Chaos in Open Flow Systems

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We discuss briefly some aspects of 'open flow systems' in the context of deterministic chaos. This note is mostly a statement of the difficulties in characterizing such flows, especially at high Reynolds numbers, by dynamical systems. Brief comments will be made on the fractal geometry of turbulence.

1. Introduction

One of the most fascinating phenomena in fluid mechanics is the transition from a steady laminar state to a turbulent state. Our concern here is a brief discussion (in the context of deterministic chaos) of this transition process (or processes), and of aspects of the fully turbulent state itself. We shall concentrate entirely on 'open flow systems', or 'unconstrained' flows, e.g., wakes, jets, boundary layers, channel and pipe flows, etc.

It is not obvious in what sense one can think of open flow systems as genuine dynamical systems. We recall from [1] that such flows could behave in generically different ways from the 'closed flow systems'. In all closed flow systems the boundary is fixed so that only certain class of eigenfunctions can be selected by the system; this does not hold for open flow systems in which the flow boundaries are continuously changing with position. Thus, while in closed flow systems each value of the control parameter (for example, the rotation speed of the inner cylinder in the Taylor-Couette problem) characterizes a given state of the flow globally, this is not true of open systems. Consider as an example the near field of a circular jet. For a given set of experimental conditions, the flow can be laminar at one location, transitional at another and turbulent at yet another (downstream) location. This usually sets up a strong coupling between different phenomena in different spatial positions in a way that is peculiar to the particular flow in question. Secondly, the nature and influence of external disturbances (or the 'noise', or the 'background or freestream turbulence') is more delicate and difficult to ascertain in open flows: the 'noise', which is partly a remnant of complex flow manipulation devices upstream and partly of the 'long range' pressure perturbations, is not 'structureless' or 'white', no matter how well controlled. Finally, it is well-known that closed flow systems can be driven to different states by means of different start-up processes; for example, different number of Taylor vortices can be observed in a Taylor-Couette apparatus depending on different start-up accelerations [2]. This type of path-sensitivity in a temporal sense does not apply to open systems, where the overriding factor is the path-sensitivity in a spatial sense (i.e., the 'upstream influence').

These remarks notwithstanding, it has been shown in Refs. 1 and 3 that it is worthwhile examining transition in open flow systems from the point of view of low-dimensional chaos. The usual way of establishing this connection is via the analysis of the time history of a single dynamical variable such as a velocity component obtained at a fixed (Eulerian) point in the flow [4]. We should stress that this procedure is inadequate especially for the open flow systems. Two remarks ought to suffice. First, since the dynamical instabilities in open flows are most often convective in nature, analysis of temporal Eulerian quantities does not carry with it much information on the evolution of the system. Deissler & Kaneko [5] have pointed out that a flow which gives every appearance of being chaotic may nonetheless have no positive Lyapunov exponents in the Eulerian frame of reference. Perhaps a more relevant method of characterizing the evolution of the flow in terms of a dynamical system would be to use the Lagrangian information obtained, say, by measuring the velocity of a fluid particle as it moves about in the flow. To say the least, accurate measurements of this type are hard to make.

The second point to be made is that most open flow systems possess strong spatial inhomogeneities in a direction normal to the flow. (Indeed, these inhomogeneities are responsible for processes that maintain the flows against viscous dissipation.) For this reason, it is *a priori* unclear to what extent the temporal information obtained at one selected point fixed in the flow can represent the global dynamics. One might think that a simultaneous measurement (at a given time or as time sequences) of a dynamic quantity such as velocity, made at many spatial points in the flow, might solve this problem. This is not so: one does not even know how to construct a dynamical system from such empirical data.

