Kolmogorov's Refined Similarity Hypotheses

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(Received 26 May 1992)

Using velocity data obtained in the atmospheric surface layer, we examine Kolmogorov's refined hypotheses. In particular, we focus on the properties of the stochastic variable $V=\Delta u(r)/(r\varepsilon_r)^{1/3}$, where $\Delta u(r)$ is the velocity increment over a distance r, and ε_r is the dissipation rate averaged over linear intervals of size r. We show that V has an approximately universal probability density function for r in the inertial range and discuss its properties; we also examine the properties of V for r outside the inertial range.

PACS numbers: 05.45.+b, 02.50.+s, 03.40.Gc, 47.25.−c

A fruitful idea in high-Reynolds-number turbulence is the phenomenological picture of local structure introduced by Kolmogorov [1]. In its simplified version, the picture relates to the probability density function (PDF) of the velocity increment $\Delta u(r)=u(x+r)-u(x)$, where $u(x+r)$ and $u(x)$ are velocities along the x axis at two points x and $x+r$ separated by a distance $r \ll L$, L being the integral scale of turbulence. According to the first hypothesis, the PDF of $\Delta u(r)$ depends (besides on r itself) only on the average rate of energy dissipation $\langle\varepsilon\rangle$ and the fluid viscosity ν. The second hypothesis is that if the Reynolds number is very large, there exists a range of scales (the so-called inertial range) for which ν becomes irrelevant, so that the only important external parameter is $\langle\varepsilon\rangle$. The consequences of these universality arguments are well known [2] and need no repetition. Reference [2] also describes the circumstances which led to the modification of these hypotheses.

Kolmogorov [3] himself introduced refinements of his 1941 theory, which abandoned local universality and introduced more restrictive alternatives to the 1941 hypotheses. The first refined hypothesis postulated a certain relation between $\Delta u(r)$ and the local average of the energy dissipation rate $\varepsilon_r(x,t)$, taken over an interval of size r centered at x. Define, following Kolmogorov [3], a velocity scale at the point (x,t) by $U_r=(r\varepsilon_r)^{1/3}$ and form the local Reynolds number as

$$\mathrm{Re}_r = U_r r/\nu . \tag{1}$$

The first refined hypothesis then takes the form of the statement that, for $r \ll L$,

$$\Delta u(r) = V(r\varepsilon_r)^{1/3} , \tag{2}$$

where V is a stochastic variable whose PDF depends only on Re_r. The second refined hypothesis states that, if $\mathrm{Re}_r \gg 1$, the PDF of V becomes independent also of Re_r and thus universal. Unlike the original hypotheses, these refinements require a supplementary statement on the PDF of $\Delta u(r)$ or of ε_r. Kolmogorov's third hypothesis is that the PDF of ε_r is log-normal with a specific form for its variance.

The consequences of Eq. (2) have been tested in various ways, but mainly by viewing it as a dimensional relation $|\Delta u(r)| \sim (r\varepsilon_r)^{1/3}$. The stochastic function V is usually disregarded, often leading to misinterpretations of Eq. (2). Indeed, V contains much of the inertial range physics. For example, if Kolmogorov's $\frac{4}{5}$ law [4]

$$\langle[\Delta u(r)]^3\rangle = -\tfrac{4}{5}\langle\varepsilon\rangle r \tag{3}$$

is understood to express the cascade of turbulent kinetic energy from large to small scales, the nonzero value of $\langle V^3\rangle$ should account for this process. Similarly, the knowledge of the PDF of V would enable one to infer the PDF of $\Delta u(r)$ from that of ε_r. Further, in the inertial range, if the PDF of V is independent of ε_r and r, it ensures the same scaling behavior for both $\Delta u(r)$ and $(r\varepsilon_r)^{1/3}$.

The principal purpose of this paper is to assess the status of Eq. (2) in some detail by constructing, from measurement at high Reynolds numbers, the PDF of V, its correlation with $\Delta u(r)$ and ε_r, and the correlation between $\Delta u(r)$ and $(r\varepsilon_r)^{1/3}$. A further purpose is as follows. In a recent analysis of numerical data, Hosokawa and Yamamoto [5] claimed to have found evidence against Eq. (2). In this paper, on the other hand, we present strong experimental evidence supporting Eq. (2).

