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Self-similar multiplier distributions and multiplicative models for energy dissipation in high-Reynolds-number turbulence

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Abstract

We begin with a brief description of the multiplier distribution for ε_r, the average over a linear interval r of the energy dissipation rate, ε. Using measured multiplier distributions obtained for atmospheric surface layer data on ε, we show that quasi-deterministic multiplicative models for bases 2 and 3 (that is, binary and tertiary breakdown processes) can be developed on a rational basis. For r in the inertial range, moments computed up to a fairly high order from these models are found to be in good agreement with experimental values. For bases larger than three, such quasi-deterministic approximations for multiplier distributions are not possible. Some applications of multiplier distributions are presented.

1. INTRODUCTION

A Gaussian process is completely described in a statistical sense by its mean and standard deviation. It is conceivable that a nearly Gaussian process can be described well by its first few moments – at least well enough for many purposes. This is the situation with respect to velocity or temperature traces obtained in high-Reynolds-number fully turbulent flows not too close to the wall. On the other hand, the situation is quite different for quantities such as the energy dissipation rate, ε, in high-Reynolds-number turbulence. Figure 1 is (effectively) a one-dimensional section through the field of ε in the atmospheric surface layer a few meters over land. In contrast to Gaussian or nearly Gaussian processes, information about the first few low-order moments does not describe the signal in any detail. Peaks which are hundreds of times the mean are not uncommon, and the signal is at other times of very low amplitude; this strongly intermittent character is a generic

