Fractal Geometry of Isoscalar Surfaces in Turbulence: Theory and Experiments

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We present a Lagrangian calculation of the area of isothermal (or isoconcentration) surfaces in a medium of fluid turbulence. We argue that such surfaces appear fractal above some inner scale which depends on the Reynolds number. The fractal dimension is estimated theoretically. The theory is compared to experiments in which a dye is used as a passive scalar in various turbulent flows. We find excellent agreement in many details.

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It is a common assertion that turbulence has fractal properties [1–3], and much of the recent work in the field has been propelled by hopes for understanding the small-scale structure of turbulence using fractal, or even multifractal, notions [4]. Essentially all the theoretical work, however, has been phenomenological in nature [5], having no basis in fluid dynamics, and offering little in comparing theory (other than dimensional analysis) with experiments. In this Letter we present a calculation of the geometric properties of isoscalar surfaces. We show on the basis of fluid mechanics that such surfaces appear fractal on scales larger than some (Reynolds-number dependent) inner length λ^*, while they appear smooth on scales smaller than λ^*. This inner scale and the Hausdorff dimension of the isoscalar surfaces are estimated theoretically. The theory is compared here with experiments that use a dye as a passive scalar in a turbulent jet [6] and other turbulent flows [4], and excellent agreement is found. In fact, the theory seems to offer an explanation for many of the experimental findings.

For our purposes a passive scalar is a field variable T that satisfies the equation of motion

$$\partial T/\partial t+\mathbf{u}\cdot\nabla T-\kappa\nabla^2 T=0\,, \tag{1}$$

where $\mathbf{u}$ is the velocity field in which the scalar is embedded, and κ is the scalar diffusivity. One can think of T as a temperature field, or a concentration field, etc. The fields are functions of space in a container of integral scale L. We shall consider the geometrical properties of surfaces on which T is constant. These are "isoscalar surfaces." In our theoretical calculation [7,8] we estimate from above the area of a piece of an isoscalar surface contained in a ball of radius r. This estimate contains two terms, one that scales like r^2 and another that scales with a higher power in r. The second term overcomes the first term if r is larger than λ^*.

Imagine that one needs to calculate the area of the piece of isoscalar surface contained in a ball B of size r, about a point $\mathbf{y}=\mathbf{y}_0$, at some time $t=t_0$. Our techniques [8] require following this ball in its Lagrangian evolution for a short time δt. This duration is determined by the requirement that the Lagrangian stretching remains of order 1. In order to simplify the exposition, and in view of the shortness of δt, we can choose coordinates

$$\mathbf{x}=\mathbf{y}-\mathbf{u}_0(t-t_0)\ \text{ for }\ 0\le t-t_0<\delta t\,, \tag{2}$$

where $\mathbf{u}_0$ is the mean velocity in the ball:

$$\mathbf{u}_0=(4\pi r^3/3)^{-1}\int_{y\in B}d^3y\,\mathbf{u}(\mathbf{y},t_0)\,. \tag{3}$$

For δt small these coordinates represent the linear approximation of the Lagrangian paths, and this suffices for the present purposes. One obvious requirement for δt is that $\mathbf{y}_0+\mathbf{u}_0\delta t$ is not larger than the integral scale L. This makes δt of the order $\delta t<L/|\mathbf{u}_0|$. The scalar in this approximate Lagrangian frame is denoted by $\tilde T(\mathbf{x},t)$, $\tilde T(\mathbf{x},t)=T(\mathbf{x}+\mathbf{u}_0(t-t_0),t)$. Its equation of motion is immediately deduced from (1), and it reads

$$\partial\tilde T/\partial t+(\mathbf{u}-\mathbf{u}_0)\cdot\nabla\tilde T-\kappa\nabla^2\tilde T=0,\quad 0<t-t_0\le\delta t\,. \tag{4}$$

We shall estimate the area of the piece of the isotherm $\tilde T=\tau$ contained in the ball B.

The main tools for estimating the area [7,8] are (i) the co-area formula of geometric measure theory [9], and (ii) a nonlinear change of dependent variable in the equation for temperature.

