# CHARACTERIZATION AND COMPRESSION OF TURBULENT SIGNALS AND IMAGES USING WAVELET-PACKETS

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**ABSTRACT**

The newly introduced Wavelet-Packet transform (Coifman & Meyer 1989, Coifman et al. 1990) allows the decomposition of a signal as a function of the scale, the position and the frequency (or wavelength) optimally. Each Wavelet-Packet coefficient provides insight into the structure of the data locally and at the appropriate scale. We have applied this transform technique to one-dimensional data and two-dimensional images and report on its ability to characterize turbulence data with a few coefficients. We find that, overall, the Wavelet-Packet transform technique performs better than its competitors. That is, significant data compression ratios can be achieved without severely distorting the signal or the image.

## 1. Introduction

Turbulent motion has traditionally been decomposed in terms of Fourier modes, and one speaks of its frequency or wavenumber components. Our intuition about 'large' and 'small' scales of motion is substantially biased by Fourier description. There are several reasons why one should think of alternative descriptions. First, most turbulent flows (except for the hypothetical case of infinitely extended homogeneous turbulence) are finite in spatial extent, at least in one or two directions, so that expansions in terms of Fourier modes depend on mode-cancellation outside the flow domain – this being uneconomical in general. One therefore needs basis functions with compact support in real space. Secondly, a description of sharp gradients by Fourier modes is inefficient. Finally, turbulent flows (even the homogeneous ones) possess spatial structure of some sort, and the question of how to represent them efficiently is gaining increasing importance. The structure is stronger in some flows than in others and has better definition at some scales than at others, but it is clear that one should make a distinction between Fourier modes on the one hand and spatial structures on the other. Indeed, descriptive efforts in turbulence have always resorted to a variety of 'turbulent eddies' (see for example, Townsend 1956, 1976). Whatever the precise meaning of the term 'eddy', *it is clear that an eddy is not a Fourier mode*!

Lumley (Tennekes & Lumley 1972) recognized this distinction and wrote as follows (p. 259): 'The Fourier transform of a velocity field is a decomposition into waves of different wavelengths; each wave is associated with a single Fourier coefficient. An eddy, however, is associated with many Fourier coefficients and the phase relations among them. Fourier transforms are used because they are convenient (spectra can be measured easily); more sophisticated transforms are needed if one wants to decompose a velocity field into eddies instead of waves ...' In particular, figure 1 shows Lumley's schematic of an eddy in both real and Fourier spaces. It turns out that this eddy, which has the desired property of being spatially compact, is an example of what are now known as 'wavelets'. It is very much to Lumley's credit that he should have introduced wavelets to turbulence, albeit without using the name, at least as early as 1972! Yet, the broad recognition that wavelets possess useful mathematical properties that are appropriate for turbulence description is quite recent.

Formally, a wavelet is a spatially localized function which can be translated and rescaled while maintaining its shape (see, for example, Daubechies 1988). Wavelets in general are not local in Fourier space, and there is the so-called 'uncertainty relation' which tells us that spatial localization and wavenumber localization are complementary, and that there is a quantifiable trade off between them. Wavelet analysis provides a means for studying scaling and transient behavior of signals by using the dilates and translates of the wavelet as a basis. Decomposition of the signal into these basis functions corresponds to identifying the local scaling behavior of the signal. Different wavelets have been used to analyze different features of turbulent signals and images. Some examples are Argoul et al. (1989), Everson et al. (1989), Liandrat & Moret-Bailly (1990), Meneveau (1990).



While wavelets are important and their use in turbulence is bound to grow in several different directions, it is useful to recognize the many advantages possessed by Fourier representation (Strang 1989). At the least, our intuition of many periodic as well as non-periodic phenomena is based on Fourier decomposition. As an example in turbulence, the distribution of small scales might to some extent be described by modulated Fourier modes. It thus becomes apparent that new transforms, which can provide information about local frequency (or wavelength) in addition to that on scales, would be very valuable. Recently, such a transform, called the Wavelet-Packet transform, has been invented by Coifman & Meyer (1989). The transform provides local frequency and local scale information about data in one, two or three dimensions. It allows a signal to be segmented into dyadic intervals (1,2,4,8, ... data points), and the segmentation is arranged so that each 'near homogeneous' piece is decomposed into the basis that suits it best. A part of the attraction is that the transform also provides a choice of basis for different segments of data. Each of the functions in a basis can be localized (to different degrees) in the Fourier domain. The localization of the segment provides the position information. Although the algorithm allows uneven segmentation of the data and a choice of basis, the decomposition is nonredundant and complete. Further, fast algorithms requiring $O(N \log N)$ operations have been written (Beylkin et al. 1989).

The Wavelet-Packet coefficients can be arranged in the order of decreasing magnitude. If it happens that some of them are large and most others small, we can discard the small ones and achieve economical representation of the data. If the memory required to store the retained coefficients is small (relative to that required to store the original signal), then we have achieved efficient data compression. Data characterization and compression are related, but not identical. For example, for each Wavelet-Packet coefficient retained, its position, scale and frequency will also have to be stored. This 'overhead' makes data compression less efficient than data characterization. On the other hand, in turbulence as well as in many other applications, information on position, scale and frequency are very useful.

The Wavelet-Packet algorithm has already been implemented in the context of acoustic signals (Wickerhauser 1989). Here, we apply the Wavelet-Packet transform both for the characterization and for the compression of turbulent data. Because of space limitations, we describe Wavelet-Packets only briefly (section 2). Some comments on the index of performance are made in section 3. We briefly describe results for one-dimensional turbulent velocity signals (section 4), and for two-dimensional concentration fields of turbulent jets (section 5). A summary and conclusions are given in section 6. For greater details, reference must be made to Zubair et al. (1990) which we plan to publish shortly.

It should be mentioned that the spirit of image and data compression here is different from that of the Karhunen-Loéve transform procedure or the proper orthogonal decomposition – yet another concept introduced to turbulence by Lumley (1970). In the latter procedure, we are given a large collection of statistically similar objects, and the purpose is to pick out eigenfunctions that garner the most energy in the ensemble-average sense. Here, on the other hand, we are given a single image or a segment of data and asked to represent it best and most efficiently by means of various shapes and scales. The two questions are complementary to each other. (Note: A single image can be broken up into blocks and these blocks can be thought of as forming various members of the ensemble. The two techniques are thus related to each other.)

