Kolmogorov's $\frac{4}{5}$th Law and Intermittency in Turbulence

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An exact relation valid for fluid turbulence at high Reynolds numbers is that the third moment of velocity increments scales linearly with the separation distance in the inertial range. Experiments show that this holds true also for absolute values of velocity increments. It is argued that inertial-range intermittency is a plausible consequence of this observation, and a model supplementing these considerations is provided.

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In high-Reynolds-number turbulent flows, velocity fluctuations that occur on scales small compared to the large scale ℓ, say, are presumed to possess universal properties. Kolmogorov's 1941 phenomenology [1], denoted by K41 for short, has provided an appealing framework for understanding these universal properties. Since the time of Landau's famous remark [2], however, it has been recognized that intermittency is one of the important ingredients missing in K41. Many experiments [3] have confirmed the existence of intermittency of dissipative scales characterized by the so-called Kolmogorov scale r_* and various phenomenological attempts [4,5] have been made to describe dissipation-scale intermittency. However, the situation is less clear with respect to intermittency in the inertial range (roughly the range of scales between ℓ and r_*). As we shall discuss shortly, doubts have been expressed recently about the very existence of inertial range intermittency. Given its fundamental importance, it seems desirable to settle this question in a rigorous way. This Letter advances the issue to some degree, though it falls short of the goal.

To understand the prevailing controversy over inertial-range intermittency, consider the so-called longitudinal structure functions defined as $\langle(\Delta v_r)^n\rangle = \langle[v(x+r) - v(x)]^n\rangle$, where $v(x)$ is the x component of velocity field, r is measured along x, and n is a positive integer. K41 requires [6] that

$$\langle(\Delta v_r)^n\rangle = C_n(r\langle\epsilon\rangle)^{\xi_n}, \tag{1}$$

where $\langle\epsilon\rangle$ is the global average of ϵ, the energy dissipation rate per unit mass, C_n are universal constants, and $\xi_n = n/3$. In contrast to K41, a variety of measurements [7,8] has consistently shown that

$$\xi_2 > \tfrac{2}{3}, \quad \xi_3 = 1, \quad \text{and} \quad \xi_n < n/3 \text{ for all } n > 3\,. \tag{2}$$

The difference $(n/3 - \xi_n)$ increases with n. While the measurements fall short of the ideal in several ways, it is generally agreed that the results (2) are correct. This nontrivial scaling of high-order structure functions is the strongest argument in favor of inertial-range intermittency. One interpretation of (2) follows from Obhukov's [9] procedure of replacing $\langle\epsilon\rangle$ in Eq. (1) by the local average ϵ_r given by $\epsilon_r = V^{-1}\int \epsilon\, dV$, where V is a ball of radius r. Because of the intermittency of ϵ, $\langle\epsilon_r^n\rangle \neq \langle\epsilon\rangle^n$ in general. Taking $\langle\epsilon_r^n\rangle$ to be of the form $\langle\epsilon\rangle^n(r/\ell)^{-\mu_{3n}}$, where the μ_n are the (unknown) intermittency exponents, Eq. (1) becomes $\langle(\Delta v_r)^n\rangle = C_n\langle\epsilon\rangle^{n/3}(r/\ell)^{n/3-\mu_n}$, consistent with Eq. (2). This relation is a consequence also of Kolmogorov's refined similarity hypothesis [4,10].

For some time now, it has been argued [11] that ϵ could essentially represent an independently intermittent field and that the K41 result for second-order structure function could well be exact. In addition, there are also claims [12] that the observed deviations from K41 could arise from artifacts of finite Reynolds numbers, finite shear, vicinity to solid boundaries, and so forth. Some recent work [13], which replaces the Fourier sum for velocity by a finite subset of wave vectors—thus allowing only restricted interaction between Fourier amplitudes to occur—shows that the scaling exponents in Fourier space are closely given by K41. Within the realm of their validity, these calculations cast doubts on the existence of intermittency in the inertial range, or Kolmogorov's refined similarity hypotheses, or both.

