Scaling of Low-Order Structure Functions in Homogeneous Turbulence

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High-resolution direct numerical simulation data for three-dimensional Navier-Stokes turbulence in a periodic box are used to study the scaling behavior of low-order velocity structure functions with positive and negative powers. Similar to high-order statistics, the low-order relative scaling exponents exhibit unambiguous departures from the Kolmogorov 1941 theory and agree well with existing multiscaling models. No transition from normal scaling to anomalous scaling is observed. [S0031-9007(96)01548-7]

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The intermittency effect of small scale dynamics on fluid turbulence has been an important issue in turbulence statistics for many decades. A lot of effort has been invested towards the understanding of the departure from Gaussian statistics and the related anomalous scaling behavior of velocity structure functions. At very high Reynolds numbers, Kolmogorov's 1941 similarity theory (K41) [1] predicts a normal (or regular) scaling for moments of the velocity increment in the inertial range: $\langle|\Delta u_r|^q\rangle \sim r^{q/3}$ for $L \gg r \gg \eta$, where the longitudinal velocity increment is defined as $\Delta u_r = u(x + r) - u(x)$, $\langle\cdot\rangle$ denotes an ensemble average, and L and η are, respectively, the integral scale and the Kolmogorov dissipation scale. For convenience, we have considered here the moments of the absolute values of velocity differences—called the "generalized structure functions" to distinguish them from the classical ones. The normal scaling has been challenged often, and the departure from K41 has been reported from both experiments and direct numerical simulations [2–4]. A recent paper [5] used data from a high-resolution simulation and demonstrated the existence of intermittency correction to K41 *without* invoking the Taylor's hypothesis used in real-life experiments.

It has been believed traditionally that the deviations from K41 tend to be more pronounced when the moment order q becomes large. This is because the non-Gaussianity is more conspicuous at the tails of the probability density function (PDF) which make increasingly significant contributions to higher order statistics. Thus, experiments and simulations have focused on structure functions with $q \geq 2$, say. With the exception of Refs. [6] and [7], little attention has been paid to the scaling of low-order structure functions ($q < 2$, say). The preliminary experimental results of a water flow in a pipe in Ref. [6] showed that exponents for moment orders as low as 0.25 departed from K41. However, a detailed study connecting large amplitude events with structure functions and scaling exponents has not yet been made.

For one-dimensional Burgers turbulence in the small viscosity limit, a bifractal structure of velocity due to the formation of shock structures leads to a normal scaling for $q < 1$ and a constant scaling exponent for $q \geq 1$ [8]. This bifractal scaling has also been reported for turbulent convection with a constant temperature gradient [9]. One cannot, in principle, rule out a somewhat similar situation in three-dimensional turbulence at very high Reynolds numbers. It is therefore both interesting and useful to examine the nature of low-order scaling exponents.

In this Letter, we present an analysis using data from direct numerical simulations for a fully developed isotropic turbulence to study low-order moments of velocity structure functions and the relation between structure function exponents and the PDF. The particular questions we address here are as follows: (1) Are there intermittency corrections to the scaling of low-order structure functions? (2) Is there a transition from regular scaling at low order to anomalous scaling at high order? (3) What aspects of turbulence make major contributions to low-order structure function statistics?

Direct numerical simulation of the Navier-Stokes equations [10] was carried out with 512^3 mesh points in a cyclic cubic box for homogeneous isotropic turbulence using the CM-5 machine at Los Alamos and the SP machines at IBM. The simulation domain was $[0, 2\pi]$ in each direction. A nominal steady state was maintained by a forcing confined to wave numbers $k < 3$. The Taylor microscale $\lambda = (15\nu v_0^2/\epsilon)^{1/2}$ and microscale Reynolds number $\mathcal{R}_\lambda \equiv v_0\lambda/\nu$ were controlled by varying the viscosity of the system. Here v_0 is the root-mean-square value of a single vector component of velocity. The analysis was carried out for forced statistically steady states at $\mathcal{R}_\lambda = 218$. A spatial averaging over the whole physical space was used to replace the ensemble average. The separation r was taken along the x direction, and the PDFs and statistics shown below were averaged over 1.5 large-eddy turnover times.