## 2. Wavelet-Packets

In general a collection of Wavelet-Packets consists of the translates, dilates and modulates of a mother Wavelet-Packet, $w$. The Wavelet-Packet $w$ can be be dilated as $\sqrt{s}w(st)$, translated as $w(t-k)$ and modulated as $e^{ift}w(t)$. In this section we describe the construction of Wavelet-Packets and some of its properties.

The starting point is a low pass filter sequence $h = h_j$, ($j = 1, 2, \ldots, N$, where N is the width of the filter). This filter $h_j$ must possess some smoothness and finiteness conditions as described by Daubechies (1988). Define a complementary filter $g = g_j$ as



$$g_j = (-1)^j h_{1-j}.$$

We denote the 'convolution and subsampling' operation with $h_j$ and $g_j$ as $H$ and $G$ given by

$$H\zeta(i) = \sum_j h_{j-2i}\zeta(j)$$
$$G\zeta(i) = \sum_j g_{j-2i}\zeta(j). \tag{1}$$

Here $\zeta(j)$ represents the discrete data array. Following Coifman & Wickerhauser (1990) we use three indices for a Wavelet-Packet $w_{f,s,k}$, where the index $f$ has a unique relationship with the frequency. The indices $k$ and $s$ describe the position and scale of the Wavelet-Packet. All the $f$ values may be obtained using the following recursive scheme:

$$w_{2f,s+1,0}(t) = Hw_{f,s,0}$$
$$w_{2f+1,s+1,0}(t) = Gw_{f,s,0}. \tag{2}$$

The position of the Wavelet-Packet is obtained using $w_{f,s,k}(t) = w_{f,s,0}(t-k)$. The $k$ index takes only the values possible at that scale. The resolution of the data limits the $f$ index that can be obtained at any scale. The recursive construction scheme given by Eqs. 2 generates a hierarchy of Wavelet-Packets. This hierarchy is shown in figure 2. We show only the $f$, $s$ indices in the figure.

The scale increases in dyadic steps $(1, 2, 4, \ldots)$ with the hierarchy. Each of the Wavelet-Packets generated by Eqs. 2 may be translated. This yields a large and redundant collection of basis functions. This collection is termed 'a library of Wavelet-Packets' by Coifman & Meyer (1989). It can be shown that the Wavelet-Packet library contains a multitude of orthonormal bases.

A measured signal is a sampled version of the original continuous variable. The sampling in data acquisition can be considered as the decomposition of the continuous variable into the Wavelet-Packet basis at the scale of the sampling interval. We set this basis as $w_{0,0,k}$

where $k$ takes integer values up to the data length$(L)$. The collection $w_{0,0,k}$ forms a complete orthonormal basis. This is the the first subset of the Wavelet-Packet library which we identify as a basis.

The property that the siblings are equivalent to their parent can be used to identify further subsets of the Wavelet-Packet library which form a basis. The siblings of $w_{0,0,k}$ are $w_{1,1,k}$ and $w_{0,1,k}$. Therefore the collection $w_{1,1,k}$ and $w_{0,1,k}$ where $k = 1, 2, \ldots, L/2$, form a second basis. (Since the siblings are at twice the scale of $w_{0,0,k}$, $k$ in $w_{1,1,k}$ and $w_{0,1,k}$ takes only values up to $L/2$.) Similarly the collection of all Wavelet-Packets at any level in the hierarchy shown in figure 2, for admissible values of $k$, form a complete orthonormal basis. In the simplest case, if we start with filter coefficients $h_0 = \frac{1}{\sqrt{2}}$, $h_1 = \frac{1}{\sqrt{2}}$, then we generate the set of Walsh functions (e.g., Beauchamp 1984) as the Wavelet-Packets at the final level. Figure 3 shows the first eight Walsh functions. Figure 4 shows Wavelet-Packets of some regularity constructed with the Coiflet filter of range 30 (Zubair et al. 1990).

We are not restricted to using Wavelet-Packets of the same scale for all the basis functions. We may choose some Wavelet-Packets at some level in the hierarchy and the siblings of the rest of the Wavelet-Packets at other levels. Extending this reasoning we can make an orthonormal basis with Wavelet-Packets from several levels. All that has to be ensured is that every 'line of heredity' in the hierarchy is represented exactly once.

The optimal representation of the data within the library of Wavelet-Packet is obtained by using the so-called 'Best Basis Algorithm' (Coifman & Wickerhauser 1990). The choice poses two questions. How is the signal to be segmented? Which basis is to be used for each of these segments? These two questions have to be answered simultaneously. We need to evaluate all possible segmentation schemes and all basis functions for each of these segments to find the optimal representation.

This gives us an immense search space. The search is started at the smallest scale. The sampling in the data acquisition can be considered as the decomposition of the continuous signal into the Wavelet-Packet basis $w_{0,0,k}$ under translation $k$ at the sampling interval. That is, we set the discrete signal as $x_{0,0,k}$. ($x_{f,s,k}$ denotes the coefficient of the corresponding Wavelet-Packet, $w_{f,s,k}$.)

We now compute the coefficients of the siblings ($x_{1,1,k}$ and $x_{0,1,k}$). Since the siblings live on a scale twice as large as the parent so that the translation step is twice as large, we still have the same number of coefficients at both levels. We compute the coefficients of the data for all the Wavelet-Packets generated by translating the siblings.

Since the representation of the data as coefficients of the parent Wavelet-Packet or as coefficients of the siblings is equivalent, we could choose one over the other. We have used an 'entropy' criterion, which is essentially a 'least distance' criterion, to choose one over the other. The entropy is equivalent to the Shannon-Weaver measure of information of a sequence (Coifman & Wickerhauser 1990).

This criterion is the natural choice for data characterization algorithm. Other criteria such as counting the bits required to code different representations may be suited for data compression.

Keeping the minimum entropy criterion, we continue the search for all k values. Thereafter we compute the coefficients at the next level of the hierarchy. We compare the entropy of the new representation to the previous minimum for the segment of data under consideration and retain the representation having the lesser of the two entropies. This search is continued to the last level. The search algorithm ensures that the entire signal is covered by disjoint segments. Each segment customizes for itself the optimal basis from the many possibilties.

The signal is completely described by the coefficients in the optimal basis so determined. No information has been lost. To compress the data we rank the retained coefficients by their magnitude. The least significant ones are discarded. The image can be reconstructed by summing all the retained Wavelet-Packets weighted by their coefficients.

For a one-dimensional record of length N, the decomposition and search algorithm has a complexity of $O(rN \log_2 N)$, where $r$ refers to the length of the filter. For images of size $N^2$ the algorithm has a complexity of $O(rN^2 \log_2 N^2)$.