Measurements were made in the atmospheric surface layer about 6 m above a wheat canopy. The microscale Reynolds number R_λ, based on the root-mean-square velocity and the Taylor microscale, was estimated to be 2000. Data were also acquired about 2 m above the roof of a four-story building ($R_\lambda \approx 1500$). Velocity fluctuations were measured using the standard hot wire (5 μm diameter, 0.6 mm length) operated on a DISA 55M01 constant-temperature anemometer. The anemometer voltage was digitized on a 12-bit digitizer at sampling frequencies of 6000 and 10 000 Hz.

For further analysis, we interpreted a time trace as a spatial cut by invoking Taylor's frozen turbulence hypothesis. The sequence $x^{(k)}$ is simply $x^{(k)}=x^{(0)}+k\tau$, τ being the sampling interval (the inverse of the sampling frequency). We take the velocity differences to be $\Delta u^{(k)}(r)=u(x^{(k+r)})-u(x^{(k)})$, where r is expressed in units of sampling intervals. We approximate the energy dissipation rate by its one-dimensional surrogate $\varepsilon(x) \sim (du/dx)^2$, and discretize it as $\varepsilon(x^{(k)}) = [u(x^{(k+1)}) - u(x^{(k)})]^2$. The local average of ε is taken to be $\varepsilon_r^{(k)} = (1/r)\sum\varepsilon(x^{(i)})$, the sum extending from $i=k$ to $k+r-1$. The inertial range is defined as the range in which Eq. (3) holds, and L as the scale at which $\langle|\Delta u(r)|^3\rangle$ asymptotes to a constant (see Ref. [6]). Both $\Delta u(r)$ and $(r\varepsilon_r)^{1/3}$ are normalized by $(L\langle\varepsilon\rangle)^{1/3}$.

For the results to be quoted below, we have used units most convenient for data handling purposes. For example, r is quoted in units of sampling intervals. In these units, the inertial range spans approximately between $r_{\min}=10$ and $r_{\max}=300$, L has a value of 10000, and $(L\langle\varepsilon\rangle)^{1/3}$ a value of 3.1. These numbers are related to their counterparts in CGS units (denoted by primes) by the following conversion factors: $\Delta u'/\Delta u = 87.3$, $r'/r = \frac{1}{15}$, and $\varepsilon'/\varepsilon = 3.86 \times 10^6$. V' is dimensionless and related to V by the relation $V'/V = 1.37$.

Figure 1 shows $r\varepsilon_r$, $(\Delta u)^3$, $|\Delta u|^3$, and V for one set of data, and corresponds to $r=100$ in the inertial range. The segment (a) shows that ε_r is a highly intermittent quantity; its scaling properties have been studied extensively [7]. It is clear from Figs. 1(b) and 1(c) that $(\Delta u)^3$ as well as $|\Delta u|^3$ are also intermittent. Since V is the ratio of two intermittent quantities, it might be thought that it is ill defined in general. This is not so because ε_r is an average over an interval r, which renders it well behaved at least when r is large enough. We have investigated velocity data sampled at different frequencies, employed different schemes of differentiation to obtain ε, and found

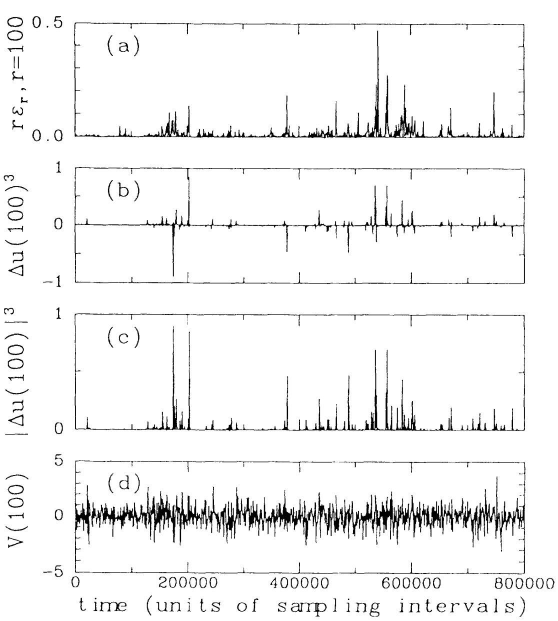

FIG. 1. Time traces of (a) $r\varepsilon_r$, the total dissipation in a box of size r, (b) $(\Delta u)^3$, (c) $|\Delta u(r)|^3$, and (d) the stochastic variable V. The "box" size $r=100$. Notice the correlation between (a) and (c). Both $r\varepsilon_r$ and $[\Delta u(r)]^3$ have been normalized by $L\langle\varepsilon\rangle$. For clarity, only every 100th point has been plotted.

that the effects on V are not large enough to change its statistics perceptibly. Figure 1(d) shows the stochastic variable V for the same duration as Figs. 1(a)-1(c).