It therefore appears worth enquiring explicitly whether, in open flow systems, attractors constructed from Eulerian point measurements, using the usual time delay techniques, are chaotic; that is, whether they are characterized by low dimensions, and possess (at least!) one positive Lyapunov exponent. This is done in section 2. In section 3, we examine the variation with the flow Reynolds number of the dimension of the attractor, and comment briefly on the dimension at large Reynolds numbers. In section 4, brief remarks will be made on two aspects of turbulence that can be ascribed fractal dimensions.

2. Chaotic attractors for open flows: low Reynolds numbers

Chaotic attractors are characterized by at least one positive Lyapunov exponent and by relatively low dimensions that do not continously increase with the embedding dimension. We have made point measurements of velocity signals in several different flows and constructed attractors using the time delay technique; we have obtained the correlation dimension ν according to the Grassberger-Procaccia algorithm [6], and the largest Lyapunov exponent according to the algorithm given in Wolf et al. [7]. (Spurred by a talk that Harry Swinney gave in Kyoto in 1983, we wrote versions of a program to calculate the largest Lyapunov exponent, but have now switched over to the method of Ref. 7.) Since both these procedures are now well-known, we shall not describe them here.

In Table 1, we list some basic information for four flows. A crucial factor in obtaining the correlation dimension is the choice of the optimum time delay τ. We simply varied τ over a wide range, and used a τ in the range where its precise value is not critical. We show in Fig. 1 the correlation dimension as a function of τ. Clearly, too large a τ will result in the increase of ν.

Figure 2 shows the convergence with the number of iterates of the largest Lyapunov exponent for the wake, calculated using an embedding dimension of 6; other embedding dimensions yield essentially the same asymptotic value, even though the initial behaviors could be quite different. It should be remarked that the dimension and the Lyapunov exponents usually converge (for the calculations typified by Table 1) relatively fast; total signal durations of the order of $2000\tau_o$, where τ_o is the zero-crossing time scale of the auto-correlation function, was found to be usually sufficient.

Table 1: Typical data for low Reynolds number open flow systems

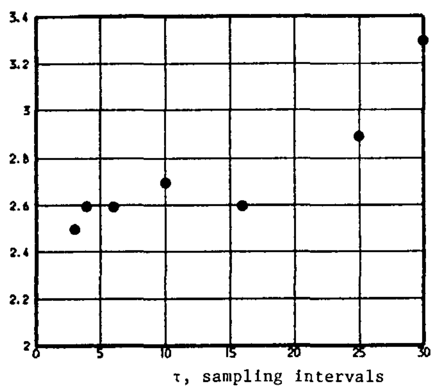
Flow	$Re = U_o d/\nu$	*Correlation* dimension, ν	*Largest* Lyapunov exponent, λ_1
wake behind circular cylinder[1]	67	2.6	0.65 bits/orbit
axisymmetric jet (unexcited)[2]	1000	6.3	0.95 bits/orbit
axisymmetric jet (excited)[3]	1000	3.2	——
curved pipe[4]	6625	6.0	0.40 bits/orbit

[1]d = diameter of the cylinder, U_o = upstream flow speed; data were obtained 10 diameters downstream, 1 diameter off-axis.

[2]d = diameter of the nozzle, U_o = nozzle exit velocity; data were obtained in the potential core 2 diameters downstream of nozzle exit.

[3]no Lyapunov exponent was computed because we lost the data sets immediately after computing the dimension.

[4]d = pipe diamter, U_o = section average velocity; the data correspond to the centerline of the pipe.



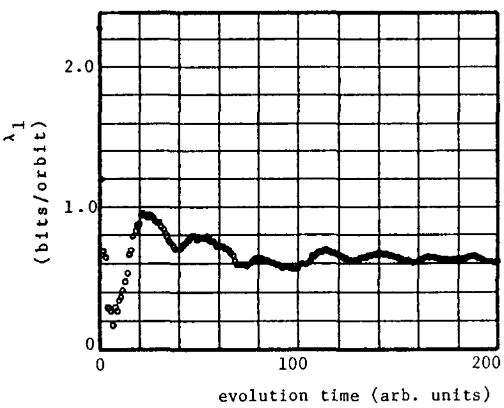


Fig. 1 The variation of the correlation dimension as a function of the time delay τ used to construct the attractor.

