Scaling Exponents near the Onset of Turbulence

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Velocity measurements made in the wake of a circular cylinder near the onset of turbulence have been analyzed to determine various scaling exponents. One conclusion is the apparent divergence of a correlation length as one approaches a "critical" Reynolds number. Other aspects analyzed are the scaling of turbulent energy and energy dissipation rate, as well as the circulation around square boxes of various sizes.

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When two distinct states of condensed matter coexist in equilibrium, the spontaneous transition from one state to another can be understood within the framework of a well-developed theory. While the situation with respect to nonequilibrium systems is far less satisfactory, one maintains a tantalizing hope [1] that some very similar ideas may be useful, if only by providing analogies, for describing symmetry-breaking processes at the macroscopic level. A comprehensive account of the recent progress made in this direction, with particular reference to pattern formation in fluid systems, can be found in Ref. [2]. Reference [3] provides a nice account of the theory and phenomenology of the structure of weak turbulence and illustrates key ideas with model examples. Mention should also be made of some prior experimental work done in a similar spirit on Rayleigh-Bénard convection [4] and on capillary ripples [5]. A more difficult problem is that involving the transition to three-dimensional "hard" turbulence; this was examined in a very preliminary fashion in Ref. [6] by noting that turbulent "spots" in boundary layers or "slugs" in pipe flows spread with the difference speed between the front and the back varying as $(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})^{1/2}$; here Re is the Reynolds number and the suffix cr stands for the critical value. In this paper, we measure some exponents when the wake behind a cylinder undergoes spontaneous transition from a laminar to a turbulent state at a "critical" Reynolds number (to be explained later): we are, however, well aware of the lack of formal and universal frameworks within which such observations can be explained.

Consider the flow past a circular cylinder of diameter D and length $\Lambda \gg D$. As the flow speed increases, a significant structural change that occurs in the wake of the cylinder is the appearance of vortex shedding; the onset value of the Reynolds number, $\mathrm{Re} = UD/\nu$, is around 47, where U is the oncoming flow speed and ν is the fluid viscosity. With some effort [7], these vortices can be maintained essentially two dimensional. Additional complexities that can arise in this state have been described, for example, in Refs. [8] and [9], but undoubtedly the next major transition is the appearance of three dimensionalities in vortex structures (see, for example, Refs. [10–12])—which carry the seed for turbulence in the form of "dislocations" or finite-sized spots of turbulence; the spots spread and coalesce as they propagate downstream and bring about transition to turbulence. This transition, which occurs at a Reynolds number of about 150, is relevant to our study here.

Fix attention on velocity fluctuations at some finite distance behind the cylinder, say, 100 diameters, and measure them simultaneously at several points along the cylinder span. Before the onset of three dimensionalities, these fluctuations are periodic (sinusoids with superimposed harmonics—see, for example, Fig. 1 of Ref. [8]) and well correlated along the span. In principle, this correlation extends over distances of the order of the cylinder span itself. As three dimensionalities develop, the translational invariance along the span is broken, and the motion on one side of a propagating spot becomes less well correlated with the motion on the other side. These spots appear apparently randomly in the flow, so a length scale defining the average extent of the spanwise correlation can be expected to be smaller than that prior to the onset of three dimensionalities. As the Reynolds number increases, the spots become more frequent and spread more rapidly [6], and the correlation scale decreases rapidly.

This qualitative description (while admittedly ignoring several finer aspects) suggests that the measurement in the vicinity of the transition Reynolds number, which we shall call critical, would be of interest. We have especially measured a correlation length scale, the mean-square fluctuation energy, energy dissipation rate, exponent of the circulation around closed contours, and extracted "scaling exponents" for each of them.