It is difficult to dispel these doubts conclusively from experiment alone: At the least, an extensive program of measurements in very high-Reynolds-number flows under various combinations of shear and boundary effects is needed. It would be more effective if the issue can be resolved by showing, if that is indeed the case, that intermittency in the inertial range is a necessary consequence of the Navier-Stokes equations. Perhaps the only inertial-range result known to follow at high Reynolds numbers from the Navier-Stokes equations is the so-called Kolmogorov's $\frac{4}{5}$th law [14] given by

$$\langle(\Delta v_r)^3\rangle = -\tfrac{4}{5}\langle\epsilon\rangle r\,. \tag{3}$$

This result [15] is obtained by (justifiably) dropping the viscous term in the full equation valid for all scales

$r \ll \ell$. The analogous equation for passive scalars, which can be found in [16], has been studied recently in some detail in Ref. [18].

The question now is whether inertial-range intermittency is contained in Eq. (3). Some preliminaries are in order. First, denote the *generalized* structure functions as $S_q(r) = \langle |\Delta v_r|^q \rangle = C'_q v_\ell^q (r/\ell)^{\zeta_q}$, where q is a real number and v_ℓ is the root-mean-square velocity. It is clear that even-order integer moments in Eq. (1) coincide with ζ_q. Experiments ([8]; see also Table I) show that $\xi_n \gtrsim \zeta_n$ for odd n and nearly coincide for $n = 3$ (i.e., $\zeta_3 \approx \xi_3 = 1$).

Second, we shall also use the *generalized sign-structure functions* defined as

$$S'_q(r) = \langle |\Delta v_r|^q \operatorname{sgn}\{\Delta v_r\} \rangle = C_q v_\ell^q (r/\ell)^{\zeta'_q}, \qquad (4)$$

where $\operatorname{sgn}\{x\} = 1$ if $x > 0$, $= 0$ if $x = 0$, and $= -1$ if $x < 0$. Clearly, both the absolute value signs and the $\operatorname{sgn}\{x\}$ function can be omitted for odd q.

Finally, define $\Delta v_\pm(x, r) = \frac{1}{2}[|\Delta v_r(x)| \pm \Delta v_r(x)] \geq 0$, and consider $\Delta v_r(x)$ at the cutoff scale, $r = r_*$, where velocity and vorticity fluctuations are smoothed out by viscosity. It is clear that $\Delta v_{r_*}(x) \approx \partial_x v(x) r_*$, so that $\Delta v_{r_*}(x)$ corresponds to a velocity derivative (same as vorticity, in one dimension). Thus, $\Delta \tilde{v}_\pm(x) \equiv \Delta v_\pm(x, r_*) = \frac{1}{2}[|\Delta v_{r_*}(x)| \pm \Delta v_{r_*}(x)]$ would be two distributions (nonnegative functions), representing positive and negative parts of the $\partial_x v(x)$ process. We can therefore consider separately the two distributions $\Delta \tilde{v}_\pm(x)$ instead of the distribution $|\partial_x v(x)|$.

Consider a unit interval and divide it into boxes of size r'. The number of boxes is $N = 1/r'$. Define the measures of ith subinterval, $\mu_\pm(r') = \langle \Delta \tilde{v} \rangle^{-1} \times \int_{x_i}^{x_i + r'} \Delta \tilde{v}_\pm(x)\, dx$. Here, we have used the fact that $\langle \partial_x v \rangle = 0$, which means $\langle \Delta \tilde{v}_+ \rangle = \langle \Delta \tilde{v}_- \rangle = \langle \Delta \tilde{v} \rangle$. Denoting $r'^{-1} \int_{x_i}^{x_i + r'} \Delta \tilde{v}_\pm(x)\, dx$ by $\Delta \tilde{v}_\pm(r')$, we have, according to [19],

$$\langle \Delta \tilde{v}_\pm(r')^q \rangle = C_q^\pm \langle \Delta \tilde{v} \rangle^q (r'/\ell)^{-(1-D_q^\pm)(q-1)}, \qquad (5)$$

where $D_q^\pm$ are the generalized dimensions for positive and negative parts of the process $\partial_x v(x)$.

