has been worked out for the onset of chaos for period-doubling [31] and quasiperiodic cases [32]. Experiments in forced Rayleigh-Benard convection [33] and the near-field of oscillating cylinders at low Reynolds numbers [34] strongly support the universality theory: the $f(\alpha)$ curve describing the non-uniform distribution of the invariant measure on the attractor at the onset of chaos agrees well with that calculated for one-dimensional circle maps. This is the power of universality.

As already mentioned, the $f(\alpha)$ curve provides a thermodynamic – and hence degenerate – description of the dynamical system. Even so, it has been possible to develop [35] a basis for extracting (up to a level of detail which depends on our knowledge of the system in terms of other statistical measures – such as the similarity structure exhibited by the power spectral density) the multiplicative process leading to the observed multifractal state. This entire exercise should be heralded as a classic application of multifractals where powerful theory and imaginative experiments have come together satisfactorily.

It should be noted that dynamical universality was indeed known before the multifractal formalism came to the fore. For example, the Feigenbaum number in period doubling bifurcations [31] was experimentally observed in convection experiments [36]. Furthermore, it was also known that the microscopic information about a deterministic dynamical system and its scaling properties could be characterized in detail by the scaling function [31]. However, the experimental measurement of the scaling function is quite difficult, and the advent of multifractals (albeit statistical) made the search for universality significantly easier.

As a final note, we wish to emphasize that chaos (which involves temporal complexity) is quite different from turbulence (which involves spatial as well as temporal complexity); however, transition to chaos is often relevant to early stages of transition to turbulence.

7. NON-REACTING AND REACTING TURBULENT FLOWS

High-Reynolds-number turbulence consists of a wide range of dynamically interacting scales. The ratio of the largest to the smallest scale increases roughly as the 3/4 power of the large-scale Reynolds number. The conventional wisdom is that statistical similarity prevails over a range of intermediate scales; the precise form of this similarity and the range of scales over which it holds are matters of much interest. It has been thought that fractals and multifractals provide proper tools for better description, and hence better understanding, of the problem of turbulence. In the following summary statements, we indicate the degree of our confidence by C (for fairly certain) or P (for provisional); the latter means that the results come essentially from one laboratory. For references to original sources and further discussions, see [7].

7.1 Fractal scaling of flames, iso-surfaces and interfaces

The main question here is whether these objects can be treated as thin surfaces with many self-similar convolutions. As a paradigm problem in turbulence that involves fractals, it is useful to quote a few results in some detail.

(a) For the scalar interface (i.e., outer boundary of scalar-marked regions in turbulent free shear flows), fractal scaling occurs over the entire interval between the integral scale and the Kolmogorov scale. The dimension in this scale range is 2.35 ± 0.05. (C)

(b) In fully turbulent parts of shear flows, iso-scalar contours possess a fractal scaling with a dimension of 2.67 ± 0.05. The scaling is smaller than for (a), and the inner cut-off occurs at a multiple of the Kolmogorov scale; it can be estimated *a priori* [42]. (C)

For both (a) and (b), data exist on the Reynolds number variation of the dimension. (P)

The original heuristic explanation [7] for these observations relied on the applicability of the Reynolds number similarity. Recently [37], by combining the 'co-area formula' culled from measure theory with the convection-diffusion equation governing the scalar evolution, it has been possible to obtain the fractal dimensions of iso-scalar surfaces and interfaces in turbulent flows. The only information about the velocity field that enters into the calculation is the scaling of the first-order structure function. The dimensions obtained are in excellent agreement with experiment.

(c) For the range between the Batchelor scale and the Kolmogorov scale, the fractal dimension of iso-scalar surfaces as well as interfaces approaches 3 as the Schmidt number approaches infinity. The finite Schmidt number correction is logarithmic. (P)

(d) The results (a) and (b) hold also for vorticity interfaces and iso-vorticity contours. (P)

(e) The dimension of flame surfaces depends on the ambient turbulence level, but the flame front in both diffusion and premixed flames has a fractal dimension of 2.35 for large turbulence levels. (C) Note that, in contrast to high-Reynolds-number isothermal flows, the scale range of convolutions in high temperature flames is small, except when the turbulence levels are high. Several fractal-based closure models have been attempted in combustion. (P)