The co-area formula is a way of relating the weighted area of isothermal surfaces to integrals in the physical space. Denote by $\mathcal{A}_\tau$ the surface on which $\tilde T(\mathbf{x})=\tau$. Assume that $\tilde T$ is at least Lipschitz; then for any ψ (Borel measurable, non-negative function),

$$\int\psi(\mathbf{x})|\nabla\tilde T|d^3x=\int_0^\infty d\tau\int_{\mathcal{A}_\tau}\psi(\mathbf{x})dH^{(2)}(\mathbf{x})\,, \tag{5}$$

where $H^{(2)}$ is the two-dimensional Hausdorff measure. ($dH^{(2)}$ equals a differential element of surface for smooth surfaces.) In particular, a good choice of $\psi(\mathbf{x})$ is the form $\psi(\mathbf{x})=\chi_B(\mathbf{x})\phi(\tilde T(\mathbf{x}))$. Here $\chi_B(\mathbf{x})$ is a smoothed-out characteristic function of the ball B (being unity within the ball and zero outside the ball), and ϕ is an arbitrary (L^2) function of $\tilde T$. This choice provides a way to calculate the weighted integral

$$I_B(\phi)=\int d\tau\,\phi(\tau)\,\mathrm{area}_B(\tau)\,, \tag{6}$$

where $\mathrm{area}_B(\tau)$ is the area of a piece of $\mathcal{A}_\tau$ contained in B. Ideally, we could choose ϕ as a delta function, to turn (6) to the area of a sharply chosen isoscalar. However, our techniques do not allow the use of a delta function. To proceed, we perform the nonlinear change of variables

$g(\mathbf{x},t)=\chi_B(\mathbf{x})G(\tilde{T}(\mathbf{x},t))$, where we set

$$G(\tilde{T})=\int_{\tilde{T}(y_0,t_0)}^{\tilde{T}}(\tilde{T}-\tau)\phi^2 d\tau\,. \qquad (7)$$

Using Eq. (4) we deduce the equation of motion for g, which reads

$$[\partial_t+(\mathbf{u}-\mathbf{u}_0)\cdot\nabla-\kappa\nabla^2]g+\kappa\chi_B[\phi(\tilde{T}(\mathbf{x}))]^2|\nabla\tilde{T}|^2=R\,, \qquad (8)$$

where

$$R=G(\tilde{T})[(\mathbf{u}-\mathbf{u}_0)\cdot\nabla\chi_B-\kappa\Delta\chi_B]-2\kappa G'(\tilde{T})\nabla\tilde{T}\cdot\nabla\chi_B\,. \qquad (9)$$

Integrating, we get

$$\frac{1}{\delta t}\int_{t_0}^{t_0+\delta t}dt\int_B\chi_B(\mathbf{x})\phi^2(\tilde{T}(\mathbf{x},t))|\nabla\tilde{T}(\mathbf{x},t)|^2 d^3x=\frac{1}{\delta t\kappa}\left(\int_B[g(\mathbf{x},t_0)-g(\mathbf{x},t_0+\delta t)]d^3x+\int_{t_0}^{t_0+\delta t}\int_B R\,d^3x\,dt\right). \qquad (10)$$

Recall that we need $I_B(\phi)=\int_B\chi_B(\mathbf{x})\phi(\tilde{T}(\mathbf{x},t))|\nabla\tilde{T}|d^3x$. The Cauchy-Schwartz inequality

$$\left(\frac{1}{\delta t}\int_{t_0}^{t_0+\delta t}dt\,I_B(\phi)\right)^2\le\left(\int_B\chi_B d^3x\right)\left(\frac{1}{\delta t}\int_{t_0}^{t_0+\delta t}dt\int_B\chi_B\phi^2(\tilde{T})|\nabla\tilde{T}|^2d^3x\right) \qquad (11)$$

allows us to estimate the left-hand side of (11) by the right-hand side of (10), multiplied by $\int_B\chi_B d^3x$. The calculation contains only routine manipulations [8], and is not reproduced here. The result is

$$\frac{1}{2\delta\tilde{T}}\left[\int_{T(x_0,t_0)-\delta\tilde{T}}^{T(x_0,t_0)+\delta\tilde{T}}d\tau\left(\frac{1}{\delta t}\int_{t_0}^{t_0+\delta t}dt\,\mathrm{area}_B(\tau)\right)\right]^2\le\tfrac{1}{2}r^4[c_3+c_2r\delta u_r/\kappa+c_1r^2/\kappa\delta t]\,. \qquad (12)$$

In (12), c_1, c_2, and c_3 are nondimensional constants. $\delta\tilde{T}$ represents the maximal temperature fluctuation registered in B in the time interval $[t_0,t_0+\delta t]$:

$$\delta\tilde{T}=\sup[|\tilde{T}(\mathbf{x},t)-T(\mathbf{y},s)|],\quad \mathbf{x}\in B,\quad \mathbf{y}\in B,\quad t,s\in[t_0,t_0+\delta t]\,. \qquad (13)$$