Note that the two sets $\Delta \tilde{v}_+ \geq 0$ (and $\Delta \tilde{v}_- = 0$) and $\Delta \tilde{v}_- \geq 0$ (and $\Delta \tilde{v}_+ = 0$) do not intersect. Averaging over these two sets separately, we obtain the formula

$$\langle |\partial_x v|^q \operatorname{sgn}\{\partial_x v\} \rangle = \{\langle \Delta \tilde{v}_+^q \rangle - \langle \Delta \tilde{v}_-^q \rangle\} r_*^{-q}. \qquad (6)$$

Then, the use of Eq. (5) considered at $r' = r_*$, together with Eq. (6), yields

$$\langle |\partial_x v|^q \operatorname{sgn}\{\partial_x v\} \rangle = \langle \Delta \tilde{v} \rangle^q r_*^{-q} \left[C_q^+ \left(\frac{r_*}{\ell} \right)^{-(1-D_q^+)(q-1)} - C_q^- \left(\frac{r_*}{\ell} \right)^{-(1-D_q^-)(q-1)} \right]. \qquad (7)$$

This formula should yield $\langle \partial_x v \rangle = 0$ for $q = 1$, thus implying that $C_1^+ = C_1^-$. However, it could give nontrivial results for $q \neq 1$, e.g., for odd moments.

It is not possible to determine the relative magnitudes of C_q^+ and C_q^- in Eq. (7) for general q, but a plausible assumption is that they are of the same order of magnitude. This assumption is tantamount to neglecting intermittency in the energy-containing eddies of size ℓ. We shall assume their equality, but note that the conclusions to follow are not sensitive to the assumption—at least for high-order moments. It then follows that Eq. (7) vanishes if $D_q^\pm = 1$; it also vanishes if these two distributions have identical multifractal structure, $D_q^+ = D_q^- \neq 1$. Only if, say, $D_q^- < D_q^+ \leq 1$, can the first term on the right-hand side of (7) be neglected, allowing us to write nontrivial power laws for the moments as

$$\langle |\partial_x v|^q \operatorname{sgn}\{\partial_x v\} \rangle = -C_q^- \langle \Delta \tilde{v} \rangle^q r_*^{-q} (r_*/\ell)^{-(1-D_q^-)(q-1)}. \qquad (8)$$

If $D_q^+ < D_q^- \leq 1$, the first term in Eq. (7) will prevail and the moments are again nontrivial. The implication is that $\partial_x v$, which is dominated by one of the two terms in Eq. (7), has nontrivial scaling given by the exponents D_q^+ or D_q^-. This is the essence of intermittency.

The physical meaning of this result can be explained in the framework of a ramp model, e.g., Ref. [20]. If the velocity field has ramplike structure (Fig. 1), its derivative has the form depicted in Fig. 2. It is clear that we have $\langle \partial_x v \rangle = 0$ while $\langle (\partial_x v)^3 \rangle \neq 0$. It is also obvious that the negative part of the structure in Fig. 2 occupies less volume than the positive. In other words, the negative part is "more intermittent," or its flatness factor is larger. If the ramp is steep enough, then the dimensions $D_q^- < 1$, whereas $D_q^+ = 1$. In a statistical ensemble, this structure

TABLE I. The odd-order scaling exponents for classical structure functions and generalized structure functions. For a discussion of the convergence of data and error estimates, see Zubair [8] for where these numbers have been obtained. Although the error bars on both ζ and ξ are substantial and increase with the order of the moment, the two sets of exponents are consistently different as shown.