Fig. 2 Variation of the largest Lyapunov exponent with the evolution time.

From many such calculations, we conclude that if one constructs attractors using a single Eulerian dynamical quantity via time delay techniques, such attractors do possess (at low Reynolds numbers) characteristics of chaotic dynamics. Perhaps, Eulerian quantities do preserve some information on the dynamical evolution, in some loose sense akin to Poincaré sections!

We shall remark that these calculations do not unequivocally establish that tubulence is chaotic (in the sense of extreme sensitivity to initial conditions). Our findings could perhaps be interpreted equally well in terms of 'external noise amplification' in the system. Much more work is needed before one can determine the extent to which

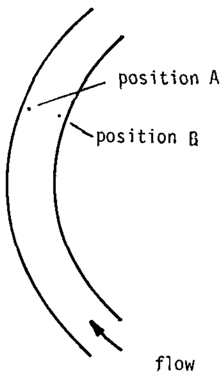


Fig. 3a Measurement stations for the curved pipe. Flow at the measurement stations is fully developed. Configuration details can be found in [3].

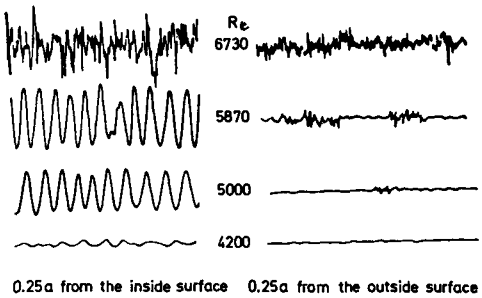


Fig. 3b Streamwise velocity fluctuations at several Reynolds numbers at position A (the right set of signals, measured 0.25 radius from the outer wall) and position B.

this last mentioned factor competes with the intrinsic sensitivity to initial conditions as the mechanism for the generation of turbulence. We should also reiterate the variation with spatial position of the characteristics of the 'Eulerian attractors'. For the curved pipe, Fig. 3 shows samples of streamwise velocity history at two spatial locations (but at the same streamwise section in the so-called fully developed region). Clearly, attractors constructed from signals at these two different locations can be expected to have different dimensions and spectra of Lyapunov exponents. For an Re of 6625, the data are as shown in Table 2. At the least, these data suggest that the interpretation of the dimension as an indicator of the dynamically significant degrees of freedom of flow needs some qualification.

Table 2: The spatial variation of the characteristics of the 'Eulerian attractors' at two different spatial positions in the same flow at the same streamwise location at the same Re. Data are for curved pipe; details as in Fig. 3.

position A		position B	
ν	λ_1, bits/orbit	ν	λ_1, bits/orbit
6.0	0.4	2.7	0.17

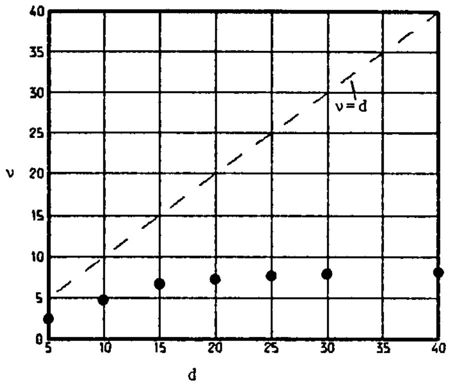
3. Dimension calculations at higher Reynolds numbers

If we persist with dimension calculations at higher Reynolds numbers — using the same technique, in spite of its shortcomings — they become uncertain because:

(a) The number of data points required for convergence, and the number of steps involved in dimension calculations go up;

(b) One cannot in general find a proper range of time delays over which the results are sensibly independent;

(c) There is no guarantee that the dimension calculations asymptote to constant values as the embedding dimension increases.



