Intermittency, the second-order structure function, and the turbulent energy-dissipation rate

Gustavo Stolovitzky[1,2] and K. R. Sreenivasan[2]
[1]Center for Studies in Physics and Biology, Rockefeller University, New York, New York 10021
[2]Mason Laboratory, Yale University, New Haven, Connecticut 06520-8286
(Received 3 April 1995)

In the context of an interpolation formula for a second-order structure function, Grossmann [Phys. Rev. E **51**, 6275 (1995)] considered various implications of the asymptotic behavior of the energy dissipation rate for inertial range intermittency. We reconsider the issue and show that the tendency of the nondimensional dissipation rate to asymptotically approach a constant is consistent with finite intermittency corrections. By extending Lohse's ideas [Phys. Rev. Lett. **73**, 3223 (1994)] put forth in a nonintermittent setting, we compute for intermittent turbulence the Reynolds number dependence of the nondimensional dissipation rate and show that the result compares favorably with experimental data.

PACS number(s): 47.27.Gs, 47.27.Jv

One of the interesting questions currently under investigation is the existence of inertial range intermittency in high Reynolds number turbulence. This issue, once thought to have been settled—at least experimentally (see Refs. [1–3])—has resurfaced in various ways recently (see, for example, Ref. [4]); the latest work in that direction is due to Grossmann [5]. In Ref. [5], Grossmann related a parametrized form of the second-order structure function to the asymptotic independence of the energy dissipation rate on fluid viscosity and made several interesting inferences (see below), although, in the end, he left open the issue of intermittency itself. In this paper we reexamine the issue much in the spirit of Grossmann's work and conclude that the asymptotic tendency to a constant of the nondimensional energy dissipation rate is consistent with finite intermittency corrections for the scaling exponent of the second-order structure function.

Let u be the x component of the velocity field, ν the kinematic viscosity of the fluid, $\langle \epsilon \rangle$ the time-averaged energy dissipation rate, η the Kolmogorov scale defined by $(\nu^3/\langle \epsilon \rangle)^{1/4}$, ζ_2 the second-order scaling exponent in the regime $r \gg \eta$, and r_c the scale at which the crossover occurs between the viscous and inertial ranges. For the second-order structure function, a useful interpolation formula [6–8,3], valid for both dissipative and inertial ranges, is

$$S_2(r) \equiv \langle [u(x+r)-u(x)]^2 \rangle = \frac{\langle \epsilon \rangle \eta^2}{15\nu} \frac{(r/\eta)^2}{[1+(r/r_c)^2]^{(2-\zeta_2)/2}} . \quad (1)$$

As shown in Ref. [3], this expression fits the data well. For $r \gg r_c$ we shall use the usual ansatz [9] for the scaling of $S_2(r)$,

$$S_2(r) = (L\langle \epsilon \rangle)^{2/3} C_2 (r/L)^{\zeta_2} . \quad (2)$$

It must be emphasized that, while some legitimate concerns do exist about the applicability of Eq. (2) over sizable scaling range at moderate Reynolds numbers, Eq. (2) forms the basis of much of what we know about three-dimensional turbulence at high Reynolds numbers.

Generalizing ideas originally proposed by Lohse [10] in a nonintermittent setting, Grossmann analyzed the relation between Eq. (1) and the expression for the energy dissipation in the form

$$\langle \epsilon \rangle L/u'^3 = C(R_\lambda) , \quad (3)$$

where L and u' are, respectively, the length and the velocity scales characterizing the large scale motion and $C(R_\lambda)$ is, in general, a function of the microscale Reynolds number $R_\lambda \equiv u'\lambda/\nu$, based on the Taylor microscale λ. For later convenience, we summarize Grossmann's observations in the following set of mutually exclusive statements.

(G1) If $\zeta_2 = \frac{2}{3}$ and r_c/η is independent of R_λ, then $C(R_\lambda \to \infty)$ is independent of R_λ.

(G2) If $\zeta_2 \neq \frac{2}{3}$ and r_c/η is independent of R_λ, the $C(R_\lambda \to \infty)$ depends on R_λ.

(G3) If $\zeta_2 \neq \frac{2}{3}$ and r_c/η is R_λ dependent (in an appropriate way), then $C(R_\lambda \to \infty)$ is independent of R_λ.

Now, provided r_c/η is independent of R_λ, (G1) states that $\zeta_2 = \frac{2}{3}$ is a sufficient condition for $C(R_\lambda \to \infty)$ to be independent of R_λ. On the other hand, according to (G2), if $C(R_\lambda \to \infty)$ does not depend on R_λ, then either $\zeta_2 = \frac{2}{3}$ or r_c/η depends on R_λ, or both. Based on these observations, Grossmann noted that the experimentally found approach to a constant level of $C(R_\lambda)$ is an argument in favor of ζ_2 being equal to $\frac{2}{3}$. However, it is clear from (G3) that the possibility exists that $\zeta_2 \neq \frac{2}{3}$, but that r_c/η depends on R_λ. This alternative was noted by Grossmann, who, however, did not favor it. Using the same kind of analysis as that used in [5], we will show that it is this last possibility that is favored by the preponderance of experimental evidence.