7.2 Results from time series analysis

The difficulty in the determination of the fractal structure of a time series lies partly with the definition of a suitable cover, and partly with proper recognition of the cut-offs between global, local and latent dimensions. These artifacts are now moderately well-understood, and the fractal structure of a time series of velocity or temperature fluctuations in high-Reynolds-number turbulent flows has been explored [38]. These time traces resemble fBm traces with the exponent $H \approx 0.35$, and its (local) fractal dimension $D = 2 - H \approx 1.65$. This is consistent with the classical theory of Kolmogorov [39]. An implication is that the dimension of iso-velocity and iso-temperature surfaces in fully developed turbulence is about 2.65 – consistent with the result (b) in section 7.1. At moderate Reynolds numbers, the scaling is better for the spatial aspects and ambiguous for temporal data with the exception of those taken for interfaces, for which modest scaling occurs even at moderate Reynolds numbers.

7.3 Multifractal scaling

The evidence is strong that multifractals are useful tools for describing scaling properties of structure functions [13,40], and of turbulent energy dissipation rate and scalar dissipation rate in turbulent flows [41,7]; provisional evidence [7] suggests that other positive definite quantities, such as the square of turbulent vorticity, can be described similarly. This type of work has produced the following results.

(a) Phenomenological models for small-scale intermittency, with outcomes consistent with experiment, have been constructed. These models have yielded realistic intermittency corrections in the inertial and dissipation ranges, produced a refinement of the scaling of the power spectra in the dissipation range [42], generated stochastic signals which do not differ from the real turbulent signals in most respects [43], and so forth.

(b) The nature of the observed near-singularities appears consistent with the mathematical result [44] on the partial regularity of weak solutions of the Navier-Stokes equations.

(c) By assuming that the scalar as well as vector fields are fractal graphs with the measured dimension, a broad theoretical apparatus has been constructed [45] to explain the

Kolmogorov scaling in the inertial range and its various modifications [46].

8. GEOPHYSICAL PHENOMENA

Geophysical fields (such as cloud radiance, rainfall, temperature and pollution records, sea surface infrared reflectivity, sea surface geometry, lightning paths, and so forth) are a result of nonlinear processes involving different fields at widely varying scales. In each case, the statistical invariance of a wide scale range suggests that fractal concepts may be useful. One could ask, for instance, if lightning paths and cloud boundaries are fractal and, if so, measure their fractal dimensions. Among the first fractal measurements made in geophysics was the dimension of cloud boundaries [47]: the dimension of fair-weather clouds as well as large clouds is about 2.35 (the same as that of scalar interfaces in turbulent flows), whereas clouds strongly affected by the mean wind shear possess smaller dimensions [7].

In geophysics, there are numerous candidates which are potentially fractals. As already remarked in section 3, it is not enough to seek information about the binary picture of whether or not there is rainfall (for example), but one needs to know something about the different rainfall rates. The variability and intermittency of rainfall records suggest that multifractals could be quite useful. Preliminary multifractal analysis has been made for rainfall rates, cloud radiance and other geophysical fields. For a survey of articles on these topics, see [8]. Because accurate computations of high-order statistics require extraordinary amounts of data, geophysical measurements have generally been restricted to low-order multifractal measures.

It is claimed that high-order moments may diverge in geophysical situations [48,8]. However, controlled measurements in the atmospheric surface layer [49] suggest that the apparent divergence is an artifact of relatively small data records.

9. APPLICATIONS FOR IMAGE COMPRESSION AND DATA INTERPOLATION

The application of fractals in the construction of natural scenery – such as mountains, clouds, lakes, and so forth is well-known (see, for example, [2,50]). In fact, this technology has found application even in ventures such as movie-making [51], art [52] and music [53]. Since these are far from fluid dynamics, they will not be discussed here.

Fractals find a useful application in image compression and reconstruction. Lovejoy & Mandelbrot [54] constructed cloud images using the model that a rain field is composed of self-similar pulses; the areas of these pulses had power-law distributions and rain rates were random. The resulting pictures looked quite realistic even though real clouds are stratified in the vertical direction, and the governing principle for reconstruction should be self-affinity rather than self-similarity.

Image compression by taking advantage of fractal structure is the subject of [55]; related applications involve fractal interpolation schemes by minimizing the Hausdorff distance. However, extensive application of these ideas to turbulence data has not been made. It is worth emphasizing that image compression on the basis of self-similarity or self-affinity is not especially the domain of fractals alone; in fact, applications based on wavelet theory work quite well [56].