In the inset of Fig. 1, we present the generalized structure functions

$$S_q(r) = \langle|\Delta u_r|^q\rangle \sim r^{\zeta_q}, \tag{1}$$

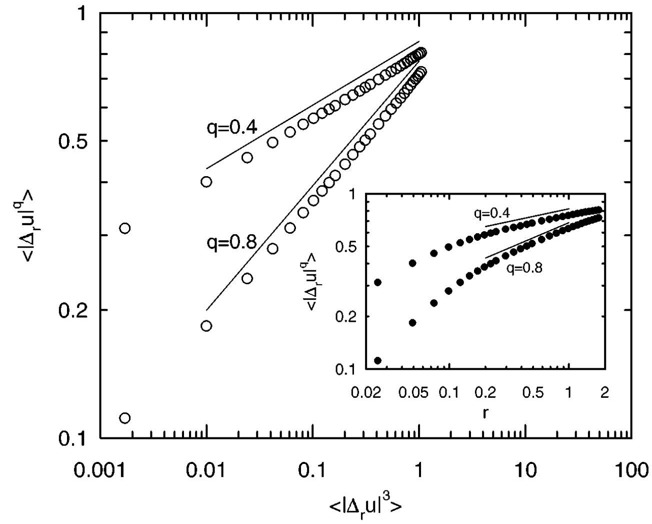

FIG. 1. Velocity structure function $S_q(r)$ for $q = 0.4$ and 0.8 versus $S_3(r)$. The inset shows the velocity structure functions versus r. The symbols are direct numerical simulation results, and solid lines indicate slopes from the SL model.

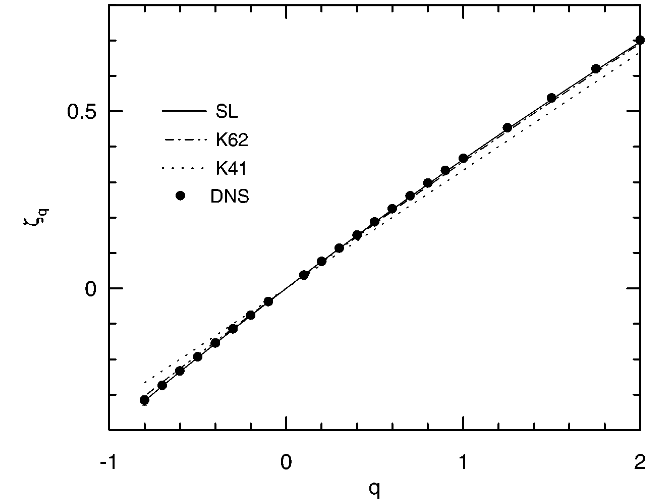

FIG. 2. Scaling exponents of structure functions as a function of the order index q compared with various models.

with $q = 0.4$ and 0.8. Here ζ_q is the scaling exponent. A power law range can be identified for $0.2 \leq r \leq 0.8$. In Ref. [6], we have shown that the third-order structure function in this region has the expected inertial range behavior [11]. The structure functions are compared with a phenomenological multiscaling model proposed by She and Leveque [12] (solid lines). To better calculate the scaling exponents, in Fig. 1 we have utilized the so-called extended scaling similarity (ESS) hypothesis advocated by Benzi *et al.* [13], i.e., plotting $S_q(r)$ against $S_3(r)$, and identified the relative scaling exponent, ζ_q. From our previous experiences [5] and the current study, we believe that the ESS works well for velocity structure functions in homogeneous turbulence—meaning that the relative scaling shows a wider region than the direct scaling region shown in the inset. We have studied many instances of direct and relative scalings for different power indexes [14], ranging from $q = -0.8$ to 12, and found the scaling relations to be very similar to those presented in Fig. 1.