δu_r is the average velocity fluctuation in the ball B during the same time,

$$\delta u_r=(1/\delta t)(4\pi r^3/3)^{-1}\int_{t_0}^{t_0+dt}dt\int_B|\mathbf{u}(\mathbf{x},t)-\mathbf{u}_0|d^3x\,. \qquad (14)$$

The left-hand side of Eq. (12) represents an average value of the square of the area of the piece of the isotherm $[\mathbf{y}|T(\mathbf{y},t)=T(\mathbf{y}_0,t_0)]$ contained in B in the interval $[t_0,t_0+\delta t]$. For sufficiently small r the first term always dominates, leading to a smooth area. On the other hand, the first two terms become comparable for $r=\lambda^*$, which is the Reynolds-number-dependent length which satisfies the equation

$$\delta u_{\lambda^*}\lambda^*/\kappa=c_3/c_2\,. \qquad (15)$$

To proceed, we need a physical estimate of the typical velocity difference across the ball δu_r. This is where the properties of the velocity field which drives our scalar T enter. For fully developed turbulence we can use the standard estimate [10]:

$$\delta u_r\sim(r/L)^\zeta U\sim(r/L)^\zeta|\mathbf{u}_0|\,, \qquad (16)$$

where U is the velocity on the integral scale L. This velocity is used to define the Reynolds number Re on the integral scale, UL/ν, ν being the kinematic viscosity. In the usual Kolmogorov estimate one finds $\zeta=\frac{1}{3}$. Using this and the estimate for δt, $\delta t<L/|\mathbf{u}_0|$, we find readily that the second term in (12) is always larger than the third. We are thus left with the competition of the first and second terms. We recognize that the left-hand side of (15) is nothing but the Reynolds number (or more strictly, the Peclet number) on the scale r. This result makes perfect physical sense. It says that isoscalar surfaces would appear fractal on scales larger than those on which the local Reynolds number exceeds a threshold, c_3/c_2. We denote the smallest scale for which (15) is obeyed by λ^*. To calculate the Re dependence of λ^*, we use (16) in (15), and find

$$\lambda^*=L\,\mathrm{Re}^{-1/(1+\zeta)}C^*\,, \qquad (17)$$

where the constant $C^*=[(c_3/c_2)\mathrm{Pr}^{-1}]^{1/(1+\zeta)}$, with Pr being the Prandtl number ν/κ. We recognize [10] that the Kolmogorov scale η has the same scaling dependence on Re, and we predict that λ^*/η is a Reynolds-number-independent ratio, $\lambda^*/\eta=C^*$.

Finally, we want to estimate the fractal dimension above the scale of λ^*. Examining (12) we see that the exponent in r of the second term on the right-hand side is $5+\zeta$. Since this exponent describes the scaling of the square of the area of a piece of an isotherm in a ball of radius r, the dimension for measurements with a yardstick larger than λ^* is $2.5+\zeta/2$. In the standard Kolmogorov-type scaling ζ is $\frac{1}{3}$, and the dimension is estimated to be 2.67. If, however, the exponent ζ differed from $\frac{1}{3}$, so would the dimension.

This theory can be tested against an experiment [6] in which a dye takes the role of a passive scalar. In Fig. 1(a) we show a typical two-dimensional image of the dye in an axisymmetric jet. The image is obtained by laser-induced fluorescence as follows. A Nd-doped yttrium-aluminum-garnet laser beam shaped into a sheet of 200–250 μm thickness was directed into a water tank into which the jet fluid was emerging from a well-contoured nozzle and standard upstream flow management; the jet fluid contained small amounts of fluorescing dye which gets dispersed downstream by turbulence. The fluorescence excited by the laser radiation was captured on a digital camera. The pixel intensity in the digital image is linearly related to the dye concentration in the jet. Different shades of grey in the figure qualitatively reflect