Records (a) and (b) reveal that regions of strong activity in $(\Delta u)^3$ and $r\varepsilon_r$ coincide quite often [8]. This suggests an instantaneous communication between inertial range scales and dissipation scales—at least when the forcing (i.e., local the energy flux ε_r) is large enough. This feature would be obscured if one blindly computed the correlation coefficient between Δu and $(r\varepsilon_r)^{1/3}$; the correlation coefficient is small (of the order 0.05; see Fig. 2) partly because Δu alternates rapidly in sign [see Fig. 1(b)], and partly because the correlation in the two signals is weak when the amplitudes are small. A consideration of $|\Delta u|$ instead of Δu eliminates the former problem, and one can expect the correlation to improve. Figure 2 shows that the correlation coefficient between $|\Delta u(r)|$ and $(r\varepsilon_r)^{1/3}$ is around 0.5 in the inertial range, and decreases weakly with r.

Let us move on now to the properties of the variable V. Figures 3(a) and 3(b) show the PDFs of V conditioned on $(r\varepsilon_r)^{1/3}$ for $r=50$ and 100, respectively, both in the inertial range. The PDFs coalesce for a range of values of $(r\varepsilon_r)^{1/3}$, and the shape is preserved in going from $r=50$ to 100, indicating an approximate independence of r and ϵ. In Figs. 3(a) and 3(b), the mean of the PDFs is zero and their variance is unity. Noting the conversion factors given above, this means that the true (or, real world) V has a variance of $1.37^2 = 1.88$.

We have examined (but not shown here) the correlation between V and $(r\varepsilon_r)^{1/3}$. Irrespective of whether we consider V or $|V|$, the correlation coefficient in the inertial range is close to zero. Since zero correlation between two random variables does not guarantee independence, we have considered the mean value of $|\Delta u(r)|$ conditioned on $(r\varepsilon_r)^{1/3}$. It follows from Eq. (2) that

$$\langle|\Delta u(r)|\,|(r\varepsilon_r)^{1/3}\rangle = (r\varepsilon_r)^{1/3}\langle|V|\,|(r\varepsilon_r)^{1/3}\rangle . \quad (4)$$

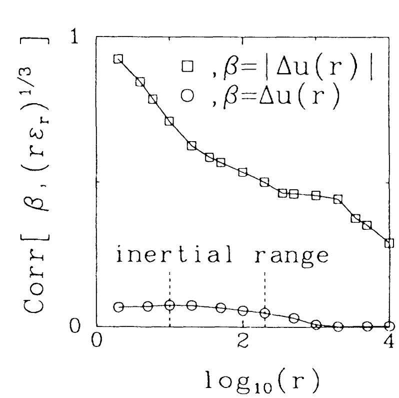

FIG. 2. The correlation coefficient as a function of r between $(r\varepsilon_r)^{1/3}$ and the quantity β where β is (a) $\Delta u(r)$ (circles) and (b) $|\Delta u(r)|$ (squares). The correlation coefficient is defined as $\mathrm{corr}(x,y) = \langle(x-\langle x\rangle)(y-\langle y\rangle)\rangle/\sigma_x\sigma_y$, where the variance $\sigma_z = \langle(z-\langle z\rangle)^2\rangle^{1/2}$.