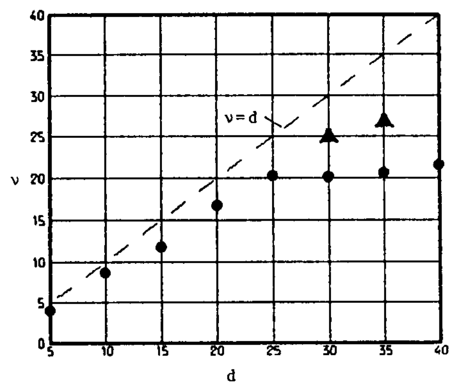


Fig. 4a The variation of the correlation dimension ν with the embedding dimension d. Re = 500, approximately 5 diameters downstream of the cylinder. A space-filling attractor is expected to have the behavior shown by the dashed line.

Fig. 4b The variation of the correlation dimension ν with the embedding dimension d. Re = 2000, approximately 5 diameters behind the cylinder. ν = d line holds for a space-filling attractor. The ▲'s indicate the values of ν computed for the random noise from a commercial random noise generator. Notice that the asymptotic value of ν is definitely below the noise data, although only by a small margin. The nearness of the noise data to the flow data shows why we cannot place too much emphasis on high dimension computations.

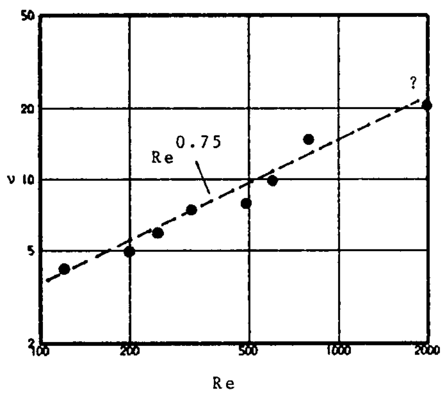


Fig. 5 The variation of the dimension with Reynolds number. Data are for the wake of a circular cylinder.

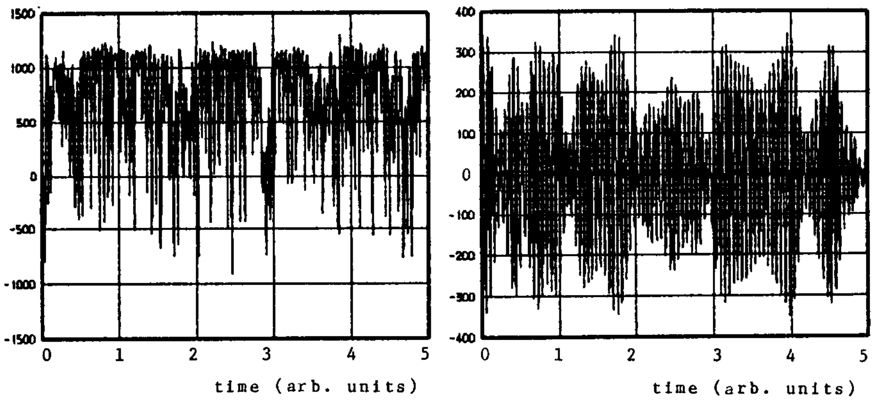
Figures 4a and b illustrate this last point; Fig. 4b is the upper limit on the Reynolds number at which some credibility (already rather low!) can be ascribed to the dimension calculations. If we believe the numbers obtained from such calculations, we may deduce that a power law relation like $Re^{3/4}$ is not unlikely (Fig. 5).

It is worth mentioning that Constantin et al. [8] have placed the upper bound on the dimension of Navier-Stokes attractors to be of order $R^{9/4}$ (and higher if self-similarity in the Kolmogorov range does not obtain!), where the Reynolds number R = u'L/ν, u' being a root-mean-square velocity fluctuation,and L is an integral state of turbulence. The precise relation between R and the Reynolds numbers Re used in Table 1 depends on the flow, but it is clear that if the present finding of a 3/4 - power law is true, it is of undoubted significance in spite of our earlier reservations on the meaning of the dimension obtained in this way.