In Fig. 2 we have compared the ESS scaling exponents for $-0.8 < q < 2$ with predictions from K41, a lognormal model by Kolmogorov (K62) [2] which gives $\zeta_q = q/3 + q(3 - q)\mu/18$ where $\mu = 2/9$ is the dissipation scaling exponent [15], and a log-Poisson model by She and Leveque (SL) [12] which gives $\zeta_q = q/9 + 2/3[1 - (2/3)^{q/3}]$. To be more quantitative, we list here five scaling exponents from direct numerical simulations: $\zeta_{-0.8} = -0.32 \pm 0.01$, $\zeta_{0.2} = 0.074 \pm 0.002$, $\zeta_{0.4} = 0.150 \pm 0.002$, $\zeta_{0.6} = 0.223 \pm 0.002$, and $\zeta_{0.8} = 0.296 \pm 0.002$. For very small q, ζ_q should be a linear function of q following a Taylor expansion. The difference of the slope of scaling exponents when $q \rightarrow 0$ between K41 and SL model (or K62) is about 12% [3]. It is evident that the scaling exponents deviate from K41 even for very small q, in favor of the intermittency models [2,12,16] for both positive and negative powers. Since the departure from K41 starts from very small q and the relative scaling exponents agree well with some intermittency models for q from -0.8 up to 8 [5], there is no evidence to support a transition of scaling exponents and the bifractal structure, in contrast to the case of Burgers equation [8] or pressureless turbulence [17].

As pointed out in [3], it is generally believed that the PDF of the velocity increment has a Gaussian core for small amplitude events and a stretched exponential shape for the tail part. The intermittency effect presented in Fig. 2 for low-order moments may imply two possibilities: (1) the low-order moments have substantial contributions from the high amplitude (tail) events; (2) the core part of the PDFs is also sufficiently non-Gaussian. To shed light on these possibilities, we have enlarged in Fig. 3 the core part PDF of Δu_r with the inset showing the whole PDF. While the deviation from the Gaussian is evident at the tails, even the core part of the PDF displays departures from Gaussianity. This departure may have significant contributions to low-order statistics.

To probe this further, we define a cumulative structure function,

$$S_q(r, \phi) = \int_{-\phi}^{\phi} |\phi'|^q P(\phi')\, d\phi', \qquad (2)$$

and the cumulation ratio

$$\gamma_q(\phi) = \frac{S_q(r, \phi)}{S_q(r)}, \qquad (3)$$

where $\phi' = \Delta u_r / S_{\rm rms}(r)$ and $S_{\rm rms}(r) = S_2(r)^{1/2}$ is the root mean square for a given separation r; $P(\phi')$ is the probability density function of ϕ'; and ϕ is the bound of truncated ϕ' domain. The ratio $\gamma_q(\phi)$ *quantitatively* defines a cumulative contribution from events with amplitude up to ϕ for the qth-order structure function. In

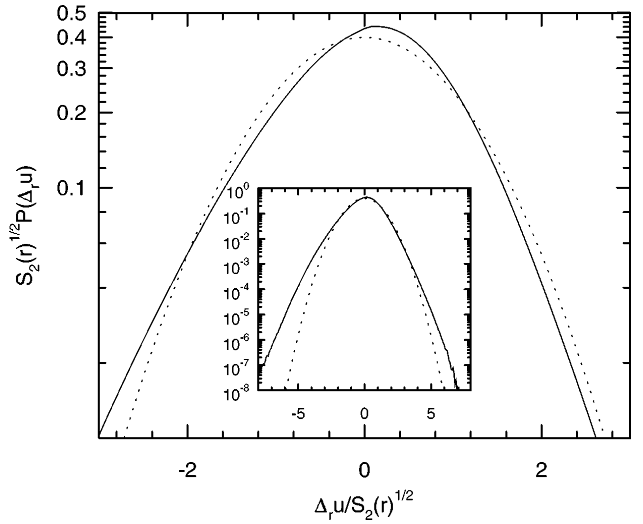

FIG. 3. The core part of the PDF for velocity increment (solid line) compared with Gaussian distribution (dotted line). The inset shows the whole distribution for the PDFs. All curves are normalized by the root mean square.