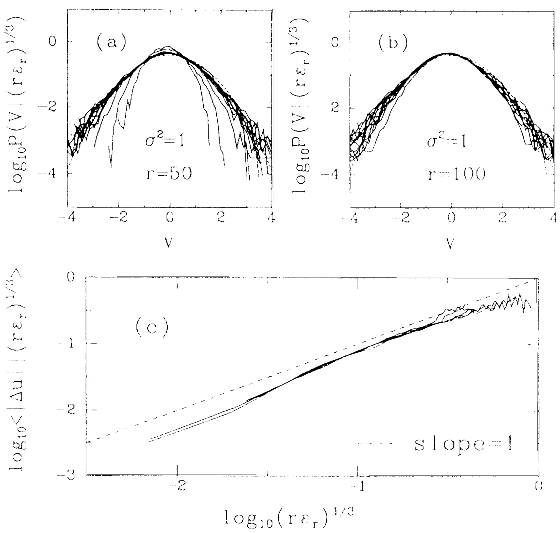

FIG. 3. (a),(b) The logarithm of the PDF of V conditioned on $(r\varepsilon_r)^{1/3}$ for different values of r. Each solid curve corresponds to a different value of the conditional parameter $(r\varepsilon_r)^{1/3}$. The value of r, the number of curves, and the minimum and maximum values of $\alpha=(r\varepsilon_r)^{1/3}/(L\langle\varepsilon\rangle)^{1/3}$ considered here are (a) 50, 14, 0.024, and 0.24; (b) 100, 15, 0.03, and 0.29. The intermediate values of α are equally spaced linearly between the minimum and maximum values. The dashed curves correspond to a Gaussian of mean zero and unity variance. In (a), the three innermost curves (which are distinct from the ones that collapse) correspond to the three lowermost values of $(r\varepsilon_r)^{1/3}$. (c) The logarithm of the mean value of $|\Delta u|$ conditioned on $(r\varepsilon_r)^{1/3}$ as a function of the logarithm of $(r\varepsilon_r)^{1/3}$. The four coalescing solid curves correspond to r in the inertial range of 20, 50, 100, and 200. The dashed line has a slope of 1.

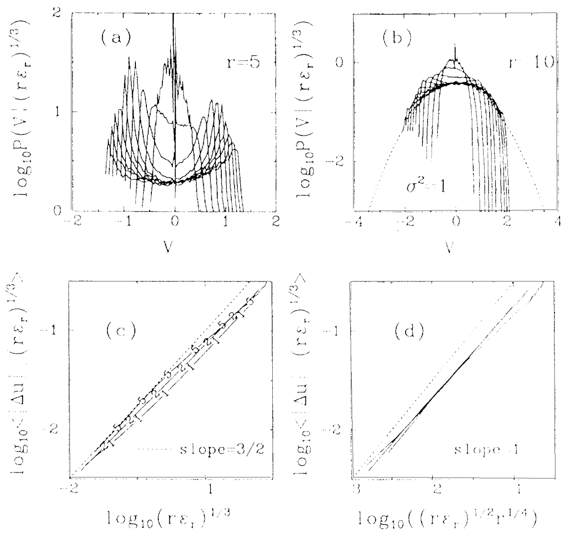

FIG. 4. (a),(b) The same quantities as in Figs. 3(a) and 3(b) for r values below the inertial range. The relevant numbers are for (a) 5, 10, 0.004, and 0.1; for (b) 10, 9, 0.007, and 0.13. The values of $(r\varepsilon_r)^{1/3}$ increase as we go from the innermost curves to the outermost. (c) The analog of Fig. 3(c) for $r=1$, 2, and 5. These curves collapse more or less but not as well as those for the inertial range. (d) They can be collapsed onto a unique line of slope unity if an *ad hoc* dependence (see text) on r is imposed by changing the abscissa to $\log_{10}[(r\varepsilon_r)^{1/2}r^{1/4}]$.

If V were independent of $(r\varepsilon_r)^{1/3}$ in the inertial range, it is clear that the left-hand side of Eq. (4) would be a linear function of $(r\varepsilon_r)^{1/3}$ for all values of r in the inertial range. Figure 3(c) shows that this is indeed the case.

The fact that emerges is that the stochastic variable V in the inertial range is independent of r and ε_r, and that it is approximately universal. It then follows from Eq. (2) that the mth-order structure functions are given by

$$\langle[\Delta u(r)]^m\rangle=\langle V^m\rangle\langle(r\varepsilon_r)^{m/3}\rangle\,. \qquad (5)$$

We have remarked that $\langle V^2\rangle=1.88$, a value quite comparable to the accepted estimates [2] of the Kolmogorov constant in second-order structure functions. Comparison of Eqs. (3) and (5) shows that $\langle V^3\rangle=-\frac{4}{5}$. Thus, appearances to the contrary, V is indeed skewed (with the normalized skewness of the order -0.3). The data records in our experiments were not long enough to obtain a converged value for $\langle V^3\rangle$, but we obtained numbers of the right order of magnitude (and, of course, the right sign).