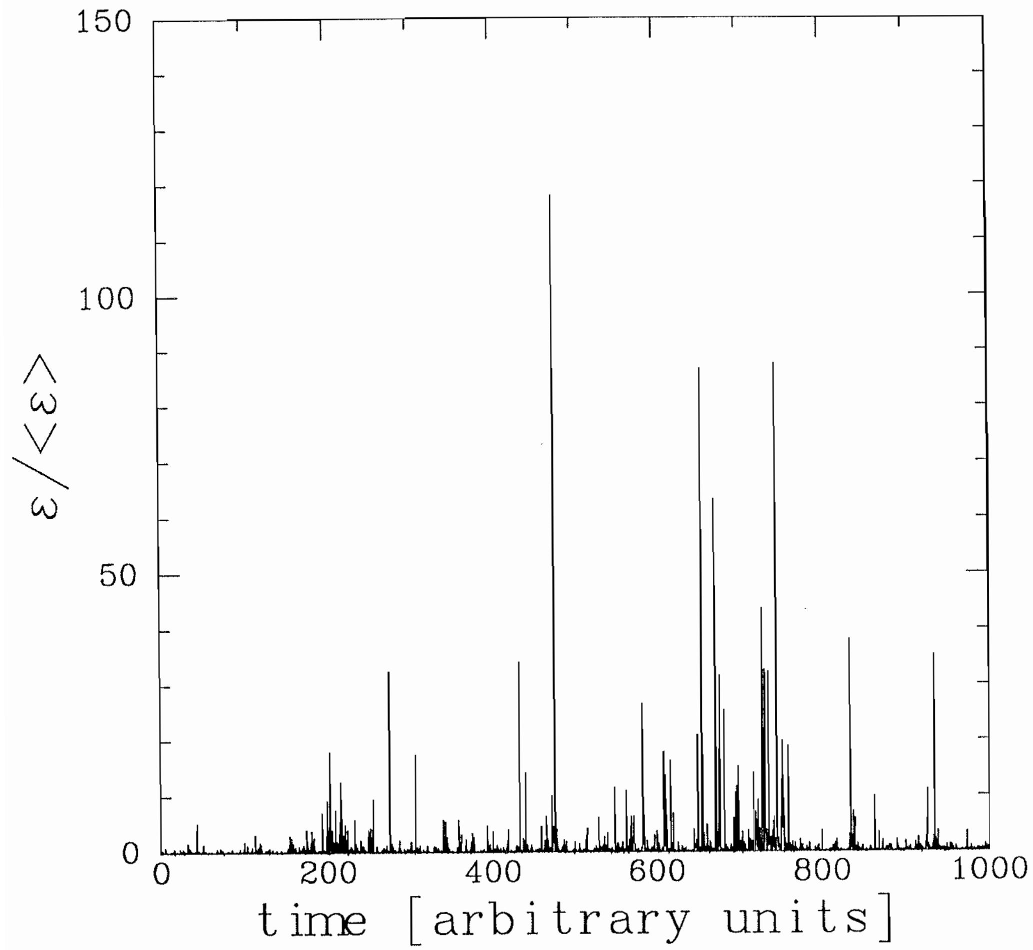

Figure 1: A typical signal of a representative component of ε, namely ε' = $(du/dx)^2 \sim (du/dt)^2$, normalized by its mean. Here, u is the velocity fluctuation in the direction x of the mean velocity U. In writing the last step of the above approximation, it has been assumed that Taylor's frozen flow hypothesis, namely that the spatial derivative can be approximated by the temporal derivative, holds. The velocity fluctuation u was obtained by a hot-wire mounted on a pole 6 m above the ground level over a wheat canopy. The microscale Reynolds number is of the order 2500.

property of ε in high-Reynolds-number turbulence. Its understanding and modeling is important to any practical scheme for computing turbulent flows. The intermittency has important implications also in contexts such as the structure of turbulent flames.

In the last few years, much work based on multifractals has occurred on the description and modeling of the intermittent character of energy dissipation rate (and other similar characteristics). For a summary, see [1]. In Refs. [2-4], several simple quasi-deterministic multifractal models were shown to describe the statistical properties of the energy dissipation rate quite accurately. Here, we provide an organized basis for developing such simplified intermittency models.

The energy dissipation rate is a positive definite quantity which is additive (in the sense that ε over two non-overlapping intervals equals the sum of ε values distributed over the sum of the two intervals). In this sense, it is convenient to think of ε as a measure distributed on an interval.

2. MULTIPLIER DISTRIBUTIONS

Consider a long data string of ε distributed over an interval which is N integral scales in extent, N being some large integer. Divide the interval into 'a' equal-sized sub-intervals, and obtain the ratios of the measures in each of the sub-intervals to that in the entire interval. These ratios, to be called multipliers, are clearly positive and lie between zero and unity. Subdivide each sub-intervals into 'a' pieces as before, and repeat the procedure. When we reach sub-intervals of the size of the integral scale of turbulence, L, there will be sufficiently large population of the ratios $M_i^{(L)}$, $1 \le i \le N$, and one can obtain a converged histogram of the multipliers $M_i^{(L)}$. Proceed with further subdivisions. At the n-th subsequent level, where each sub-interval is of size $r/L = a^{-n}$, there are $N.a^n$ multipliers $M_i^{(r)}$. Construct the histogram of the multipliers at each level. Repeat the procedure until the smallest sub-interval reached is of the order of the Kolmogorov scale.

The thought behind this hierarchical construction is that the nonlinear processes occurring in the inertial range of scales may be abstracted by a breakdown process in which each eddy subdivides into 'a' pieces, with the energy flux redistributed in some unequal fashion without loss among the sub-eddies; since the energy flux, as it cascades down to smaller scales, is ultimately converted into energy dissipation, the two quantities are equal on the average. It is further thought that this unequal distribution among sub-eddies is the heart of the observed intermittency. The reality is, of course, more complex. For instance, it is not obvious what the appropriate value of 'a' (that is, the base for the cascade process) must be, or whether it remains the same from one step of the cascade to another. In spite of this ambiguity, it is clear that if there is a scale-similar breakdown occurring in the cascade process, the histograms of the multipliers should be identical at each step of the subdivision or cascade.