## 3. Index of Performance

An index of performance used to quantify the compression achieved is the 'Coefficient Compression Ratio' (CCR). It is defined as the ratio of the total number of coefficients in the Wavelet-Packet decomposition to the number of coefficients in the retained basis. The compression ratio measures the ability of the transform to characterize data. Another index is the Bits-Per-Pixel (BPP) ratio. It is defined as the average number of bits needed to code each pixel. This index includes the overhead bits and is a true measure of the compression achieved.

To implement any algorithm an allowable distortion criterion has to be established. Both the information content and the subjective appearance of the image are degraded by the distortion. The information lost can be quantified by a measure such as the Signal-to-Noise-Ratio (SNR) or the Normed Mean-Square Error (NMSE) between the orignal and the reconstructed image. These are defined as :

$$NMSE = \frac{1}{p^2 NM} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} [f(i,j) - f'(i,j)]^2 \tag{3}$$

$$SNR = 10 \log_{10} \frac{1}{NMSE} \tag{4}$$

where $f$ refers to the original image and $f'$ to the reconstruction, and $p$ refers to the peak value of the data. Definitions for one-dimensional data follow trivially.

## 4. Compression of One-Dimensional Data

The top trace in figure 5 shows the velocity measured in the atmospheric surface layer about 6.5 meters above the ground over a substantial stretch of a wheat field at the Connecticut Agricultural Experiment Station. The microscale Reynolds number is about 2000. Thus we expect a large scaling range. We have analyzed several data segments, and present results for one typical segment of 16384 points. The Wavelet-Packet transform was applied and the best basis functions for the data were determined as described in section 2. Reconstructions of the top trace were obtained by retaining some small number of coefficients. The second trace from the top was reconstructed using only about 5% of the coefficients. That is, the Coefficient Compression Ratio (CCR) was about 21. The reconstruction shows excellent fidelity to the original signal. The following traces are reconstructions with approximate CCR's of 41, 234 and 497 respectively. It is seen that the algorithm is able to capture local events efficiently and preserve local edges in the signal. The primary effect of increasing CCR appears to be the increasing absence of small scales; even for CCR's as high as 234 and 497, the large scale features are preserved. If the reconstruction were attempted by means of Fourier techniques, the sharp local variations would be smoothed out completely. Because of the Gibbs phenomenon, Fourier transform characterizes abrupt changes rather poorly.



Figure 6 is a plot of SNR as a function of CCR. As expected the quality of reconstructions degrades with CCR, but is still acceptable even at CCR's of order 500.

Similar reconstructions have been attempted for a variety of signals in laboratory flows, in particular those close to the wall in the boundary layer. The results are comparable in quality to those in figure 5. The effectiveness of such relatively high CCR's suggests the ubiquitous presence of strong structure in the signal. Notice that the energy of the reconstructed signals is a very large fraction of the original; even for CCR's of the order of a few hundreds, typically more than 90% of the energy is retained. Reconstructions with high CCR are by definition low-dimensional, and it is instructive to note that low-dimensional reconstructions contain most of the energy.

Computations performed on highly intermittent quantities such as the energy dissipation (strictly, $(\frac{du}{dt})^2$ ) yield poorer reconstructions. This is not surprising because a highly intermittent signal has poor spatial correlation (or structure). The Wavelet-Packet transform can only pick out the structure in signals, in the absence of which its performance will diminish.

Typically, decomposition and reconstruction of one dimensional records considered here take on the order of a few minutes on a VAX II/GPX.

## 5. Compression of Two-Dimensional Images

The extension of the Wavelet-Packet analysis to higher dimensions is quite straightforward and is described by Coifman & Meyer (1989). The two-dimensional basis functions are constructed as tensor products of one-dimensional Wavelet-Packets. For convenience of computation, we restrict attention to tensor products of Wavelet-Packets of the same scale. The rules for choosing different orthonormal basis functions are direct extensions of the one-dimensional case. As before, data compression is achieved by ranking the coefficients of the 'best basis' and discarding the least significant ones.

Figure 7 shows a two-dimensional map of the concentration field in a turbulent jet at a nozzle Reynolds number of about 4000. The gray level indicates the concentration of the fluorescing dye, and the image has been obtained by the laser-induced-fluorescence technique whose details can be found in Prasad & Sreenivasan (1989, 1990). The jet pictures obtained by Prasad and Sreenivasan are typically 1300 x 1030 pixels in extent, and each pixel is a twelve-bit word. To economize on the memory and computational requirements, we have analyzed various fractions of these images using Wavelet-Packets; for example, figure 7 is 1024 x 512 pixels in extent. We have also analyzed images at a lower Reynolds number of 2000; these latter images have a steady laminar region near the nozzle exit, transitional region somewhat downstream, and the turbulent regime far downstream. The analysis helps us to understand the downstream evolution of various scales in the concentration field. For brevity, we present results only for the higher Reynolds number jet. The figures are presented in gray scales. A representation via color is more instructive (and these were presented at the meeting), but exorbitant reproduction costs preclude this option.

Figure 8, 9 and 10 show reconstructions of figure 7 for various CCR's. The reconstructions are obtained by keeping smaller and smaller number of coefficients. The CCR values are 50, 100 and 160 respectively. It is clear that the reconstructed images retain many structural features tolerably well even at the highest CCR's.

The behavior of SNR as a function of CCR is similar to that for one-dimensional signals (figure 11). It may be helpful to include data on the the Signal-to-Noise Ratio as a function of BPP, or the average number of bits used to code each pixel (see section 3). This is given in figure 12.

We have by no means optimized the computer programs, and it is perhaps not as useful here to quote typical computation times as it is for the one-dimensional case. Yet the following number may give some indication. In the current version of our programs, decomposition and reconstruction of images of the size of figure 7 takes



## 6. Summary and Conclusions

Turbulent signals and images possess some temporal and spatial structure. There is always some correlation between neighboring elements which, in some sense, implies that there is redundancy of information. This in principle allows data compression. We may also be willing to tolerate some specified degradation in data compression and reconstruction. Often, as in the case of turbulent jet images discussed in section 5, the degradation is not perceptible except for high CCR's. This is because the precision in data coding may surpass the discrimination of human eye. In any case, some degradation may indeed be tolerated depending on applications.