Measurements have been made in two nominally identical wakes. First, a wake was created behind a cylinder of diameter 0.318 cm in low-turbulence wind tunnel with a 30 cm × 30 cm working cross section, and 1 m in length. Well-resolved single-point velocity fluctuations were obtained along and (in some instances) across the wake width, using a standard hot wire approximately 5 μm in diameter and 0.6 mm long, operated at constant temperature on a DANTEC 55M01 anemomenter. In another experiment, the wake was created behind a cylinder of diameter 0.48 cm placed in a water tunnel comparable in size to the wind tunnel. In this wake, spatial data [13] in two

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dimensions were acquired by the so-called particle image velocimetry (PIV) method. The PIV technique involves the illumination of a plane of the seeded flow field by a pair of laser pulses separated by a small but finite time interval and capturing both pulses on a single frame of photographic film. Each particle pair on the frame conveys information on the local velocity field. When the developed film is interrogated with a beam of He-Ne laser, the spacing and orientation of the fringes can be converted to velocity data; here this was done using a software from FFD Inc. The accuracy of velocity data so obtained varied between 1% and 8%, and a typical value is about 5%. The velocity vectors were obtained on a grid size 44 × 66 pixels (a pixel was 1.36 mm in some and 1.8 mm in other experiments); the grid was centered at 50 cylinder diameters downstream [14]. The pixel resolution varied from about 2η at the lowest Reynolds number to about 5η at the highest Reynolds number, where η is the estimated Kolmogorov scale. Velocity data were obtained in both the transverse (x-y) and longitudinal (x-z) planes, where x is the flow direction upstream of the cylinder, y is the direction of maximum shear, and z is along the cylinder span.

Before considering the exponents, it would be useful to examine briefly the dramatic change occurring around a Reynolds number of ~150. While this has been known since the seminal measurements of Roshko [10], we may demonstrate this explicitly in the following way. In the Reynolds number range covered here, after an initial adjustment for a few diameters downstream, the mean-square velocity fluctuation monotonically decays with distance away from the cylinder, as shown in Fig. 1 for two typical Reynolds numbers. The decay is exponential for Re < 150 and can be written as [15]

$$u' \equiv \langle u^2 \rangle^{1/2} = A \exp\left(-\lambda \frac{x}{D}\right), \tag{1}$$

where A and λ are functions of the Reynolds number. The variation of λ as a function of the Reynolds number is shown in the inset to Fig. 1. The downstream spatial decay for Re < 150 is largely due to the fact that vortices of opposite signs—emerging with opposite signs from the two sides of the cylinder—annihilate each other; this effect can be estimated to be an order of magnitude or so stronger than the viscous effect. As the Reynolds number increases, the rate of decay, viz. λ, decreases (presumably because the vortices become more compact). At Re ≈ 150, λ fluctuates strongly from one experiment to another—and depends on a variety of conditions, not all of which are understood. For Re > 150, the downstream development is more complex. However, the peaks in the power spectrum at the vortex-shedding frequency can be fitted roughly by an exponential, at least for short distances (say, between 15 and 45 diameters at Re ≈ 1000), but the decay constant now *increases* with Re. This is presumably so because the flow becomes intermittently turbulent via spots, thus augmenting the usual decay by turbulent dissipation. Thus Re = 150 marks a change in the flow character.

We shall now return to scaling exponents near this critical Reynolds number of ~150. Figure 2 shows the variation of four other quantities of interest for Re between ~190 and ~1000. The correlation length L in Fig. 2(a) is defined as

$$L = \int_0^{\xi} dr_z \frac{\langle u(z_0)u(z_0 + r_z)\rangle}{\langle u(z_0)^2\rangle}, \tag{2}$$

where ξ is the first zero crossing of the integrand and the separation distance r_z is in the spanwise direction z; note that z is the direction of homogeneity (away from the walls of the flow facility), and so the integrand depends only on r_z. The data for Re < 1000 can be fitted adequately by

$$L/D \approx 40(\mathrm{Re} - 150)^{-0.42}. \tag{3}$$

The uncertainty in the exponent is of the order ±0.05. Figure 2(b) shows that u'/U_0 can be fitted by [16]

$$u'/U_0 \approx 0.2(\mathrm{Re} - 150)^{-0.16}. \tag{4}$$

The uncertainty in this exponent is of the order ±0.03. Needless to say, the square of the left-hand side of Eq. (4) is proportional to the fluctuation kinetic energy.