The probability density P(M) of the multipliers M - here and subsequently, we omit the indices on the $M_i^{(r)}$ and denote them simply by M - have been obtained for different stages of subdivision of the interval. Since the value of 'a' is not known *a priori*, Chhabra & Sreenivasan [4] obtained P(M) for various bases. Figure 2 shows the results for 'a' = 2, 3 and 5. The shape of each of the distributions is invariant over a certain range of scales, suggesting that some type of self-similarity occurs in this scale range, whatever the assumed base. This range of scales over which P(M) is self-similar agrees quite well with the inertial range of scales determined by the scaling range in spectra and structure functions. The larger symbols show an average over steps involving comparisons between boxes of size 'm' and those of size 'm*a', where m ranged from 50 to 1000 in units of sampling intervals. (For the very smallest scales, the distributions have a concave shape. This concavity is related to the divergence of moments [5] and will be discussed elsewhere. For very large box-sizes, multiplier distributions approach a delta function centered around 0.5, as would be the case for random measures.)

The scale-invariant multiplier distributions obtained in figure 2 are fundamental to the understanding of the observed multifractal scaling [2]. One can compute [4] from them not only the asymptotic scaling properties such as the multifractal spectrum (or the f(α) curve [6]) of a measure, but also finite-size fluctuations of scaling properties [4]. In addition, even in instances where high-order moments diverge, P(M) remains well-defined. Finally, the f(α) function may extend over $(-\infty,\infty)$ whereas P(M) is a compact function defined on $M \in [0,1]$.

A disadvantage of P(M) is that it is base-dependent. However, if the cascades giving rise to the observed intermittency are randomly multiplicative, then the multiplier distributions corresponding to different bases are related by convolution, and one can scale out this base-dependency [4]. If the multiplicative process is random (i.e., successive multipliers are uncorrelated) several base-independent functions can be constructed from these multiplier distributions. In particular, for any two bases 'a' and 'b', we have

$$\log\langle (M_a)^q\rangle/\log(a) = \log\langle (M_b)^q\rangle/\log(b) = -[\tau(q) + D_0] \qquad (1)$$