Several data compression techniques are now available. We have outlined one of them and briefly presented some results. The technique allows the definition of the structure in terms of its shape, size and frequency (or wavenumber). A comparison between the performance of the present technique and several others has been made. The exact comparison can be carried out at several levels, and depends on the precise parameters held fixed. For example, one can specify the maximum mean-square error permissible in the reconstruction, and ask questions about the speed of the method and the computer memory required. One can keep a fixed CCR and determine the SNR of reconstructions. Furthermore, when comparing with the Fourier representation, details of image segmentation will be crucial. For all these reasons, a proper comparison requires a lot of detail and will not be reported here; it can be found in Zubair et al. (1990). It suffices to say that, overall, the Wavelet-Packet transform performs somewhat better than its competitors. We believe that the greatest advantage of Wavelet-Packets lies in the explicit identification of spatial position, frequency as well as the scale of structures. This immediately suggests several possible applications in turbulence. These studies are currently under way.

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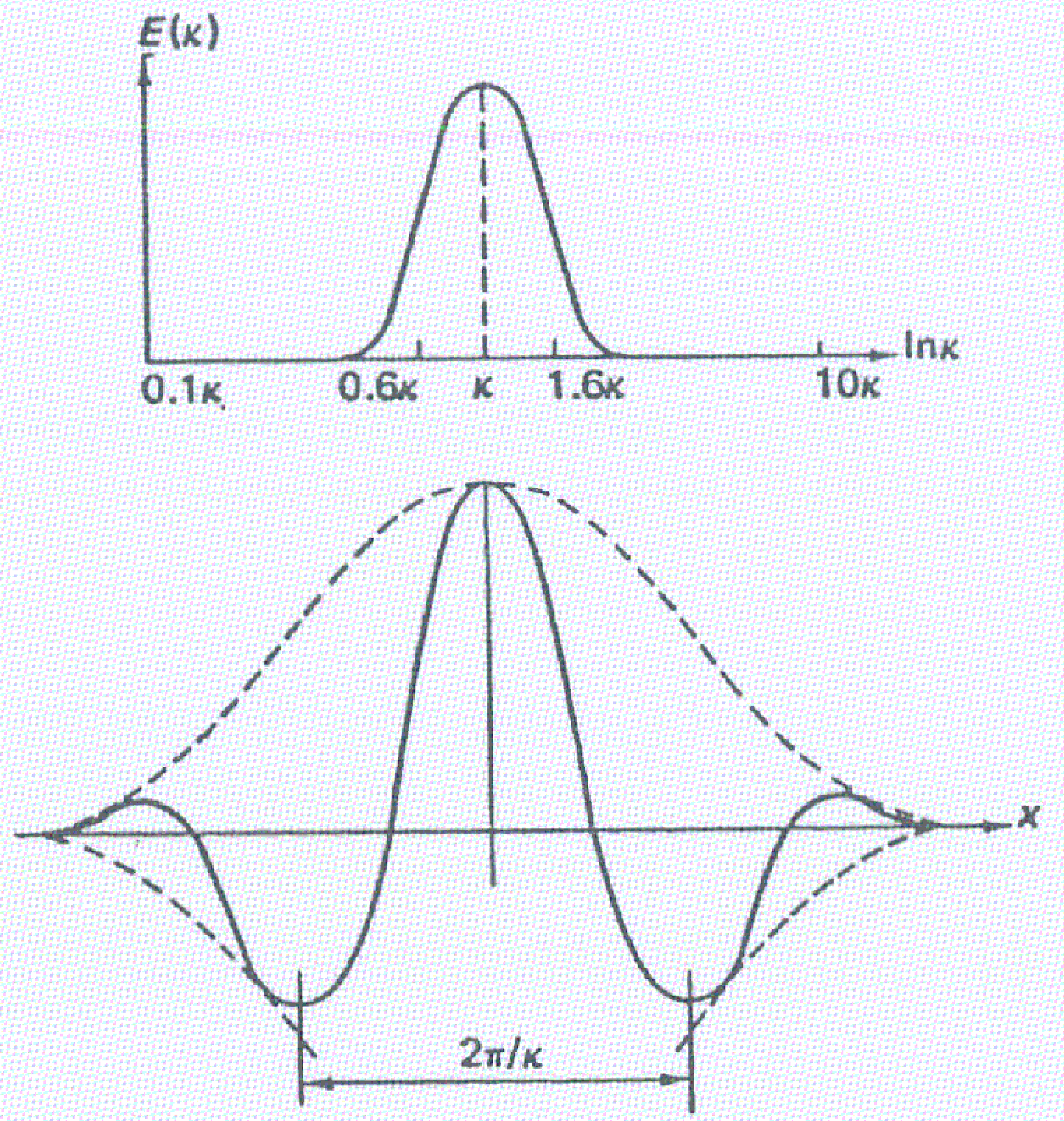


Figure 1: An eddy of wave number $\kappa$ and wavelength $2\pi/\kappa$. From Tennekes & Lumley (1972).