10. A QUALITATIVE ASSESSMENT OF PAST ACCOMPLISHMENTS

It is sometimes implied that fractals embody one of the basic truths about complexity: all known truths about Nature are expressible in the form of some generalized concepts, and that fractals represent one such generalized concept. *This is the appeal of fractals.*

This philosophical appeal has occasionally produced an exaggerated response, as pointed out in [57], and the 'wheat' cannot always be separated from 'chaff'. *Note, however, that other glamorous concepts such as catastrophe theory never came close, even in their haydays, to enjoying the degree of appeal that fractals possess.* (Whether popular appeal is always correlated with scientific significance is another matter.)

Fractals have been taken seriously in mainstream science for about a dozen years or less. We hope that the summary given above makes clear the impressive fact that *fractals have had tremendous impact on providing descriptive and incremental understanding in many fields.*

To the examples already cited, we can add a few more here. Suppose we need to model the spread of forest fires, or the seepage of ground water or radioactive substances. While exact formulation of these problems is in principle possible, it would be futile to take this approach because of their extreme complexity. Instead, simulations of the type carried out in percolation studies can be quite useful in providing an overall picture (even if incomplete). Similarly, a host of other phenomena such as rainfall rates can be modelled by multifractals. *Fractals provide tools for modeling a variety of complex systems with some realism.*

Even though the evidence is still not compelling, it is strong enough to think that fractals are well-suited for handling scaling phenomena. *There is a certain tangible benefit of unity that fractals have brought to apparently unrelated areas of science.*

Much of the work using fractals has so far occurred in the physics community (or those small pockets of fluid mechanics community with relatively strong ties to physics), and the focus has not been engineering applications. It can be said that a serious beginning has been made. However, *for fractal-based models to solve complex problems at the level of engineering utility, such as producing new materials or new numerical codes for computing high-Reynolds-number turbulent flows, it is essential for the engineering community to take sustained interest in these tools.*

Wherever fractal (and multifractal) scaling is observed in fluid flows, it is nearly always statistical. This means that they provide only partial information – even if valuable and unique. *Therefore, as with all partial information, the degree to which one can make sense of an observation depends on the ingenuity of the individual trying to extract it. This is the ultimate constraint.*

We noted earlier that fractal-related work falls into three broad classes: description, explanation and prediction. Description and measurements of fractal dimensions consumed most of the energy in the early days. This effort was not trivial because it involved an understanding of the potential of fractals when they were still not commonplace, as well as the improvisation of theoretical, experimental and computational techniques. As for the second aspect, in percolation, multifractal measures, scalar interfaces in turbulence, and in other areas, a variety of results exist in which phenomenology and rigor play complementary roles. It appears to us that more such results are likely to emerge as fractals integrate increasingly with applications-oriented research. As regards predictions, a few powerful ones have already been pointed out in section 6, even though their engineering consequence is unclear. *On the whole, the past efforts have been quite rewarding.*[5]

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[5] In terms of applications of the new wave of ideas, it is the considered opinion of several people with whom I have discussed the matter that fractals have been more successful than chaos.

11. FUTURE NEEDS

The question of what constitutes future opportunities in fractals may well be thought to belong to the domain of mathematics. That, however, is not very useful here; an important mathematical result was recently proved [58] but it produced hardly a ripple elsewhere. On the other hand, opportunities in specific fields of fluid mechanics will have to be assessed in the context of those specific fields. That would be a Herculean task for this brief report, and it therefore seemed best to restrict it to a few broad questions common to most applications of fractals.

(a) The situation *typical* of most fractal-related studies is that much of our knowledge comes from experiment and simulations. While simulations and experiments are very useful, they are often limited by approximations, finite size effects, noise, and other artifacts, and cannot supplant theoretical results. *There is at present a big backlog of observations without the backup of solid explanations.*

(b) Much of the physics in problems that possess scale-similar phenomena is hidden in the cut-off scales. *It is essential in practice to pay greater attention to cut-off scales and cross-over phenomena.*

(c) It would be essential to know, at least for one hard problem like turbulence, the relation between dynamics on the one hand and fractal geometry and multifractal measures on the other. (As already remarked in section 6, this is known for some simple cases.) What aspects of the hydrodynamic equations yield the fractal structure of its solutions? How may one show, without empirical input, that the turbulence structure is fractal, obtain dimensions of its various facets and generate a closed list of fractal dimensions to define turbulence uniquely? *Without such knowledge, it is difficult to make a case for the inevitability of fractals as* the *tool of choice for studying large classes of nonlinear problems.*

(d) There are practical issues that appear to be within the realm of near-term possibility. For example, it appears possible to construct a much better model for flame speeds in premixed flames; model the gross spread rates of jets; generate a robust large-eddy-simulation model based on multifractals. Similarly, one should be able to shed light on conditions under which flame extinction occurs because of excessive local stretch; generate synthetic signals which, if used as initial conditions for direct numerical simulations for turbulence, yield rapid convergence; generate good data compression techniques. *Such efforts should be driven more by expertise in the respective fields rather than in fractals* per se.