Consider now the average energy dissipation rate per unit mass $\langle \epsilon \rangle$. For Re < 150, this quantity can be

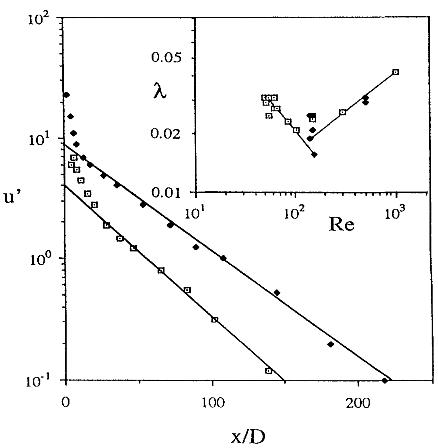

FIG. 1. Typical data on the spatial decay of root-mean-square velocity fluctuation (expressed in arbitrary units) as a function of the distance downstream of the cylinder (normalized by its diameter); Re = 62 and 95, respectively, for the lower and upper curves. Measurements were made along a line where the fluctuation intensity was the largest. The spatial decay is closely exponential, except for the vicinity of the cylinder. The spatial decay constant λ in Eq. (1), obtained from measurements at various Reynolds numbers, is plotted (see inset) as a function of Re. The inset shows a nearly sudden change in flow behavior at the critical Reynolds number of about 150. Different symbols correspond to marginally different experimental conditions. The fluctuation energy integrated across the wake width, obtained for a few Reynolds numbers, also shows exponential decay; so, too, does the root-mean-square velocity fluctuation in the direction y.

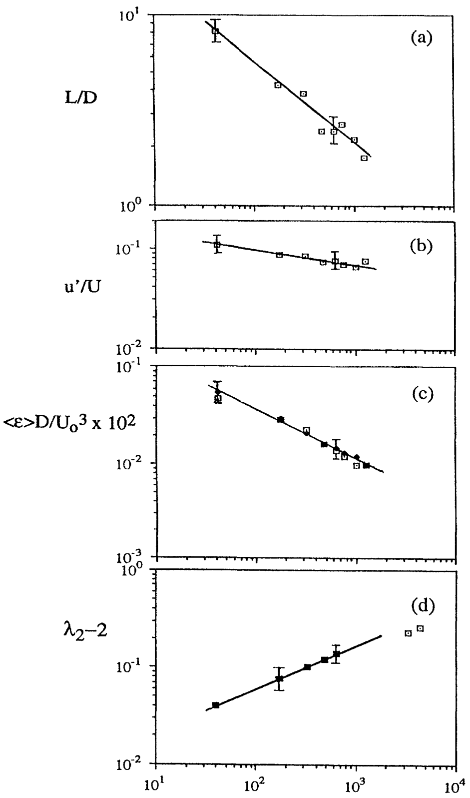

FIG. 2. As the Reynolds number increases upward of ~150, the length scale, velocity scale, and energy dissipation rate, as well as the scaling exponent for mean-square circulation, behave as shown, respectively, in (a)–(d). All data are obtained from PIV measurements. Detailed measurements for Re > 1000 necessitated the use of a bigger cylinder, thus changing (unfortunately) experimental conditions such as the aspect ratio; they are, therefore, not included here. In general, deviations from power laws do set in beyond Re ≈ 1000. In (d), $\lambda_2 = 2$ would be a trivial value, and so only the departures from that value would be of interest. Where explicitly not marked, the uncertainty in the data is indicated to some degree by the scatter; a few related issues of accuracy have been explored in Ref. [25].

evaluated easily from the exponential decay illustrated in Fig. 1 and can be shown to be rather small. The decay rate for Re > 150 has been measured in several different ways. First, the hot-wire data have been used to get the time derivative of velocity, which itself was converted to spatial derivative by the use of Taylor's hypothesis, and the dissipation was obtained using the assumption of local isotropy [viz., $\langle\epsilon\rangle = 15\nu\langle(\partial u/\partial x)^2\rangle$]. Second, the PIV data were also used to obtain dissipation. While the PIV data obviate the need for Taylor's hypothesis, the resolution of velocity measurements was not adequate, and so $\langle\epsilon\rangle$ was obtained as follows. A crude estimate was first obtained by calculating the velocity derivative from the insufficiently resolved data. Assuming a universal form of the dissipation spectrum [17], an assessment was made of the amount by which the dissipation was underestimated due to insufficient resolution, and a new estimate was obtained. The procedure was repeated once more. The final dissipation estimates were comparable to those obtained using good-resolution hot wires in the wind tunnel and to those obtained from Kolmogorov's $\frac{4}{5}$th law [18]. Figure 2(c) suggests that the dissipation rate can be fitted moderately well as

$$\langle\epsilon\rangle D/U_0^3 \approx 4 \times 10^{-3}(\mathrm{Re} - 150)^{-0.5}, \qquad (5)$$

with an uncertainty in the exponent of the order ± 0.05.