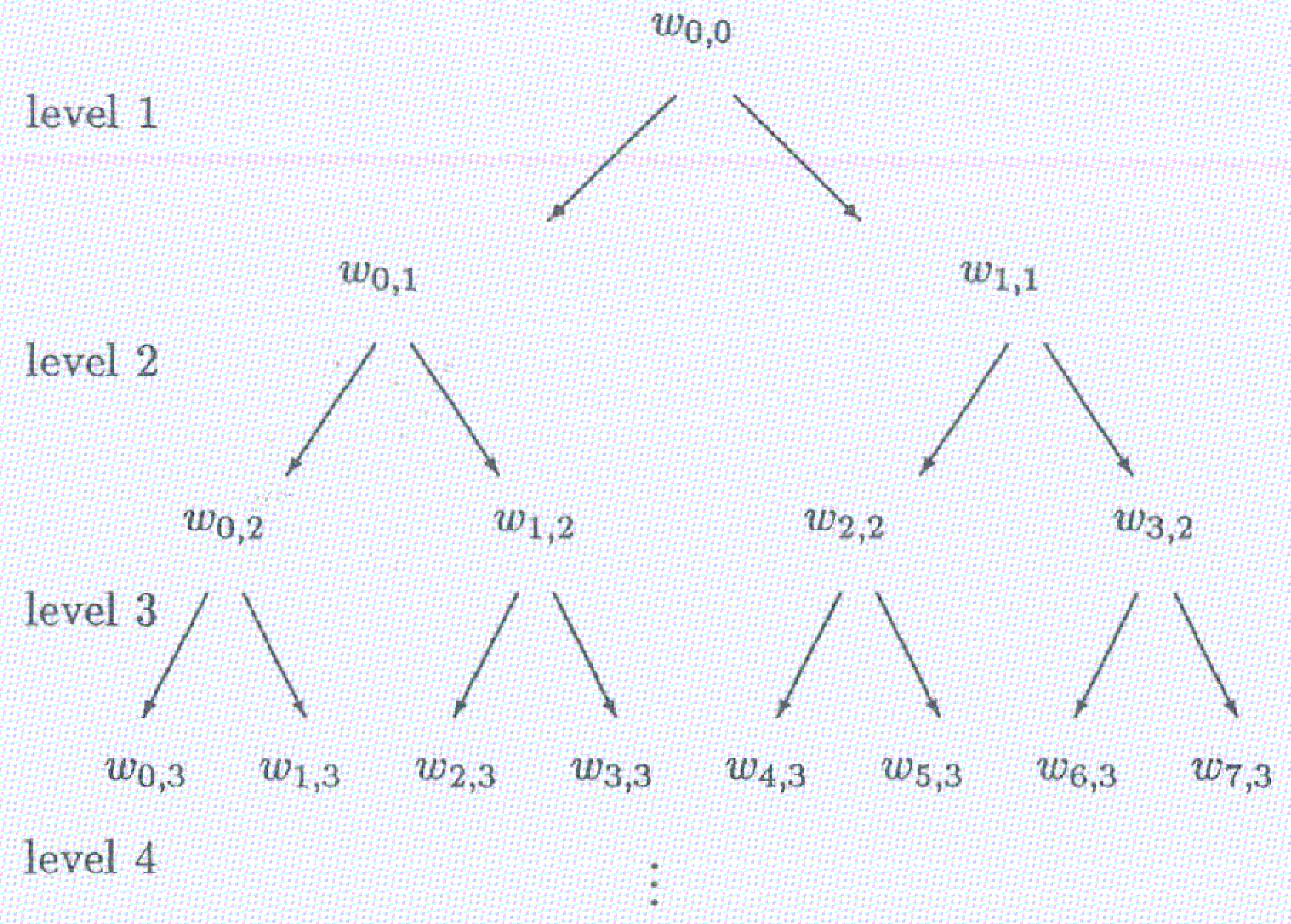


Figure 2: Hierarchy of Wavelet-Packets

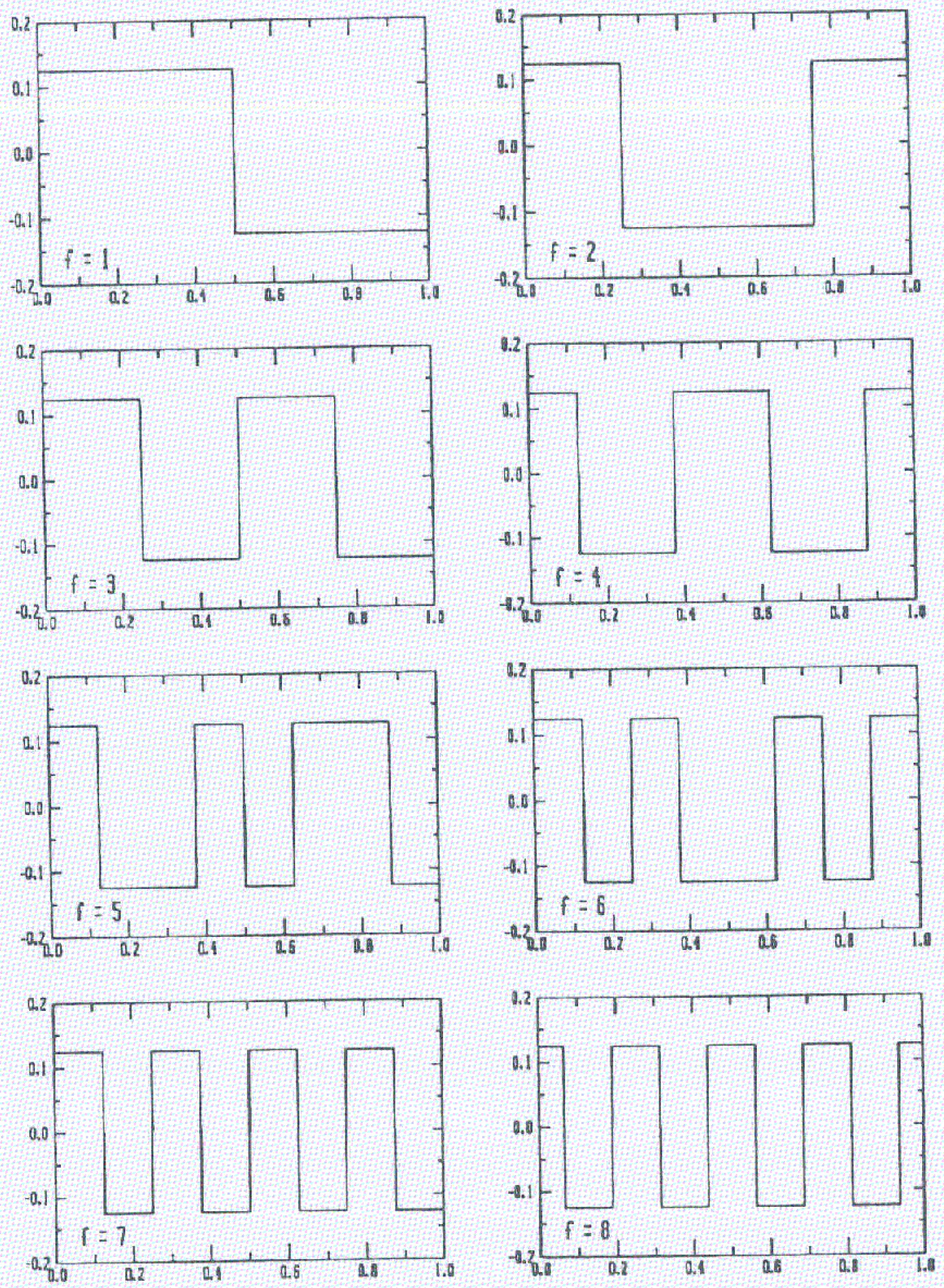


Figure 3: Walsh functions (see Beauchamp 1984).