The situation for scales below the inertial range is shown in Fig. 4. Figures 4(a) and 4(b) present PDFs of V for $r=5$ and 10, respectively; each curve corresponds to different values of $(r\varepsilon_r)^{1/3}$. The uncertainty is somewhat high (because the averaging intervals are small). Figure 4(a) shows a bimodal distribution exhibiting a strong dependence on $(r\varepsilon_r)^{1/3}$. For $r=10$, the bimodality becomes weaker but the dependence on $(r\varepsilon_r)^{1/3}$ continues to persist [see Fig. 4(b)]. According to the first refined hypothesis, the dependence can occur only through Re_r [or on $r(r\varepsilon_r)^{1/3}$—since in our measurements the fluid viscosity is fixed parameter]. This is evidently not true because Figs. 4(a) and 4(b) together show that fixing $r(r\varepsilon_r)^{1/3}$ does not guarantee the same shape for the PDF of V.

Figure 4(c) shows a plot of $\log_{10}[\langle|\Delta u(r)|\,|(r\varepsilon_r)^{1/3}\rangle]$ as a function of $\log_{10}[(r\varepsilon_r)^{1/3}]$. Although all three curves (corresponding to $r=1$, 2, and 5) have roughly the same slope of $\frac{3}{2}$, they do not collapse on each other well. They can be made to coalesce by empirically postulating a quarter power dependence of $\langle V|(r\varepsilon_r)^{1/3}\rangle$ on r. Writing $\Delta u(r)=\int_x^{x+r}(du/dx)dx$, and using the Cauchy-Schwarz inequality we find that

$$|\Delta u(r)|\le\left[\int_x^{x+r}\left(\frac{du}{dx}\right)^2dx\right]^{1/2}r^{1/2}=K(r\varepsilon_r)^{1/2}r^{1/2}\,,$$

so that $\langle|\Delta u(r)|\,|(r\varepsilon_r)^{1/3}\rangle=K[r(r\varepsilon_r)]^{1/2}\phi(\varepsilon_r,r)$, where the function ϕ has a value less than or equal to unity. For $r=1$, ϕ is obviously equal to unity and independent of ε_r. Assuming some continuity in ϕ, it may be expected that ϕ for slightly larger r would be essentially independent of ε_r but slightly smaller than unity. A possible form for ϕ

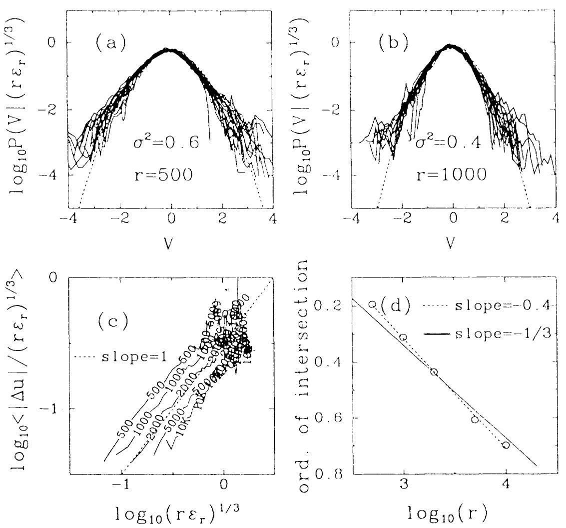

FIG. 5. In (a) and (b) are plotted the same quantities as in Figs. 3(a) and (b), but for r values above the inertial range. The corresponding numbers are (a) 500, 18, 0.07, and 0.49; (b) 1000, 19, 0.11, and 0.65. The dashed lines correspond to Gaussians with $\sigma^2=0.6$ for (a) and $\sigma^2=0.4$ for (b). In (c), which is the analog of Fig. 3(c) but for the r values 500, 1000, 2000, 5000, and 10000, the curves are approximate straight lines with a slope of unity (dashed curve). The ordinates of the straight line approximation to these curves organize themselves in a straight line as shown in (d); a simple argument can be given in favor of $-\frac{1}{3}$ slope, but a slope of -0.4 seems to fit the data better.

satisfying these requirements is $1/r^\delta$, where δ is a real positive number. It turns out from Fig. 4(d), which shows that $\log_{10}[\langle|\Delta u(r)|\,|(r\varepsilon_r)^{1/3}\rangle]$ is a unique function of $\log_{10}([r(r\varepsilon_r)]^{1/2}/r^{1/4})$, that a suitable value of δ is $\frac{1}{4}$. We thus have the result that

$$\langle|\Delta u(r)|\,|(r\varepsilon_r)^{1/3}\rangle = K[r(r\varepsilon_r)]^{1/2}/r^{1/4}. \tag{6}$$