Fig. 4, we show $\gamma_q(\phi)$ as a function of ϕ for a separation distance in the inertial range ($r = 0.29$). For fixed $\gamma_q(\phi)$, we see that the larger the q the larger the ϕ, consistent with the fact that the peak in $\phi^q P(\phi)$ shifts to larger values of ϕ for increasingly high-order moments. We provide in the inset the complementary data on ϕ [the inverse function of γ in Eq. (3)] as a function of q when $\gamma_q(\phi) = 0.9$.

The aspect just discussed has been made more specific in Fig. 5 for the flatness and skewness of the velocity increment ($r = 0.29$). The data are compared with those of a Gaussian field. The departure from Gaussian statistics for the flatness starts from $\phi \simeq 1$, which might therefore usefully define the boundary between the core and the tail of the PDF. Both the skewness and the flatness converge to constants (3.6 and -0.32, respectively) when $\phi \simeq 4$, consistent with the inset in Fig. 4. It is interesting to note that the skewness has a crossover from positive to negative at $\phi = 2$, indicating that the negative skewness basically comes from events of intermediate amplitudes between about 2 and 4 standard deviations.

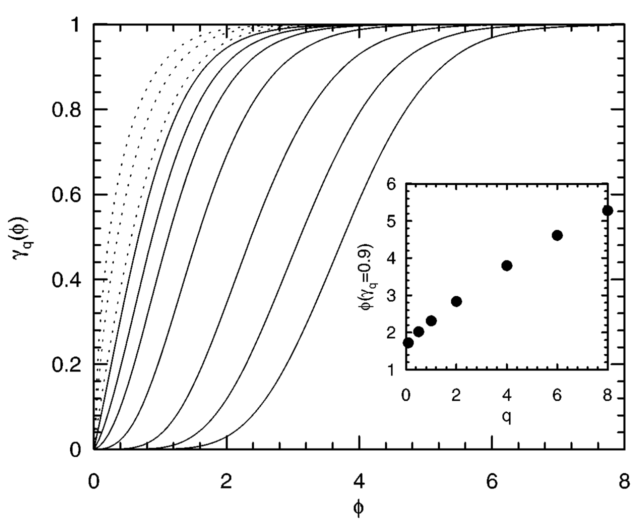

FIG. 4. The cumulation ratio $\gamma_q(\phi)$ as a function of ϕ for different q (from left to right): $q = -0.8$, -0.5, -0.1, 0.1, 0.5, 1, 2, 4, 6, 8. The separation distance $r = 0.29$ lies within the inertial range. The inset shows ϕ values corresponding to the intersections of the curves by the horizontal line given by $\gamma_q = 0.9$.

In order to see the effects of the PDF on scaling exponents, in particular, on the variation of scaling exponents with the variation of the upper bound, ϕ, it is useful to rewrite Eq. (1) as $\zeta_q(r) = d \ln S_q(r)/d \ln r$. We first note that the scaling exponents represent a *relative variation* of the structure function as a function of the separation distance. As in Eq. (2), we can now define the qth-order cumulative scaling exponent as

$$\zeta_q(r, \phi) = \frac{d \ln S_q(r, \phi)}{d \ln r}.$$

In Fig. 6, we show $\zeta_q(\phi)$, the average of $\zeta_q(r, \phi)$ over inertial range, as a function of ϕ for $q = 0.4$, 2, and 8. Two features in this plot are worth noting: First, for $\phi \simeq 1$, the scaling exponents for all three cases are close to K41. Second, all three cases converge to the SL multiscaling model when $\phi \simeq 6$. This result implies that—as far as the scaling exponents are concerned—there is no fundamental difference in the relation between structure function and the PDF of the velocity increment for high-order and low-order statistics. Note that the transition in Fig. 6 from K41 to the SL (or other) multiscale results is smooth, which seems to support the multifractal picture in which the correction to anomalous scaling comes from all amplitudes with different fractal dimensions.