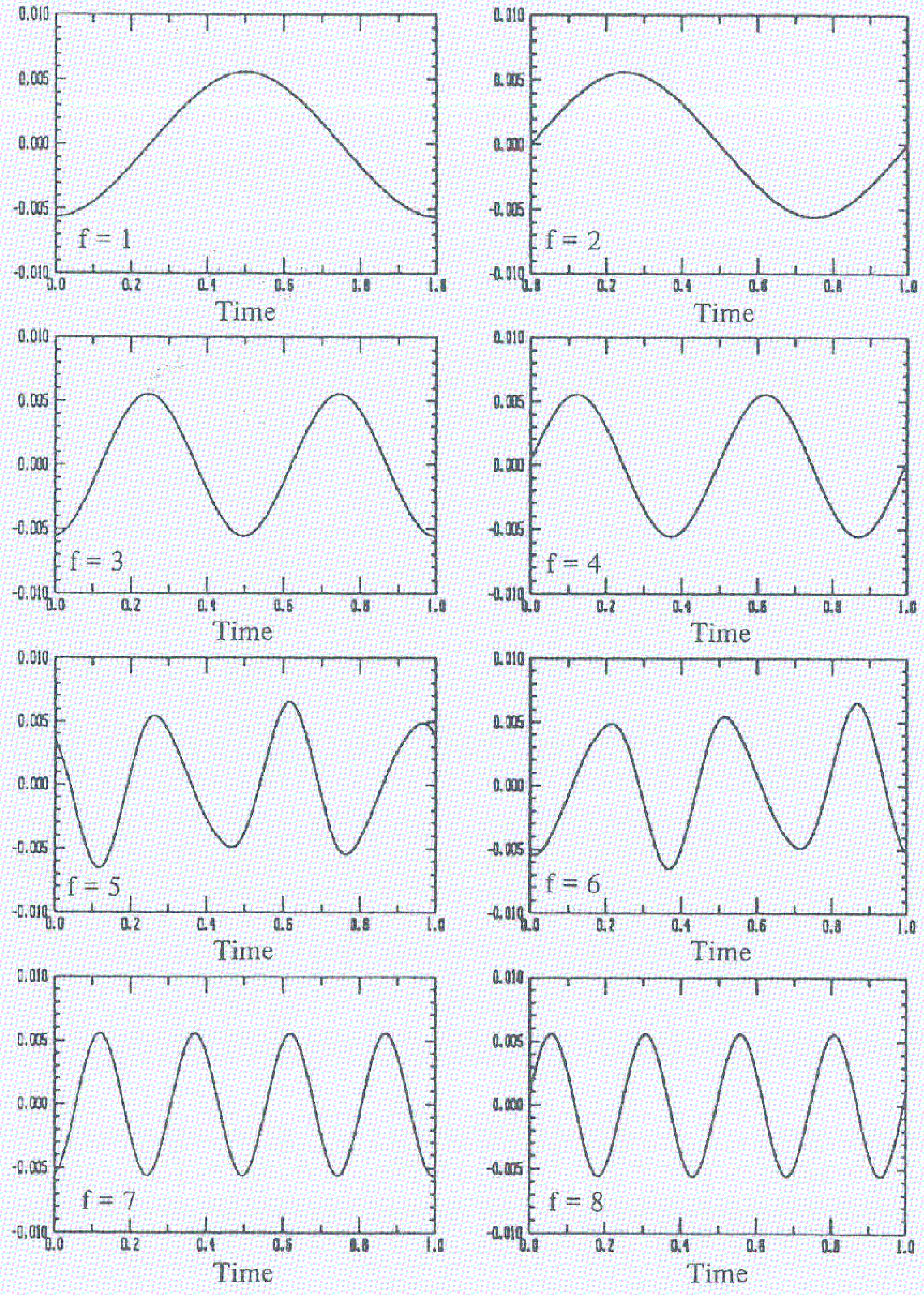


Figure 4: Coiflet functions (see Zubair et al. 1990).

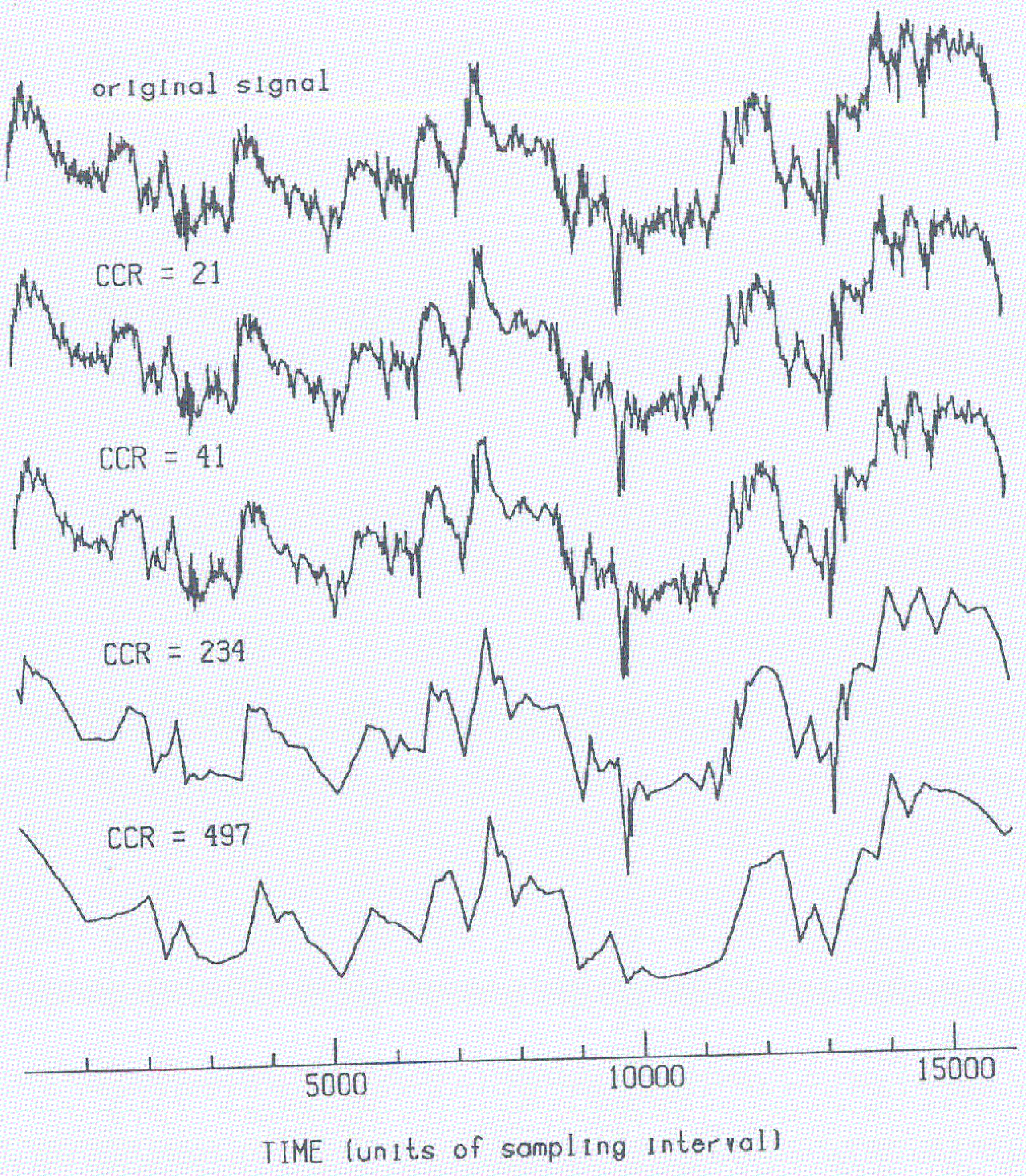


Figure 5: Compression and reconstruction of atmospheric turbulence velocity using Wavelet-Packet transforms.