We start by noting that when $r = L$, $S_2 \equiv \langle [u(x+r) - u(x)]^2 \rangle \approx 2u'^2$, given that the velocity decorrelates at a distance given by the integral scale, at least for isotropic turbulence. Therefore, from Eq. (2) with $r=L$ and Eq. (3) we obtain the important relation [10]

$$C(R_\lambda)\approx\left[\frac{2}{C_2}\right]^{3/2}. \tag{4}$$

This relation allows us to draw the first conclusion as follows.

(a) For large enough R_λ (such that the limit $r\to L\gg r_c$ can be considered) $C(R_\lambda)$ is independent of R_λ if and only if C_2 is R_λ independent.

There is empirical evidence to suggest that $C(R_\lambda)$ becomes asymptotically independent of R_λ (see, for example, Ref. [11]); in particular, if L is the velocity autocorrelation length and u' is the root-mean-square velocity fluctuation, according to Ref. [11], $C(R_\lambda\to\infty)$ is almost exactly unity for grid turbulence. [For sheared turbulence, the constant $C(R_\lambda\to\infty)$ seems to depend weakly on nondimensional shear (see Ref. [12]), but that will not be considered here.] Therefore, according to Eq. (4), C_2 has to be a constant, independent of R_λ. This is consistent with the direct experimental observation of the constancy of the prefactor C_2 with R_λ. Reference [1] puts the numerical value between 1.8 and 2.2 and a more recent and extensive assessment [13] supports this conclusion.

We now use the constancy of C_2 to find consequences on r_c/η and ζ_2. For this purpose, we note that for $r\gg r_c$, Eq. (1) asymptotically approaches

$$S_2(r)=(L\langle\epsilon\rangle)^{2/3}\frac{1}{15}\left[\frac{r_c}{\eta}\right]^{2-\zeta_2}\left[\frac{L}{\eta}\right]^{\zeta_2-2/3}\left[\frac{r}{L}\right]^{\zeta_2}. \tag{5}$$

Comparing Eq. (5) with Eq. (2) we conclude (see also [5]) that

$$C_2=\frac{1}{15}\left[\frac{r_c}{\eta}\right]^{2-\zeta_2}\left[\frac{L}{\eta}\right]^{\zeta_2-2/3} \tag{6}$$

and extract the following statement.

(b) If C_2 is indeed independent of R_λ, then only one of the following two possibilities can be correct: (b′) $\zeta_2=\frac{2}{3}$ and r_c/η is independent of R_λ or (b″) $\zeta_2\neq\frac{2}{3}$ and r_c/η is R_λ dependent in an appropriate way.

Note that both (b′) and (b″) contain separate statements (about ζ_2 and r_c/η) that have to be met simultaneously. To make further progress we first recall [7,3] that the inertial-dissipative crossover scale r_c is related to the skewness of the velocity derivatives $S=-\langle(\partial u/\partial x)^3\rangle/\langle(\partial u/\partial x)^2\rangle^{3/2}$ through the relation

$$(r_c/\eta)^2=12\sqrt{15}(2-\zeta_2)\frac{1}{S}. \tag{7}$$

Equation (7) can be obtained from Kolmogorov's structure equation [14]

$$\langle[u(x+r)-u(x)]^3\rangle=-\frac{4}{5}\langle\epsilon\rangle r+6\nu\frac{\partial}{\partial r}\langle[u(x+r)-u(x)]^2\rangle \tag{8}$$

by expanding its left- and right-hand sides in powers of r and matching the coefficients of r^3. If we use Eq. (1) to represent the second-order structure function in Eq. (8), the definition of S, as well as the result valid for isotropic turbulence, namely, $\langle(\partial u/\partial x)^2\rangle=\langle\epsilon\rangle/(15\nu)$, we obtain Eq. (7).

Now, most measurements indicate that the skewness of the derivative has some dependence on the Reynolds number (see, for example, Fig. 1 of Ref. [15]). The one exception appears in the recent measurements of Ref. [16], but there are several unanswered questions about those data. It appears therefore that the preponderance of data supports an increasing trend of the skewness with respect to Reynolds number. This dependence naturally imposes an R_λ dependence on the crossover ratio r_c/η. It follows that (b′) cannot be true, leaving (b″) as the only possibility. Our conclusion, then, is that $\zeta_2\neq\frac{2}{3}$, which implies the validity of the inertial range intermittency picture.