Finally, we consider the scaling exponent λ_2 of the mean-square circulation around a box of side r, defined as

$$\langle\Gamma_r^2\rangle \sim r^{\lambda_2}, \qquad (6)$$

where the line integral

$$\Gamma_r = \oint \mathbf{u} \cdot d\boldsymbol{l} \qquad (7)$$

is the circulation around a box of size r. The exponents λ_2 (as well as higher-order moments) were measured for a range of Reynolds numbers in Ref. [19]. Data in Fig. 2(d) show that

$$\lambda_2 - 2 \approx 7 \times 10^{-3}(\mathrm{Re} - 150)^{0.46} \qquad (8)$$

is a reasonable fit. (Note that the circulation scaling exponent can be defined with less ambiguity than those for structure functions; see Refs. [19] and [20].)

Note must be made of some unsatisfactory aspects of the results just summarized. The length scale data are vulnerable to several uncertainties, indicated partially by the scatter: the difficulties in exercising precise control on transitional wakes, large amounts of data needed to compute the integrand in Eq. (2), the very definition of the length scale when the covariance becomes negative for large separation distances, and so forth. The numerical values of the exponents show some dependence on the precise value of the critical Reynolds number assumed. While we made efforts to determine an optimal value for the critical Reynolds number by requiring the best power-law fits, we finally thought it better to determine the critical Reynolds number independently (as in Fig. 1) and retain that value throughout.

In conclusion, we have shown that near the critical Reynolds number of ~150 in the cylinder wake, various quantities of interest follow power laws over a nontrivial Reynolds number range. Of particular interest is the apparent divergence of a correlation length scale. This feature is by no means unique to the wake and has already been noted in two other flows [4,5]. As an additional illustration, consider a jet of fluid exiting a circular orifice, and fix attention on the flow at a distance of, say, one or two orifice diameters. When the Reynolds number is small, the flow rolls up in the form of vortex rings and the velocity fluctuations are correlated over the entire azimuthal angle of 2π, but the correlation diminishes as the Reynolds number is increased because of the irregular appearance of turbulence activity [21].

It may well be true that the present observations do not have a simple interpretation in terms of critical phenomena. This issue will not be clear until an appropriate theory based on the fluid equations has been advanced. We believe, however, that the present line of inquiry is useful at least in the following context. The scaling exponents traditionally explored in the turbulence literature correspond to high Reynolds numbers with $\mathrm{Re} = \infty$ assumed to be the "critical point" [22]. However, all varieties of high-Reynolds-number scaling have an upper cutoff at the so-called integral length scale, which is closely related to the correlation scale defined by Eq. (2). This cutoff scale, in conjunction with an appropriate velocity scale, is responsible for much of the turbulent transport. In fully developed turbulence, these are, in fact, the scales which control the rate of energy dissipation. A natural question to ask, then, is: What sets these length and velocity scales? If one views that they are related in some way to their counterparts at the onset of turbulence, the question to ask is: How do the length and velocity scales near the onset vary with the Reynolds number? This is precisely the issue explored here.

To illustrate the usefulness of these measurements, note by extrapolation of Eq. (8) that λ_2 reaches the Kolmogorov value of $\frac{8}{3}$ at a Reynolds number of about 2×10^4. This should be treated as a rough estimate for the asymptotic Reynolds number for which circulation (and perhaps other quantities as well) approach Kolmogorov characteristics. It follows from Eqs. (3) and (4) that $u'L = 0.026 U_0 D$ at this Reynolds number. This should be compared with the asymptotic result that $u'L = 0.021 U_0 D$, obtained by self-preservation analysis [23] in the wake. Further, at this same Reynolds number, we obtain from Eqs. (3)–(5) that $\langle \epsilon \rangle L / u'^3 = 0.26$, which should be compared with the asymptotic result, obtained empirically, that $\langle \epsilon \rangle L / u'^3 = 0.4$ for this flow [24]. We believe that the reasonable agreement observed here is not fortuitous.

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