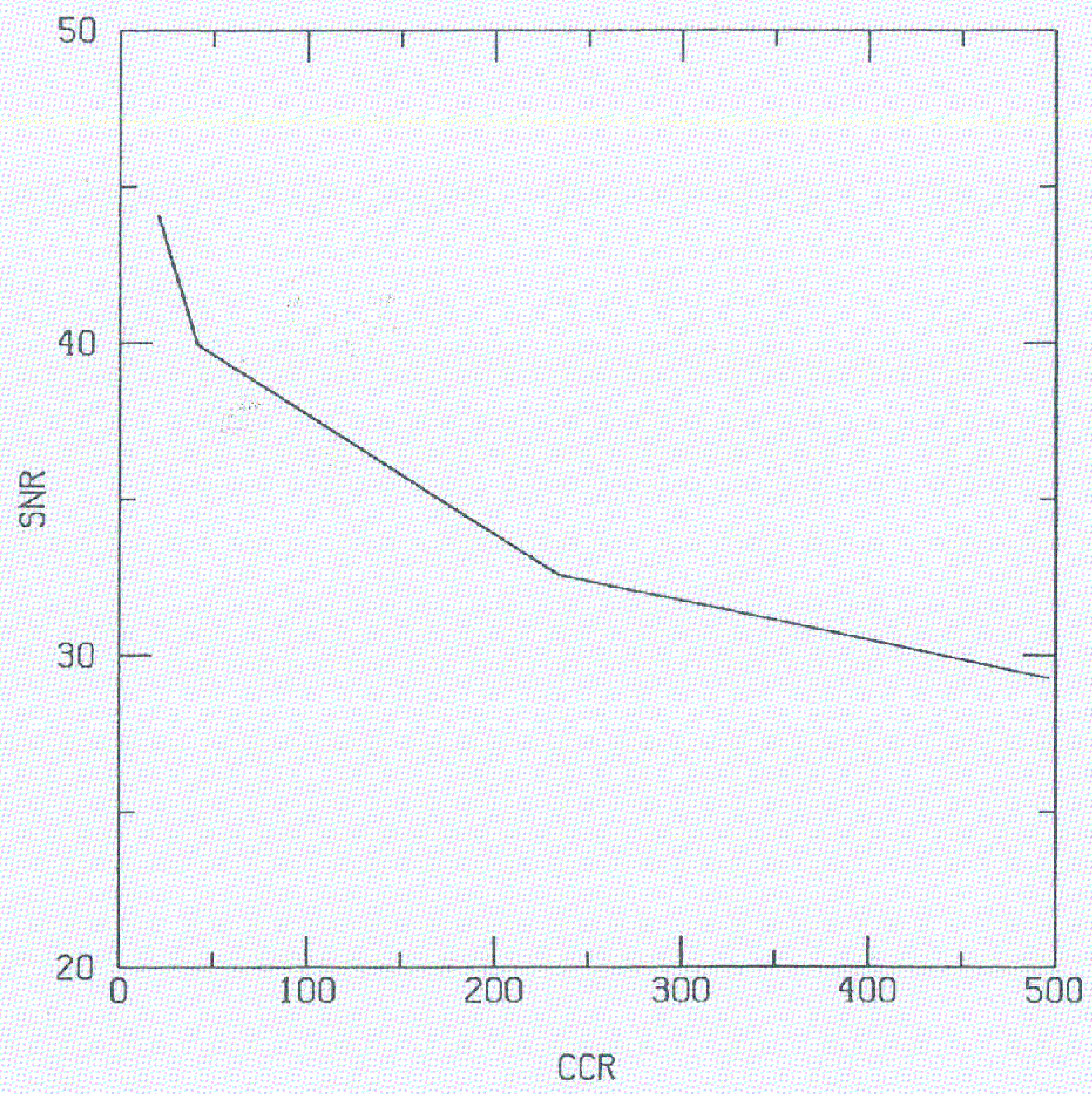


Figure 6: SNR as a function of CCR for one-dimensional velocity data

Figure 7: A two-dimensional map of the concentration field in an axisymmetric jet. A thin slice of the field has been obtained by laser-induced-fluorescence (see Prasad & Sreenivasan 1989, 1990).



Figure 8: Wavelet-Packet transform reconstruction of figure 7 for a CCR = 50.

Figure 9: Wavelet-Packet transform reconstruction of figure 7 for a CCR = 100.



Figure 10: Wavelet-Packet transform reconstruction of figure 7 for a CCR = 160.