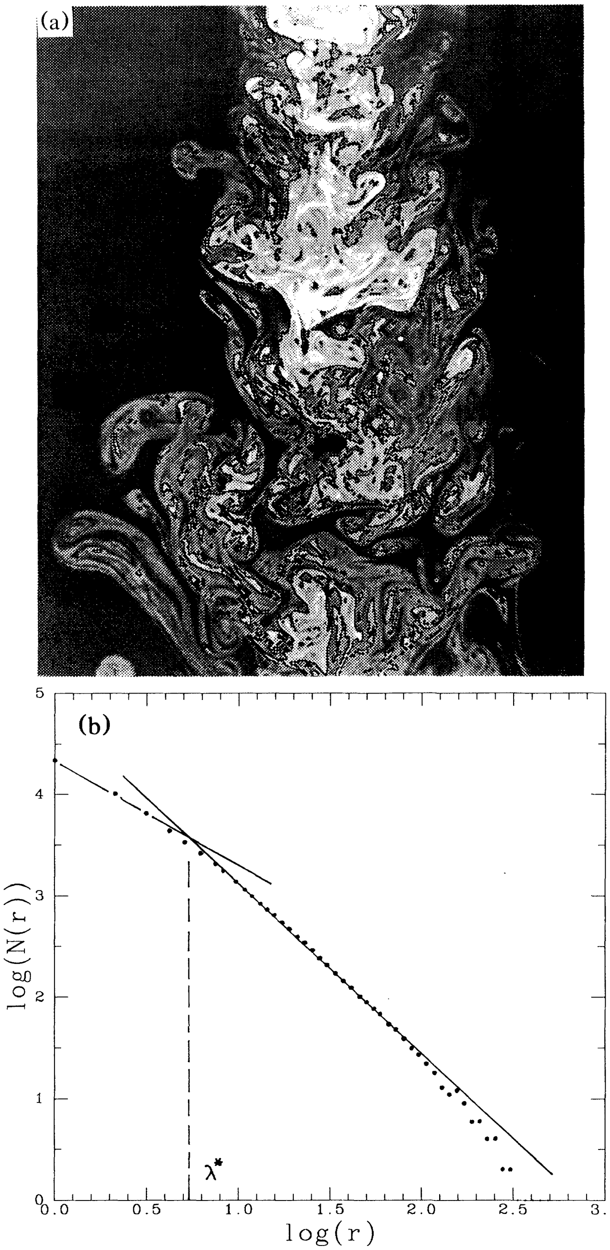

FIG. 1. The image of the dye and the determination of the dimension of an isoscalar. (a) A planar intersection of an isoscalar surface believed to reside largely in the fully turbulent part of the jet. (b) The log-log plot of the number $N(r)$ of the square area elements of size r containing the boundary marked in (a), vs the size r. The slope of the continuous line is -1.67. The dashed line has a slope of -1. The crossover is the scale λ^*, which here is about 10 Kolmogorov scales.

different concentration levels. The jet Reynolds number was about 4000. The sheet thickness was between η and 2η, where here η is the Kolmogorov scale averaged over the image. For more details see Ref. [6]. The image is digitized, and any intersection of isoconcentration surface can be defined at will. The dimension of such a section is estimated by covering it with squares of edge r, and counting the number $N(r)$ of these squares. Figure 1(a) also shows the section of an isoconcentration surface corresponding to the mean concentration level in the image (marked by solid outline). The plot of $\log N(r)$ vs $\log r$ for this section of the surface is shown in Fig. 1(b). Here, since the section resides nearly entirely in the fully turbulent part of the jet, the ζ value can be taken to be $\frac{1}{3}$. The plot shows clearly the two expected scaling regimes, one below λ^* with a slope of 1, and one above λ^* with a slope of 1.67 ± 0.04. We do not have at this time enough data to test the Re dependence of λ^*.

When the conditions are not appropriate for the use of (16), modifications in the dimension of the isoscalar surface are expected. The simplest such situation occurs at the outer boundary of the jet, at which a fractal interface of dimension D_I separates the scalar marked regions from the ambient. We shall assume that the largest contribution to the integral (14) for a ball B near the boundary comes from the boundary itself [11]. If this is so, then near this boundary we expect δu_r to scale like $r^{\zeta+D_I-3}$. Using this estimate, Eq. (12) predicts $D_I = 2.5 + \zeta/2 + (D_I - 3)/2$. The only solution is

$$D_I = 2 + \zeta . \tag{18}$$

If we can still use $\zeta = \frac{1}{3}$, we predict $D_I = 2.33$. This value is in good agreement with the result from box counting [6] that the dimension of this boundary is 2.35 ± 0.05. Notice that the crossover scale here, say λ_B^*, is smaller than λ^*. Repeating the estimate leading to (17) we find that for $\zeta = \frac{1}{3}$, $\lambda_B^*/\lambda^* = \lambda^*/L$. This significant reduction in the crossover scale appears to be confirmed by the box-counting calculations.