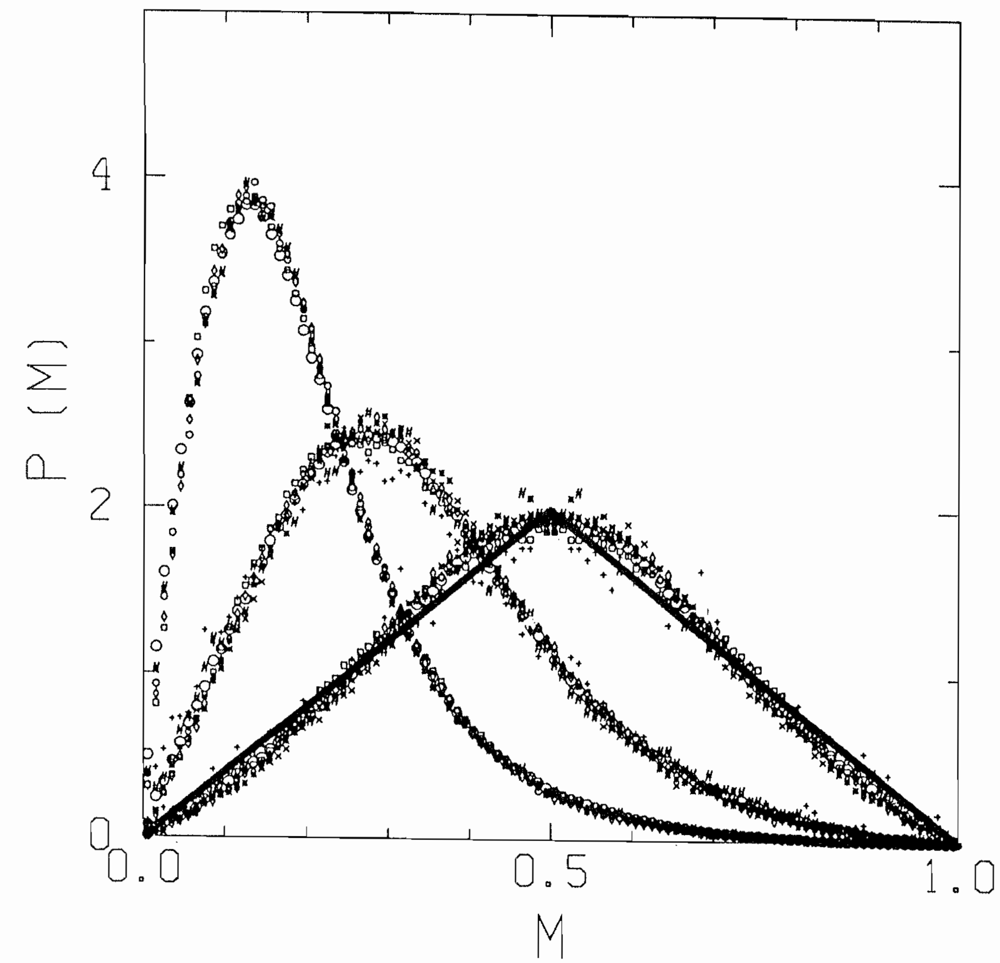

Figure 2: Multiplier distributions P(M) for bases (from right to left) 'a' = 2, 3 and 5. The larger symbols show averaged multiplier distribution, which are the mean of multiplier distributions obtained by comparing measure in boxes of size 'm' to those of 'm*a' , where m ranged from 50 to 1000 in units corresponding roughly to the Kolmogorov scale. The smaller symbols show the distributions obtained for m = 50, 80, 150, 200, 400 and 1000. The solid line is the triangular approximation to the binary case. The figure is adapted from [4].

where $\tau(q) = (q-1)D_q$, D_q being the so-called generalized dimensions [7] of q-th order, and $D_0 = D_{q=0}$ is the fractal dimension of the support of the measure. The $f(\alpha)$ function can also be easily derived from the multiplier distribution. In Ref. [4], it was shown that the $f(\alpha)$ functions computed from these different distributions were in good agreement with each other as well as with those obtained from direct methods such as box-counting. This agreement indicates the existence of a probablistic cascade where no single base is preferred.

Incidentally, a good approximation for P(M) in the binary case is the triangular distribution shown in figure 2. In Ref. [4] it was shown that the $f(\alpha)$ function for this model is in excellent agreement with that obtained directly [3]. Further, the triangular model displays the correct behavior with respect to sample-to-sample fluctuations in $f(\alpha)$. It also reproduces the stretched exponential tails, $P(\varepsilon) \sim \exp(-\beta(\varepsilon)^{1/2})$, observed in Refs. [3,8] for the probability distribution of ε.

3. SIMPLE MODELS

The multiplier distributions shown in figure 2 are extracted directly from the experiment and their analytical forms are yet to be found from the theory. The question meanwhile is a simple representation of these distributions in a way that permits one to evaluate most of the measured properties quite accurately. The goal is to seek models that are simple enough to be tractable mathematically and realistic enough to represent the spirit of the underlying physics. We already mentioned the triangular distribution as a good approximation. An even simpler possibility is the p-model [2], which is a model for a binary cascade ('a'=2). We first discuss the p-model and show how it can be obtained as a rational approximation to the measured multiplier distribution for the binary case. We will then discuss how models in the same spirit can be obtained for the tertiary case ('a'=3). The limitations of the procedure for high order subdivisions ('a'>3) will be highlighted.

From a physical point of view, the cascading process with 'a'=2 can be thought of as the break-up of a structure (the parent structure or eddy) into two sub-structures. For the one-dimensional case corresponding to figure 1, a pertinent question is the following: is there any difference between the left and right offsprings in terms of the energy flux they receive from the parent structure? One can determine experimentally that left and right are statistically indistinguishable. (This is not true for the velocity signal itself, as can be concluded from Kolmogorov's 4/5 law [9]). Now, for the sake of simplicity and modeling, let us assume that one of the two sub-eddies always receives a fixed fraction p of the