	Odd-order exponents	
n	ξ_n	ζ_n
3	1.00	0.97
5	1.53	1.48
7	1.96	1.89
9	2.40	2.30
11	2.82	2.67

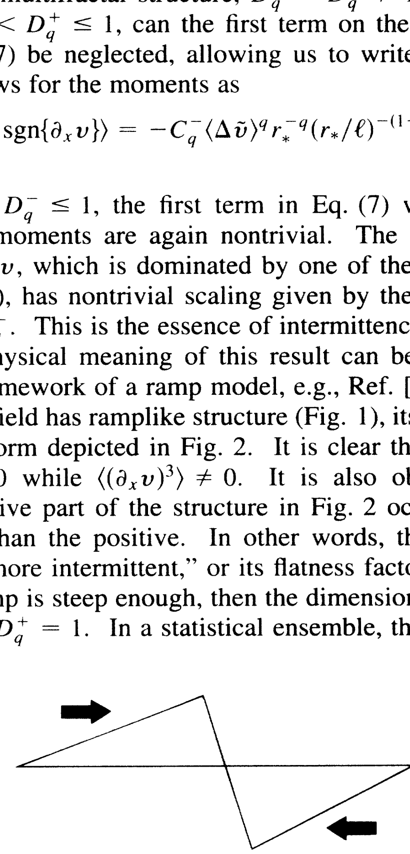

FIG. 1. The ramp structure. The arrows indicate the directions of motion. The left part of the ramp is moving to the right and the right part to the left, approaching the stagnation point at the origin.

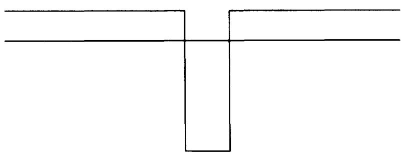

FIG. 2. The derivative of the ramp.

presumably occurs more often than the "antiramp" for which the motion is away from the steep part. In this model, the term in Eq. (7) corresponding to D_q^- should prevail in the limit, thus accounting for the nonvanishing of odd-order structure functions. In particular, the case with $q = 3$ accounts for nonvanishing skewness.

Using Eq. (6) with + instead of −, one obtains, analogous to Eq. (7),

$$\langle |\partial_x v|^q \rangle = \langle \Delta \tilde{v} \rangle^q r_*^{-q} \left[C_q^+ \left(\frac{r_*}{\ell} \right)^{-(1-D_q^+)(q-1)} + C_q^- \left(\frac{r_*}{\ell} \right)^{-(1-D_q^-)(q-1)} \right]. \quad (9)$$

Again the first term can be neglected for $D_q^- < D_q^+ \le 1$, yielding an expression that coincides with (8). This might explain the experimental fact (see Table II) that the ratio

$$R = \frac{|\langle (\partial_x v)^n \rangle|}{\langle |\partial_x v|^n \rangle} \lesssim 1, \quad (10)$$

for odd values of n, whereas, in principle, it could have been much smaller than unity. Table II also shows that R tends to unity for large n (although the approach is slow), consistent with the physics of the ramp model. Note that the expression (9) contains only one power law. This is obvious for $D_q^+ = D_q^-$. Otherwise only the dominant term prevails; for example, if $D_q^- < D_q^+$, one would have

$$\langle |\partial_x v|^q \rangle = C_q^- \langle \Delta \tilde{v} \rangle^q r_*^{-q} (r_*/\ell)^{-(1-D_q^-)(q-1)}. \quad (11)$$

We now replace $\langle \Delta \tilde{v} \rangle r_*^{-1}$ by $\langle |\partial_x v| \rangle$ in Eq. (11) and note that

$$\langle |\partial_x v| \rangle = (v_\ell/\ell)(r_*/\ell)^{-\kappa}. \quad (12)$$