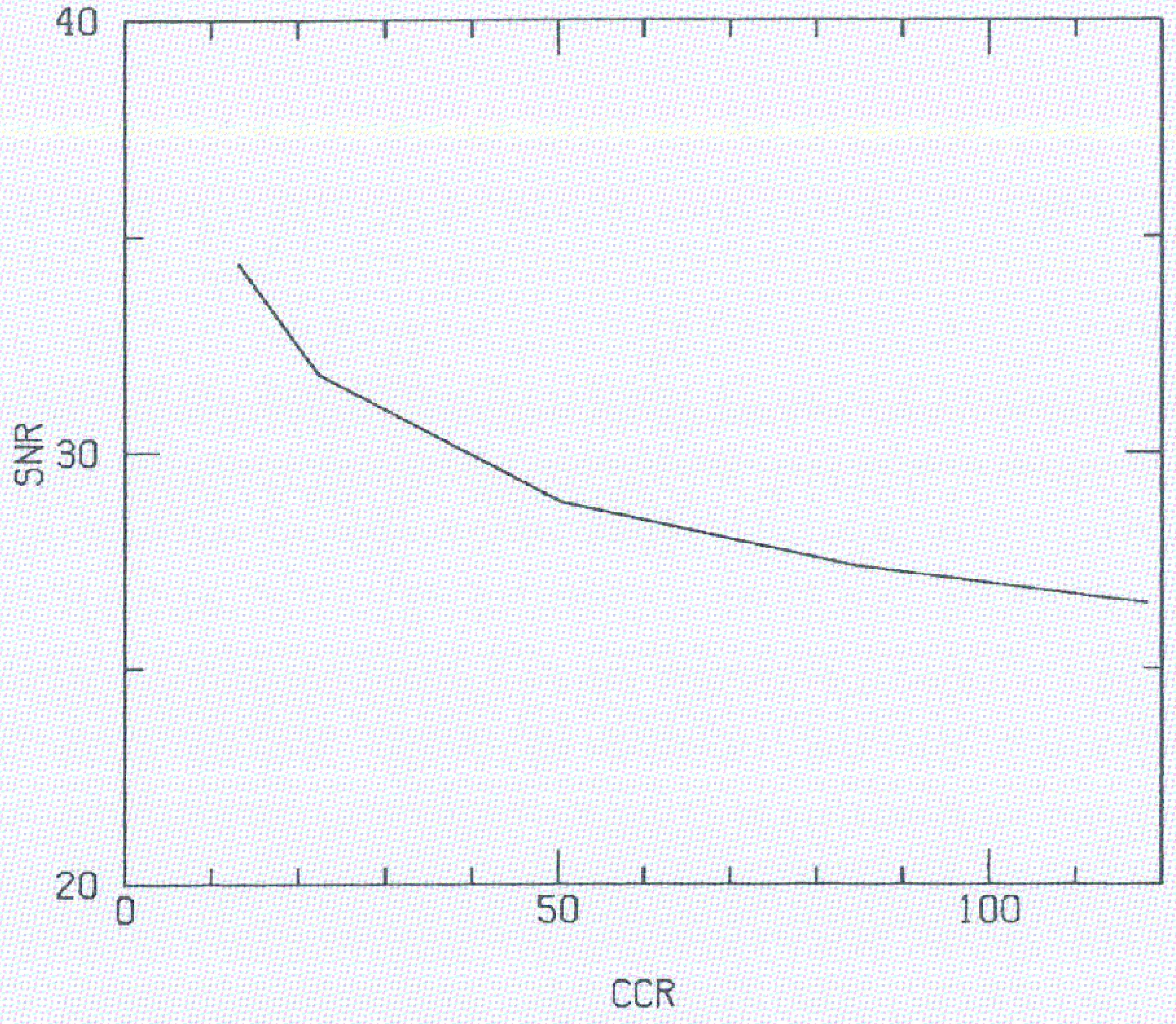


Figure 11: SNR as a function of CCR for a two-dimensional concentration field



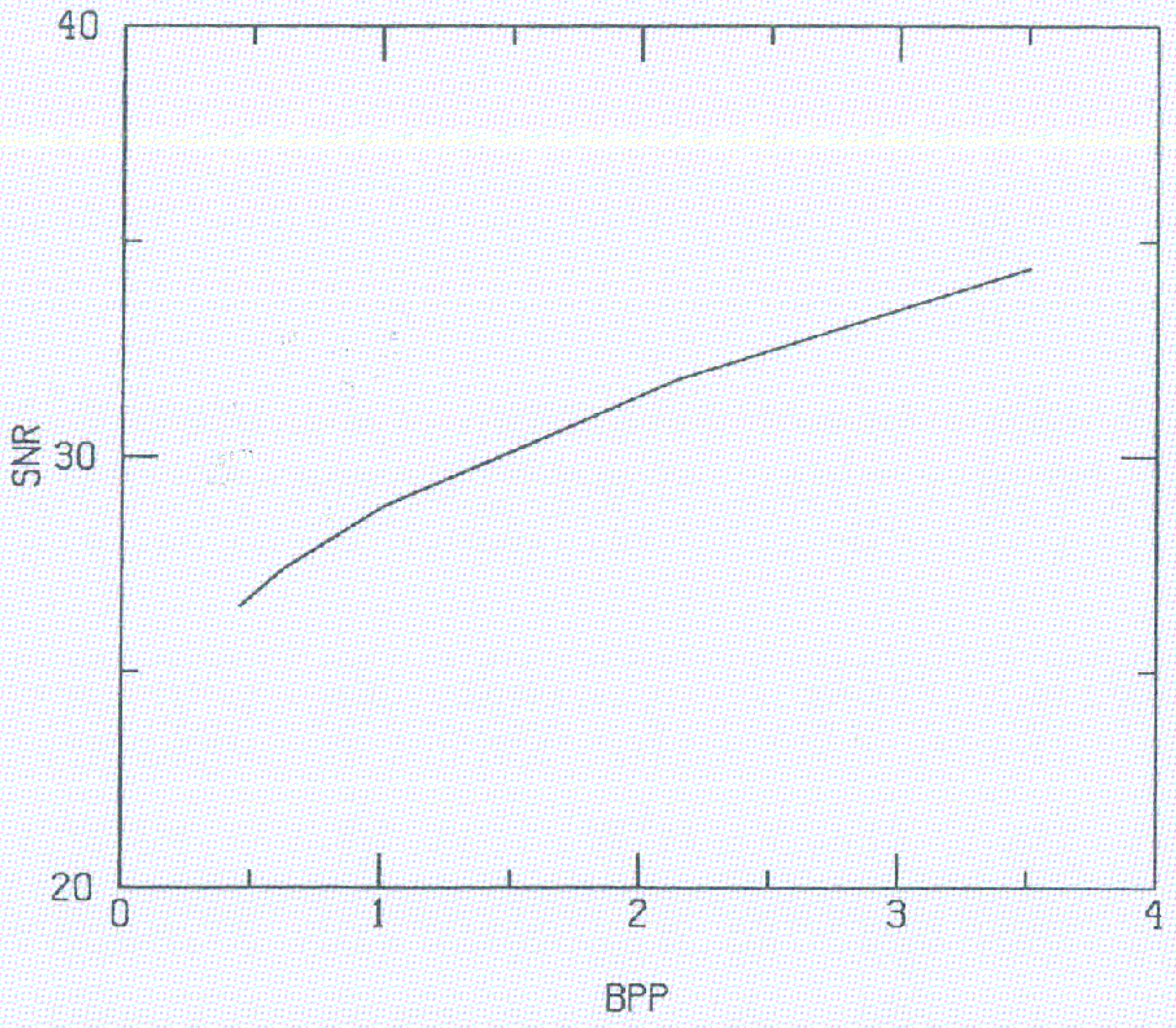


Figure 12: SNR as a function of BPP for a two-dimensional concentration field