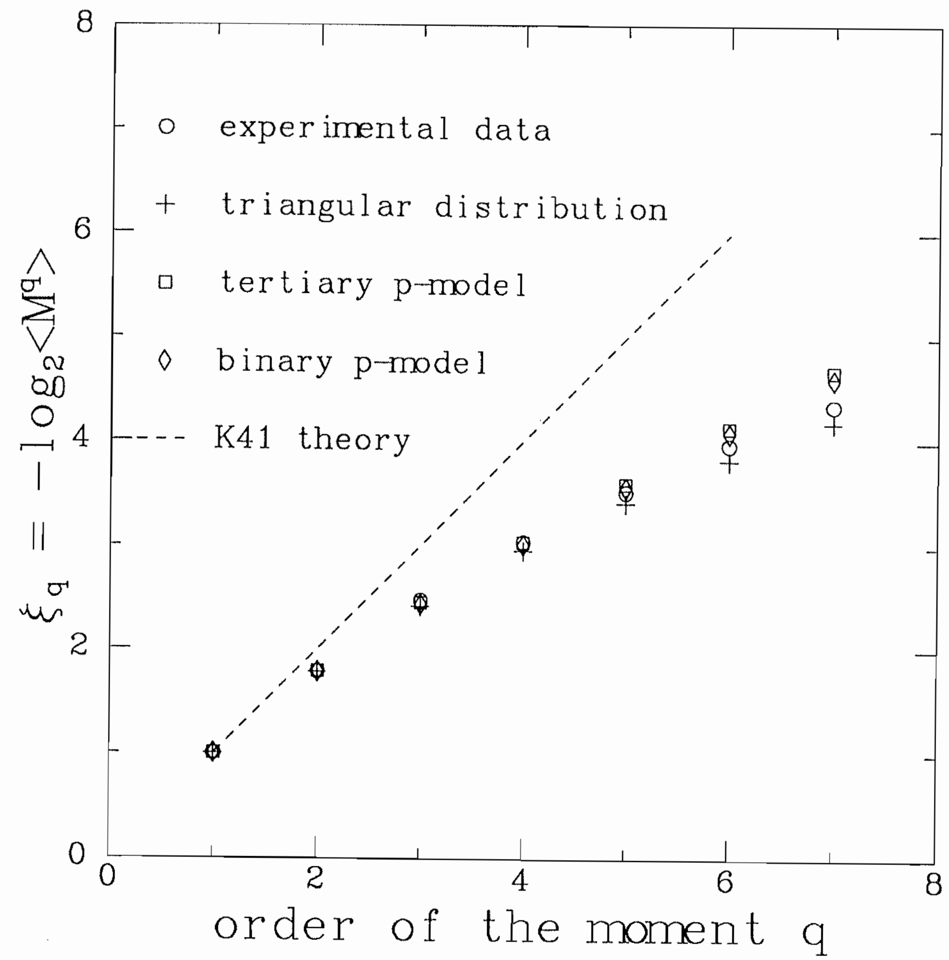

Figure 3: A comparison between moments computed from the measured multiplier distributions and those computed for the different models considered in the text. Experimental data were obtained from a record length of 810,000 data points. The convergence of moments was reasonable; for example, in the last half decade of the record length, the variations observed were smaller (in the log scale) than the symbol size.

energy contained in the parent eddy; naturally, the other will receive 1-p. In this sense, this model has some determinism. However, it is only quasi-deterministic in the sense that either one of the two eddies could receive the fraction p; because of the left-right symmetry mentioned above, a given piece will receive p as often as 1-p. Then, the multiplier distribution for the p-model becomes

$$p_{'a'=2}(M) = \frac{\delta(M-p) + \delta(M-(1-p))}{2}. \quad (2)$$

If p=1/2, there is no intermittency and the physical situation corresponds to Kolmogorov's 1941 theory [10]. To obtain intermittency, we should have a value of p different from 1/2.