Next we can try to apply this theory to situations [12,13] in which the Reynolds number is too low to justify the value $\zeta = \frac{1}{3}$. Our basic estimate can be written as

$$\text{area}_B(r) \sim C r^2 \{C' + [\tilde{\text{Re}}(\mathbf{x},t,r)]^{1/2}\} , \tag{19}$$

where $\tilde{\text{Re}}(\mathbf{x},t,r)$ is the local Reynolds number given by

$$\tilde{\text{Re}}(\mathbf{x},t,r) = \frac{r^{-2}}{t} \int_0^t ds \int_B |\mathbf{u}(\mathbf{y},t) - \mathbf{u}_0| d^3y . \tag{20}$$

Assuming a scaling law $\text{area}_B(r)/r^2 = Q(r/L)^{D-2}$, with Q being a constant, we find

$$D - 2 = \frac{\log[CC' + C\tilde{\text{Re}}(x,t,r)^{1/2}] - \log Q}{\log(r/L)} . \tag{21}$$

Assume now [13] that $\tilde{\text{Re}}(x,t,r) = \text{Re}(r/L)^{\zeta(\text{Re})}$, where Re is the outer scale Reynolds number. Define λ^* as before, via $C' = \tilde{\text{Re}}(x,t,\lambda^*)^{1/2}$; we derive after some algebra

$$D - 2 = \zeta(\text{Re}) \frac{\log(Q/CC')}{2(\log \text{Re}^{1/2} - \log C')} . \tag{22}$$

Notice that D is measured here at the scale λ^*, rather than as a derivative with respect to r.

So far, $\zeta(\text{Re})$ has been measured in two flows, the wake behind a circular cylinder [4] and grid turbulence [14]. The data, obtained by two different methods, show

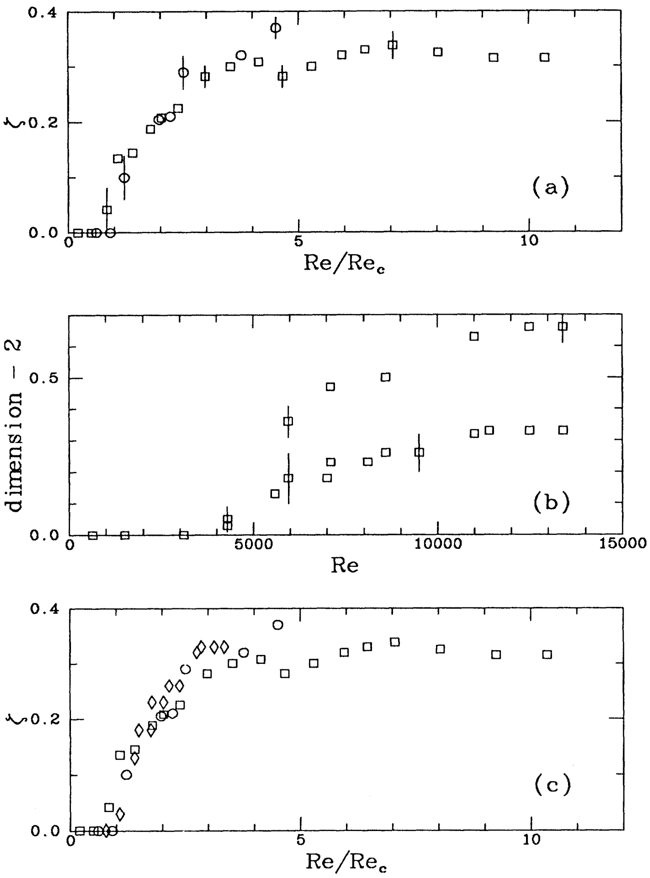

Figure 2

FIG. 2. (a) The scaling exponent ζ in $\delta u_r \sim (r/L)^{\zeta} U$, as a function of Re. Data from two different flows measured independently in two different laboratories. ○, wake behind a cylinder, $\mathrm{Re}_c \sim 165$; □, grid turbulence generated in a pipe, $\mathrm{Re}_c \sim 300$. (b) The dimensions of the isoscalar surfaces as functions of Re in temporally developing countercurrent shear layer. The symbol □ corresponds to the outer interface, whereas the symbol ○ corresponds to a surface embedded in the fully turbulent part of the flow. (c) Data from (a) and (b) plotted together. To compare $\zeta(\mathrm{Re})$ and $D_I(\mathrm{Re})$ from two different flows, we have plotted them as functions of $\mathrm{Re}/\mathrm{Re}_c$. It is seen that the two functions appear very similar. ◇ corresponds to D_I and the other symbols as in (a).