This defines the so-called cancellation exponent κ [21]. In addition, we note that $\langle |\partial_x v|^q \rangle = S_q(r_*)/r_*^q$ and suppose that there is only one range of scaling, corresponding to the inertial range where Eq. (3) is fulfilled, and that is $r_* \lesssim r < \ell$. Then, there is only one scaling that fits $S_q(\ell) = C_q v_\ell^q$ on the one end and (11) and (12) on the other [22]. This scaling is given by

$$S_q(r) = C_q^- v_\ell^q (r/\ell)^{(1-\kappa)q-(1-D_q^-)(q-1)}, \quad (13)$$

an expression obtained in [23].

Similar calculations in regard to Eq. (8) yield

$$\langle |\Delta v_r|^q \operatorname{sgn}\{\Delta v_r\} \rangle = v_\ell^q \left(\frac{r}{\ell} \right)^{(1-\kappa)q} \times \left[C_q^+ \left(\frac{r}{\ell} \right)^{-(1-D_q^+)(q-1)} - C_q^- \left(\frac{r}{\ell} \right)^{-(1-D_q^-)(q-1)} \right]. \quad (14)$$

Thus, again, in order to have nonvanishing (14)—or, in particular, nonvanishing odd moments—one of the terms on the right-hand side of Eq. (14) should prevail. This is equivalent to the intermittency in either the positive or the negative part of the structure function in the inertial range. It was assumed earlier that $C_q^+ \approx C_q^-$; this detail was not important for the estimation of (8) from Eq. (7). The situation is different with Eq. (14), however. The ratio r/ℓ is not necessarily small and approaches unity towards the upper end of the inertial range. It implies that this expression has poorer scaling, as $r \to \ell$, in comparison with generalized structure functions (13). The second power law in (14) could also explain the depletion of odd-order structure function exponents, see Table I. Indeed, attempting a power law with a single exponent results in a decrease of the slope due to the second term in Eq. (14).

We might now reiterate our arguments to support the connection between inertial-range intermittency and Kolmogorov's $\frac{4}{5}$th law. We have shown that the nonzero value of the generalized third-order structure function implies that either negative or positive part of Δv_r possesses scaling *at all scales of the inertial range* and that the scaling exponents are nontrivial. In a formal sense, this is intermittency. We have argued that this picture is consistent with the nonzero values of the skewness and the ramp model for turbulence structure.

The physical picture can be expanded as follows. The ramp depicted in Fig. 1 could result from the steepening of the velocity profile, analogous to the steepening of a shock front. We may then say that in Fig. 1 the two shocks [24] are moving in opposite directions to meet at the origin, which is the stagnation point. It is at the stagnation point that the steepening occurs. Not only does the nonvanishing of the third-order structure function imply turbulent cascade but it also implies the presence of sharpening structures. In other words, along with local interaction resulting in large eddies decaying into smaller ones, the vortex structures are stretched out and compressed into sheets and filaments (analogous to the steepening of the front). This later decreases the

TABLE II. The ratio R [see Eq. (10)] for a typical scale in the inertial range. This ratio approaches unity as the order of the moment increases, but the approach is slow.

n	Ratio of moments R
1	0
3	0.17
5	0.35
7	0.49
9	0.59
11	0.63

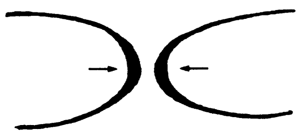

FIG. 3. Formation of intermittent structures due to the interaction of the two fluid thick parabolic lines. This leads to the formation of ramp structures in velocity distribution and sharpened vorticity distribution.

dimensions of the turbulence or, equivalently, increases the intermittency. A conjectural scenario of a filament formation in incompressible flow, corresponding to the formation of the ramp in Fig. 1, is depicted in Fig. 3. The figure corresponds to two eddies which are approaching each other. The front is steepened at the stagnation point between them, and the occurrence of the large strain rates near the stagnation point could produce intermittent structures.

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