How can we choose p? A natural way is to match the moments of $p_{'a'=2}(M)$ with those of the real p(M). For both distributions, the zero-order moment (normalization) and the first-order moment (mean value) coincide, and are 1 and 1/2, respectively. The first non-trivial condition is to match the second order moment. When this is done, we obtain the value p=0.697, or 1-p=0.303, which can be rounded off to excellent accuracy by 0.7 and 0.3, respectively. This is the p-model of Ref. [2]. It turns out, purely by luck, that high-order moments computed for the p-model also agree with those computed for the real data (see figure 3). It had been shown in [2] that the f(α) spectrum for the binary p-model with p = 0.7 fit the experimental data quite accurately.

We now discuss a general scheme for developing for all 'a'≠2 quasi-deterministic models of the sort developed above for the p-model. Again, we attempt to do this by matching moments. The general multiplier distribution for any 'a' in the p-model scheme is

$$P_{'a'}(M) = \frac{\sum_{i=1}^{a} \delta(M-p_i)}{a} \quad (3)$$

where

$$\sum_{i=1}^{a} p_i = 1, \quad (0 \le p_i \le 1) \quad (4)$$

We may now equate the moments of $P_{'a'}(M)$ to the moments of the real multiplier distributions. Since multiplier distributions for any base yield

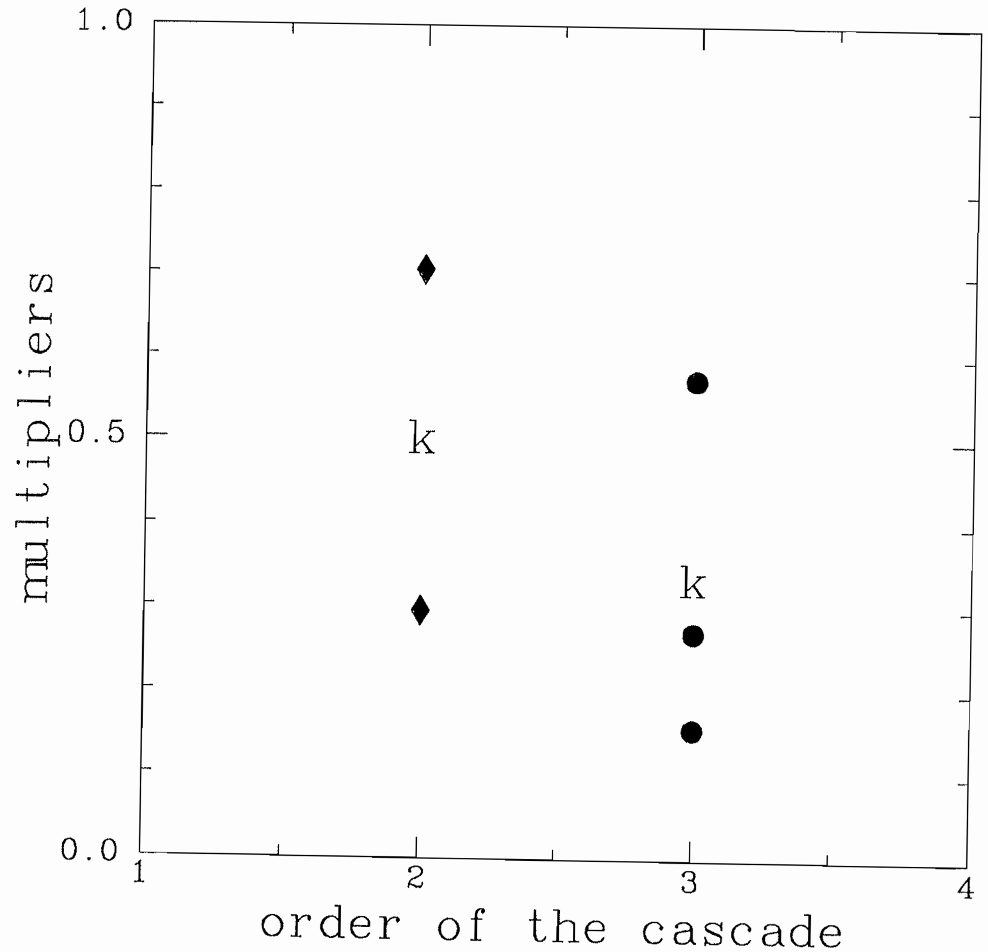

Figure 4. Values of the multipliers for the binary (diamonds) and tertiary (circles) p-models. The letter K indicates the location of the multipliers if there were no intermittency, consistent with Kolmogorov's 1941 theory [10].