Note that $(r\varepsilon_r)^{1/2}r^{1/2}=[r(r\varepsilon_r)^{1/3}]^{1/2}(r\varepsilon_r)^{1/3}$, which is in the form of the right-hand side of Eq. (4) and shows that the dependence of $\langle|V|\,|(r\varepsilon_r)^{1/3}\rangle$ is of the form $\mathrm{Re}_r^{1/2}/r^{1/4}$.

For scales above the inertial range, the PDFs start to accumulate roughly onto a Gaussian shape [Figs. 5(a) and 5(b) for $r=500$ and 1000]; they appear to be independent of $(r\varepsilon_r)^{1/3}$ but dependent on r. To a first approximation, this dependence may be thought to enter through the variance of the distribution which decreases with increasing r. In Fig. 5(c), we have plotted $\log_{10}[\langle|\Delta u(r)|\,|(r\varepsilon_r)^{1/3}\rangle]$ as a function of $\log_{10}[(r\varepsilon_r)^{1/3}]$ for $r=500$, 1000, 2000, 5000, and 10000. These curves are all straight lines with a slope of 1. This confirms that $\langle|V|\,|(r\varepsilon_r)^{1/3}\rangle$ does not depend on $(r\varepsilon_r)^{1/3}$ but depends on r. The straight lines of Fig. 5(c) can all be collapsed if we take $\langle|V|\,|(r\varepsilon_r)^{1/3}\rangle=K/r^{2/5}$, as shown in Fig. 5(d). The $\frac{2}{5}$ dependence is, of course, purely empirical.

The following summary statements can be made. First, regions of strong activity in Δu and ε_r, for r in the inertial range, are well correlated—suggesting an instantaneous communication between the inertial range scales and dissipation scales. Even so, the correlation coefficient between Δu and ε_r is close to zero because of the rapid oscillations in the sign of Δu and the relatively poor correlation during weak events. Second, the basic tenets of the second refined hypothesis, namely, the existence in the inertial range of a universal stochastic variable V, is approximately true. The variance of V is approximately 1.88, and consistency with Kolmogorov's $\frac{4}{5}$ law demands that $\langle V^3\rangle=-\frac{4}{5}$. Finally, the measurements reveal that, on either side of the inertial range, there are different dependences of V on r. These dependences have been determined empirically.

We thank Robert Kraichnan, Mark Nelkin, Philippe Similon, and Ira Bernstein for their helpful comments. The work was supported by DARPA.

Note added.—After this work was submitted for publication, two related preprints [9] were received. R. H. Kraichnan and M. Nelkin have mentioned to us the existence of another related preprint by A. Praskovsky. We had first thought that the contrary conclusion of Ref. [5] might have been a consequence of the low Reynolds number of the flow considered there. However, a repeat of the present analysis for a low-Reynolds-number laboratory boundary layer has shown that this is not the case. This was also emphasized to us by Kraichnan.

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941).

[2] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1971).

[3] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).

[4] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **32**, 16 (1941).

[5] I. Hosokawa and K. Yamamoto, Phys. Fluids A **4**, 457 (1992).

[6] P. Kailasnath, K. R. Sreenivasan, and G. Stolovitzky, Phys. Rev. Lett. **68**, 2766 (1992).

[7] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991); K. R. Sreenivasan and C. Meneveau, Phys. Rev. A **38**, 6287 (1988); K. R. Sreenivasan, Annu. Rev. Fluid Mech. **23**, 539 (1991).

[8] When the time axis is stretched and details are observed, we find the following. While, as suggested by the compressed data in Figs. 1(a)–1(c), the activities in ε_r and $|\Delta u|^3$ do coincide, there is more fine structure to $|\Delta u|^3$ than there is to ε_r. This is to be expected because the latter is an average (over r) while the former is not. A suitable coarse graining of $|\Delta u|^3$ improves the detailed correspondence. The correct statement is that the regions of activity in the two signals coincide more or less, but the detailed structure in each of the signals is different in detail.

[9] S. T. Thoroddsen and C. W. Van Atta (to be published); S. Chen, G. D. Doolen, R. H. Kraichnan, and Z. S. She (to be published).