It is possible to derive quantitative expressions for the dependence of S and r_c/η with R_λ within the present framework. Inserting Eq. (7) in Eq. (6) and using that $L/\eta=C(R_\lambda)R_\lambda^{3/2}/15^{3/4}$ [which follows from Eq. (3) and the definition of η], we obtain

$$S=\left[\frac{[C(R_\lambda)]^{4/3}}{15}\right]^{(3/2)(\zeta_2-2/3)/(2-\zeta_2)}\times\frac{12\sqrt{15}(2-\zeta_2)}{(15C_2)^{2/(2-\zeta_2)}}R_\lambda^{3(\zeta_2-2/3)/(2-\zeta_2)}. \tag{9}$$

(For $\zeta_2\approx0.7$, the R_λ dependence of S is of the order 0.07, comparable to the experimental value of about 0.12; see Ref. [15].) This expression had already been found in Ref. [3], following a similar derivation. Alternatively, the same reasoning leads to the expression

$$\frac{r_c}{\eta}=C_2^{1/(2-\zeta_2)}\frac{15^{(3/4)(\zeta_2+2/3)/(2-\zeta_2)}}{[C(R_\lambda)]^{(\zeta_2-2/3)/(2-\zeta_2)}}R_\lambda^{(3/2)(2/3-\zeta_2)/(2-\zeta_2)}. \tag{10}$$

For large enough R_λ, Eq. (10) shows that r_c/η exhibits a simple scaling with R_λ, whose scaling exponent coincides with the one found in Ref. [5] in the context of (G3) stated above.

This information can be used to obtain the R_λ dependence of $C(R_\lambda)$ within the picture of inertial range intermittency, thus extending previous work of Lohse [10] carried out in a nonintermittent setting. Grossmann [5] made similar calculations in the context of (G1) and (G2) (see Fig. 1 of Ref. [5]). Our derivation here corresponds essentially to the case (G3), which Grossmann did not consider. To this end, we observe that when $r=L$, $S_2(L)\approx2u'^2=2L\langle\epsilon\rangle=2L\langle\epsilon\rangle/[C(R_\lambda)]^{2/3}$; using Eq. (1) and again the result $L/\eta=C(R_\lambda)R_\lambda^{3/2}/15^{3/4}$, we arrive, after some algebra, at an implicit equation [17] for $C(R_\lambda)$

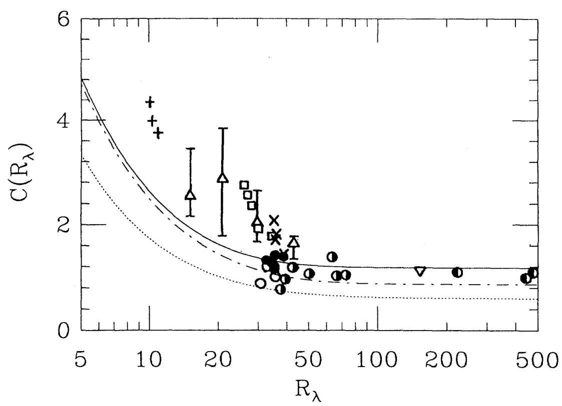

FIG. 1. Normalized dissipation rate in grid turbulence experiments as compiled in Ref. [11], where the various symbols have been explained. The dotted line is due to Lohse [10], appropriate to nonintermittent turbulence. The other two curves representing the present work use different values of C_2, with the solid line corresponding to $C_2=1.8$ and the dot-dashed line to $C_2=2.2$. These are the experimental bounds on C_2 according to Ref. [1].

$$1=\frac{R_\lambda^2[C(R_\lambda)]^2}{2\times 15^2}\times\left\{1+\left[\frac{15^2 C_2}{R_\lambda^2[C(R_\lambda)]^{4/3}}\right]^{2/(\zeta_2-2)}\right\}^{(\zeta_2-2)/2} . \tag{11}$$

In the limit of $R_\lambda \gg 1$, $C(R_\lambda)$ tends to the constant given by Eq. (4). The numerical solution to Eq. (11) is plotted in Fig. 1 for $\zeta_2=0.7$. (The solution with $\zeta=\frac{2}{3}$ is almost indistinguishable from that plotted.) Because of the experimental scatter in the values of C_2, we have plotted the solution to Eq. (11) corresponding to $C_2=1.8$ (solid line) and $C_2=2.2$ (dashed line), which form the accepted range of values according to Monin and Yaglom [1]. Also plotted in Fig. 1 are the experimental data (circles) for grid turbulence (from Ref. [11]) and the expression from Lohse's calculation (dotted line). The present expression yields a moderately better fit to the experimental data.

In summary, within the framework given by the parametrization of the second-order structure function given by Eq. (1), inertial range intermittency follows from the asymptotic constancy of the energy dissipation rate and the (weak) R_λ dependence of the crossover scale r_c/η. This R_λ dependence of r_c/η is supported by Eq. (7), relating r_c/η and the skewness of the velocity derivative S, and by the experimental observation (as far as can be said at present) that S depends (also weakly) on R_λ. The experimental dependence of r_c/η on R_λ can be used to obtain an equation for the function $C(R_\lambda)$ that agrees reasonably well with the experimental data.

We thank Siegfried Grossmann for sending us a copy of his work prior to publication. To him and to Detlef Lohse, we are grateful for useful comments on the draft.

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