the same results, we may take the distribution corresponding to the binary cascade of figure 2. Computing the moments $<M^q>$ from it, we are led to the equations

$$\sum_{i=1}^{a} p_i^q = a <M^q>^{\log(a)/\log(2)}, \quad q=1, 2, ..., a. \tag{5}$$

This is a system of 'a' equations with 'a' number of p_i's to be determined. Using Girard's rule [11], it is easy to find a polynomial of degree 'a' whose roots are the desired p_i's. The problem thus reduces to the determination of the roots of the polynomial. It turns out *a posteriori* that this problem has physical solutions only for 'a'=2 and 'a'=3: for larger values of 'a', some of the roots turn out to be complex, and have no physical meaning. The values of p_i for the tertiary cascade ('a'=3) are p_1=0.155, p_2=0.283 and p_3=0.562. We designate this as the tertiary p-model. In this scheme, the classical Kolmogorov theory would yield $p_1= p_2= p_3=$ 1/3.

The values of p_i for the binary and tertiary cascade are shown in figure 4; also marked by K are the classical non-intermittent values applicable to Kolmogorov's 1941 theory. Although the binary and tertiary p-models, respectively, are generated to possess the first three and four moments correctly, it is remarkable that the binary and tertiary p-models and the measured distributions have approximately the same high order moments up to, say, about 7 (see figure 3).

Other models have also been proposed. For example, Novikov [12] proposed a uniform distribution for P(M). At that time, however, the multiplier distribution had not been obtained experimentally. It is now clear, however, that a uniform distribution is not a good model for any of the curves in figure 2. For example, a good approximation to the binary cascade is the triangular distribution shown by a solid line in figure 2.

4. CONCLUSIONS

The multiplier distributions are a basic tool for understanding many of the scale-similar properties of energy dissipation in turbulence. In the absence of an *ab initio* theory that yields these distributions in a deductive way, analytical progress can be made only by modeling them with reasonable schemes. Here, we have summarized the attempts

made in the last few years. Below, we mention some applications of this work.

The first application is in the calculation of E_r, the energy dissipation over an interval of size r; $E_r = r\varepsilon_r$. According to the definition of the multipliers,

$$E_r = E_L \prod_{i=1}^{n} m_i, \tag{6}$$

where $n(r)=\log_a(r/L)$, and L is the large-eddy (integral) scale. Within the p-model scheme for a binary cascade, the probability density function (PDF) for the ratio $x = E_r/E_L$ is

$$p(x)=\frac{A}{x}\frac{r}{L}\frac{\Gamma(\log_2(r/L)+1)}{\Gamma(k(x)+1)\,\Gamma(\log_2(r/L)-k(x)+1)} \tag{7}$$

where A is a normalization constant, Γ is the gamma function and

$$k(x) = \frac{\log(x\,(r/L)^{\log_2(1-p)})}{\log(p/(1-p))}. \tag{8}$$

A second application is in computing the exponents ξ_q, defined as

$$<E_r^q> \sim (r/L)^{\xi_q}. \tag{9}$$

The result for the triangular distribution is

$$\xi_q = -\log_2\left\{4\left(1-\frac{1}{2^{q+1}}\right)\frac{1}{(q+1)(q+2)}\right\}. \tag{10}$$

Corresponding results for the p-model are given in [2]. The PDFs of the velocity increments can also be computed. On using the second refined similarity hypothesis [13], the velocity increments can be written as

$$\Delta u(r) = V\, E_r^{1/3}. \tag{11}$$

In Ref. [14], we obtained the probability density of V. The PDF of Δu can be computed if some model is assumed for E_r. In Ref. [8], those PDFs were computed using the binary p-model.

The tertiary p-model has been used to generate a signal that shares many features of a real turbulent velocity trace. This issue will be addressed elsewhere.

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