the same qualitative behavior which is an abrupt rise above zero beyond a "critical Reynolds number" Re_c for the appearance of three-dimensional turbulence. $\mathrm{Re}_c = 165$ for the wake (based on the diameter) and 300 for grid turbulence (based on the mesh size). When plotted as a function of $\mathrm{Re}/\mathrm{Re}_c$, it appears [Fig. 2(a)] that $\zeta(\mathrm{Re})$ is a unique function—although it would clearly be useful to have more data. In Fig. 2(b), we show $D-2$ and D_I-2 for a temporally developing countercurrent shear layer. Here $\mathrm{Re}_c \sim 4000$ (based on the global thickness of the shear layer and the velocity difference across the layer). Figure 2(c) is a direct test of Eq. (18) for these flows, and the agreement seems very close. Equation (22) is well supported by Fig. 2(b), once we observe that $D/D_I \sim 2$ in Fig. 2(b). Indeed, disregarding the weak $\log(\mathrm{Re}^{1/2})$ dependence in (22), we expect $D-2$ to be proportional to ζ.

In summary, we have presented a dynamical theory of the geometric properties of isoscalar surfaces in turbulence. This seems to be in good agreement with the existing data. We believe that this approach has implications for other issues in turbulence. For applications of this approach to understanding transitions in convective turbulence, the reader is referred to Ref. [8]. The implications of this approach for the general theory of turbulence will be expounded elsewhere.

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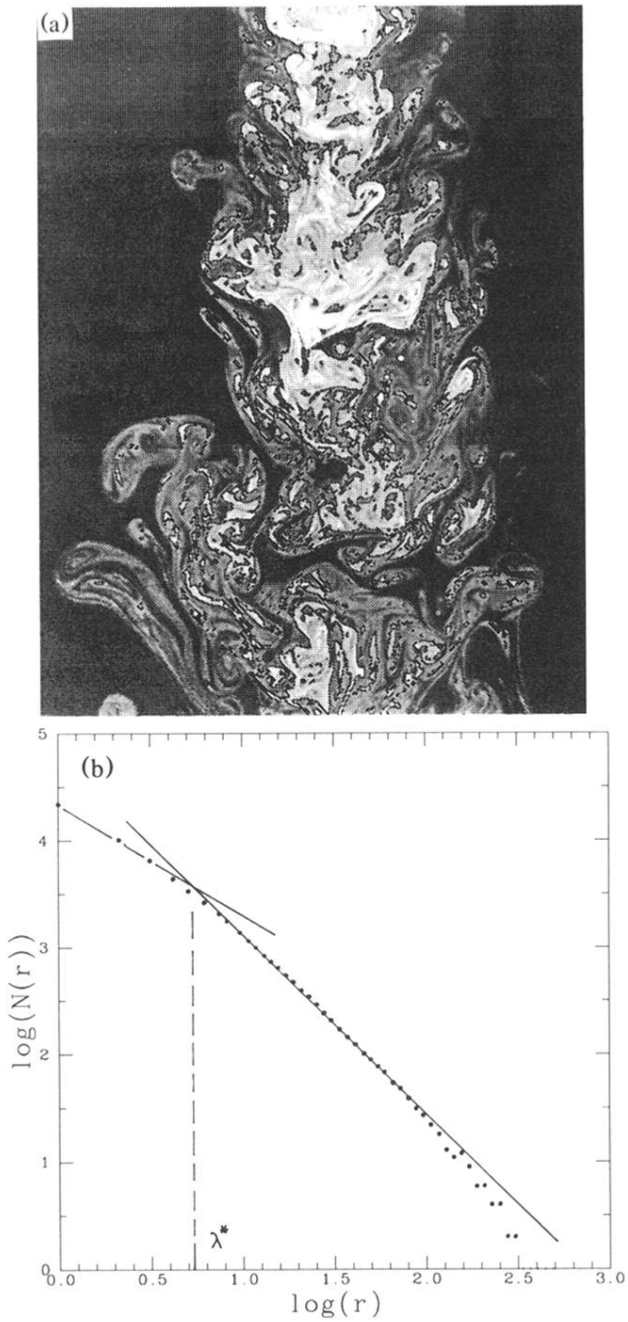

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