Fully turbulent flows are characterized by temporal and spatial chaos. Temporal dynamics is thus merely a part of the whole story; this in itself is hard to come to grips with, even if the dimension were to increase 'only' according to a 3/4 power of the Reynolds numbers. Is there then any connection between real turbulent flows and finite-(and low-) dimensional dynamical models which one hopes one can construct? (That, presumably, is the practical motivation for studies of this type.) The answer would have been an unequivocal 'no' were it not for the fact that some (perhaps strong?) spatial coherence appears to exist at least in some classes of fully turbulent flows. One might, in some way that remains unclear, be able to decompose the motion into two components, one of which consists of this coherent element and the other, its complement. One can then think of a low-dimensional attractor characterizing the coherent motion, the attractor being made fuzzy by the small scale motion whose effect is to reduce the correlation. Unfortunately, it is not clear whether this loosely worded picture is consistent with facts.

Elementary tests of this hypothesis can be made if one is able to separate the incoherent motion from the coherent part. This might be possible, for example, by some kind of ensemble averaging methods such as used in [9]. The simplest (by no means the most correct) way is to filter out linearly in the frequency domain the coherent motion from the rest. To avoid many conceptual difficulties associated with filtering as the technique for separating the coherent and incoherent motions we choose a (relatively) high Reynolds number flow where the coherent part is clearly contained within a narrow band of frequencies; we then enquire whether the motion associated with this narrow band is low dimensional.

Figure 6a shows the streamwise velocity fluctuation in the wake of a circular cylinder, measured about 2 diameters behind the cylinder and a diameter off-axis; the flow Reynolds number of 10,000 is considered moderately high. Computing the dimension of the attrac-



Figs 6a,b: The total (unfiltered) and the coherent part respectively of the streamwise velocity fluctuation in the wake of a cylinder; Re = 10,000. Both the ordinate and abscissa are arbitrary but the same in the two figures.

tor constructed from this signal is doomed to be meaningless in view of the remarks made earlier. (If the $Re^{3/4}$ dependence is valid, the extrapolated estimate for ν is of the order of 30!) We do know from power spectral measurements that this signal has a peak at a frequency f of about 550 Hz; this peak, corresponding to a Strouhal number fd/U_o = 0.21, characterizes the coherent part of the motion. If we band-pass filter this signal between, say, 500 and 600 Hz, the resulting signature is given in Fig. 6b. Calculations show that the corresponding attractor has a dimension of about 5.5!

It is appropriate to end this discussion with the statement that the coherent part, as we defined it here, contains a significant fraction of energy.

4. The fractal geometry of turbulence: a brief note

We have indicated that measurements of attractor dimensions are beset with increasing uncertainties at increasingly high Reynolds numbers. But there are other fractal dimensions whose measurement becomes increasingly definitive as Reynolds number increases. It is to a mention of two of these aspects that this section is devoted; more details should be forthcoming in [10]. The results of this section are essentially spurred by Mandelbrot's remarks on several occasions that many facets of turbulence are fractal.

4a. The fractal dimension of the turbulent/non-turbulent interface

Observations suggest that in high Reynolds number free shear flows (i.e., open flow systems with no constraining boundary) a sharp front or interface demarcates the turbulent and non-turbulent regions. Although a completely accepted view of the detailed nature of this interface does not seem to exist, a visual or spectral study suggests that contortions over a wide range of scales occur. This leads one to the natural expectation that the interface is a fractal surface.

By illuminating a thin section of a flow, and by digitizing the resulting picture, one can evaluate the fractal dimension of the curve that separates the turbulent from the non-turbulent regions; a threshold set on the intensity of illumination separates the two regions. The fractal dimension of the surface bounding turbulent regions is then one more than that of the curve.