12. EPILOGUE

We may end this note with a statistical extrapolation. Figure 2 shows the number of entries in Science Citation Index on fractals and multifractals. The increase in interest in these topics has been nothing short of phenomenal. Various curve fits were tried for the data, and the best fit was the Landau equation describing supercritical bifurcation [59]. As is well-known, the symptoms of such a bifurcation are the initial exponential growth and subsequent saturation due to nonlinear effects. At the risk of being proven wrong, the inference that can be drawn is that a continued exponential-like growth cannot be sustained without further stimulus in the form of major new ideas or some spectacularly successful application.

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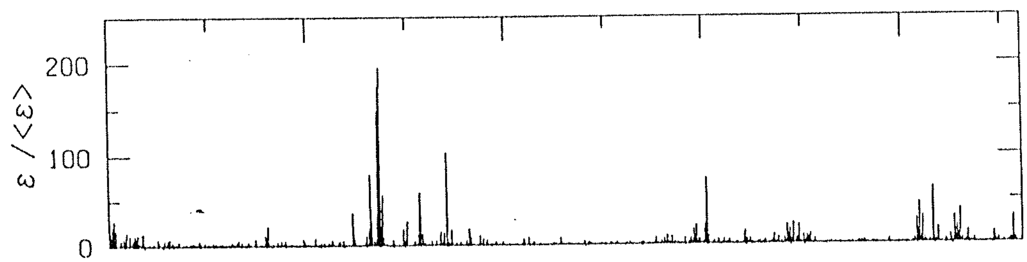

Figure 1: A typical time trace of a representative component of the energy dissipation rate $\varepsilon \sim (du/dt)^2$ normalized by its mean. It is conventional to interpret this as a spatial cut in the direction of mean motion, x, so that the absissae may be taken as a line cut in the x-direction. The data were obtained in the atmospheric surface layer about 6 m above the ground. The so-called microscale Reynolds number was about 2500.

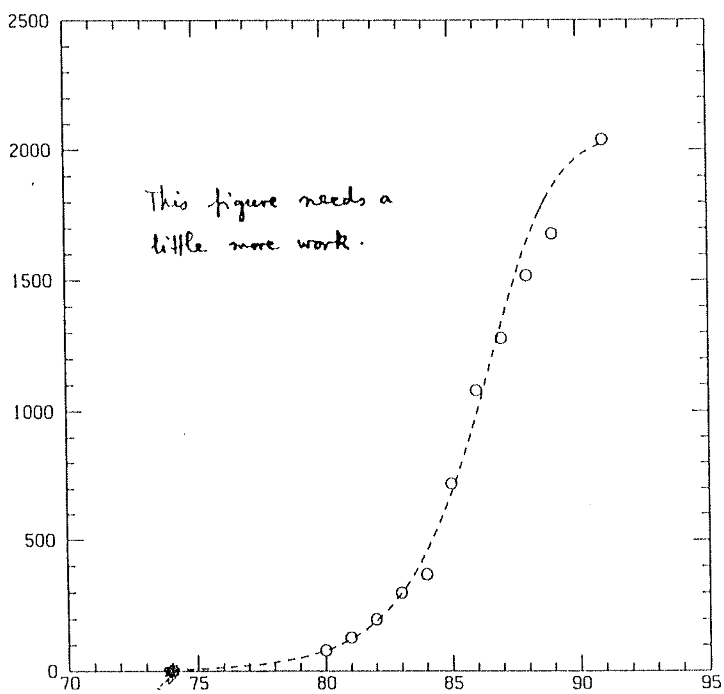

Figure 2: The number of entries in the Permutation Index on Fractals and Multifractals, as a function of time since the invention of the word 'Fractal' [2] in 1972 (check!) and the word 'Multifractal' in 1984 [13]. Note that these entries do not all refer to independent papers, and so should be treated only as representative (correct to within a factor 2, say) of the number of related publications.