It is not often appreciated that the non-Gaussianity of PDF of velocity increments is related to the phase coherence among velocity mode in Fourier space ("structure" in real space): when a pseudovelocity field is generated

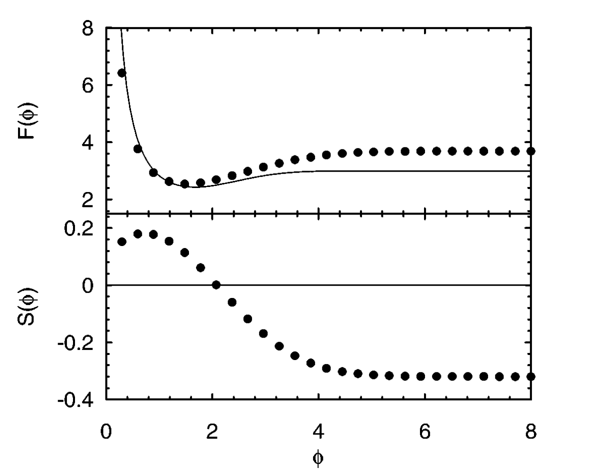

FIG. 5. The skewness and flatness factor as function of ϕ for $r = 0.29$ calculated from the data (bullets), compared with Gaussian values (solid lines).

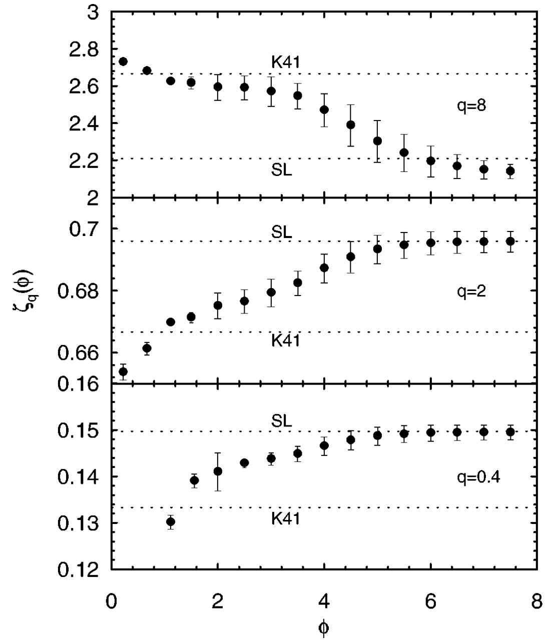

FIG. 6. Scaling exponents $\zeta_q(\phi)$ versus ϕ for $q = 0.4$, 2, and 8 for truncated structure functions. The values for complete structure functions from K41 and SL are also plotted for comparison.

by randomizing the phase for each mode (but not altering its amplitude), the resulting scaling exponents agree with K41. Although this result is not surprising, it confirms the direct link of intermittency and anomalous scaling to real space structures [18].

To summarize, we have reported a study of low-order scaling for the direct numerical simulation data on isotropic turbulence. The scaling exponents for low-order structure functions clearly deviate from K41 but agree well with existing multiscaling models. No transition from regular scaling to anomalous scaling can be identified. The dependence of structure functions and scaling exponents on amplitudes of the velocity increment are investigated. While the low- and high-order structure functions have primary contributions from small and large amplitude events, respectively, the scaling exponents for structure functions, regardless of the power index q, seems to have some contribution from all events.

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