Several methods can be adopted to measure the fractal dimension [11]. We shall describe only one rather briefly. Assign to each point in the digitized image of the flow a number 1 when the point lies within the turbulent region, and a number 0 when it lies within the non-turbulent region. Let the boundary shown in Fig. 7 represent

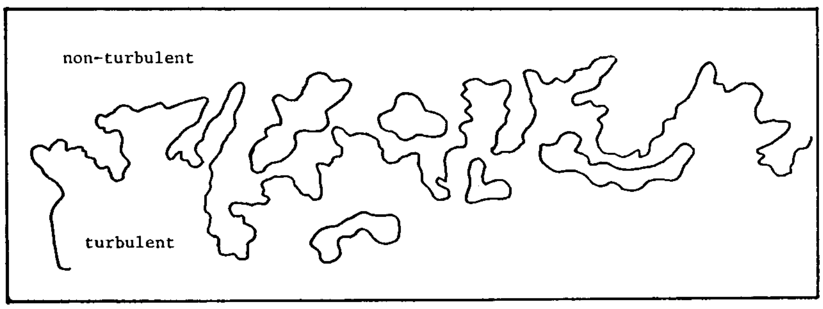


Fig. 7 The boundary between the turbulent and non-turbulent regions. If a circle of radius ε drawn around a given point in the digitized image crosses the boundary, the point is considered to be within a distance ε from the boundary.

the boundary between the 1's and the 0's. Count the number $N_b(\varepsilon)$ of the digitized points which are within a distance ε from the boundary. If this boundary is a fractal of dimension D, then it is easily shown from the basic definition of D that

$$N_b(\varepsilon) \sim \varepsilon^{2-D} \quad . \tag{1}$$

Measurements to be described in [10] show that (1) holds for scales ranging from the Kolmogorov scale to a fraction of the integral length scale (but excluding scales of the order of the integral scale and higher). The measured value of the fractal dimension for the interface varies between 2.3 and 2.4; there is no identifiable variation from one type of flow to another.

4b. The fractal dimension of the velocity and scalar dissipation fields

Another aspect of turbulence that is a candidate for fractal behavior is its dissipative (or internal or small) structure. It has been well-known for some time that the small structure of turbulence is intermittent. The essence of scale-similarity arguments in this context is the following. Within a given field of (fully developed) turbulence, consider a cube with sides of length L_o, where L_o is an integral scale of turbulence. If we divide this cube into arbitrarily large number (n>>1) of smaller cubes of length $L_1 = L_o n^{-1/3}$, the density of dissipation rate in each of these smaller cubes is distributed according to a probabilistic law. Further subdivision of these cubes into second-order ones of length $L_2 = L_1^{-1/3}$ leaves the probability distribution unaltered. This similarity extends to all scales of motion until one reaches sizes directly affected by viscosity. Clearly, this case cries out for fractal description.

Using methods discussed in [11], we have obtained the results shown in Table 3.

One concludes from here that the dissipation field is not space-filling (less space-filling in the high Reynolds number regime) and that (c) is less space-filling than (b) — a result consistent with observations in oceanography. Note that the result (b) is only at slight variance with Mandelbrot's [11] original estimate of 2.6.

Table 3: Summary of the fractal dimensions of the dissipation fields

	Field	Fractal dimension
(a)	Kinetic energy dissipation (low Reynolds number)*	2.9
(b)	Kinetic energy dissipation (high Reynolds number)	2.7
(c)	Passive scalar (e.g., temperature) dissipation (high Reynolds number)	2.6

* The boundary between the low and high Reynolds number regimes is not well-defined. A convenient boundary occurs at a microscale Reynolds number of about 150.

Theoretical explanations of these fractal dimensions, as well as of the connections that might exist among them, would be of fundamental interest.

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