Determination of intermittency from the probability density function of a passive scalar

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The probability density functions of temperature in the intermittent regions of heated jet and wake flows show a strong spike associated with the temperature of the unheated fluid. The fine structure of this spike fits a Gaussian curve quite closely, and the area under this Gaussian gives an accurate measure of $1-\gamma$ where γ is the intermittency factor. The standard deviation of the Gaussian is a measure of the noise due to residual temperature fluctuations in the free-stream and the electronic noise in the measurement system. The accuracy with which the intermittency can be determined is limited by the signal-to-noise ratio.

INTRODUCTION

The intermittency factor γ is defined as the probability that a given flow is turbulent at a given point. The determination of intermittency in free turbulent flows and boundary layers has been of considerable interest and study.[1-6] When working with velocity signals from hot-wire anemometers, the discrimination of nonturbulent fluid from turbulent fluid is complicated by potential motions induced in the nonturbulent fluid by the pressure fluctuations. When working with temperature signals, there are no such induced temperature fluctuations in the nonturbulent fluid, but discrimination is still complicated by the noise due to residual temperature fluctuations originating from the source of this fluid and to electronic noise. (Here we leave aside the interesting question of whether the interface for a scalar contaminant is, in fact, the same as that for turbulence, i.e., vorticity fluctuations.) Analog circuits used to discriminate between turbulent and nonturbulent fluid employ a voltage threshold which must be exceeded for a specified hold time. Various methods have been used to determine correct values for the threshold voltage and hold time, but they all involve some degree of subjectivity on the part of the investigator. Antonia *et al.*[6] indicate that when working with temperature directly only the threshold should be used.

The digital processing of turbulence signals has allowed accurate measurement of the probability density functions of scalar quantities, such as temperature and concentration (see Bilger,[7] for a recent review of this data). In the intermittent region of free turbulent flows the probability density functions of temperature show a strong spike associated with the occurrence of unheated (nonturbulent) fluid. Here, we examine the structure of this spike and show how it can be used to make an accurate determination of intermittency.

PROBABILITY DENSITY FUNCTION STRUCTURE

Figures 1 and 2 show the data of LaRue and Libby[8] for the probability density functions of temperature in the wake of a slightly heated cylinder at an x/d of 400 and at $y/l_c = 0.349$ and 0.431 ($l_c = \sqrt{xd}$, d is the cylinder diameter). LaRue and Libby give values of the intermittency, $\gamma = 0.509$ and 0.175 for these two cases. The insets in the figures show the spike structure. It can be seen that the spikes follow a Gaussian curve very closely. In the absence of free stream temperature fluctuations and electronic noise we would obtain a Dirac delta function for that part of the probability density function associated with the free stream; the integral of this delta function would be equal to the probability of occurrence of unheated fluid, $1-\gamma$. In the presence of free-stream temperature fluctuations and electronic noise the delta function will be broadened and a Gaussian shape will result if the noise due to these sources is Gaussian; the area under this Gaussian will be equal to $1-\gamma$. The Gaussian curves fitted in Figs. 1 and 2 yield values of γ of 0.565 and 0.232, respectively. The threshold settings required to obtain these values of the

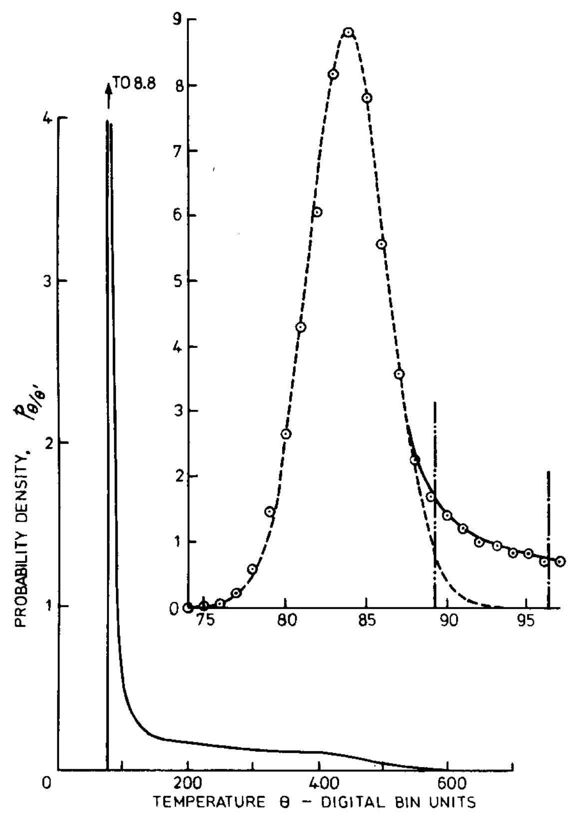

FIG. 1. Probability density function of temperature in wake of heated cylinder showing detail of spike structure. Data of LaRue and Libby.[8] $x/d=400$, $y/l_c=0.349$. One digital bin corresponds to $\Delta\theta/\theta' = 8.12\times10^{-3}$; -------- fitted Gaussian; -·-·-· threshold for $\gamma=0.509$; -··-·-· threshold for $\gamma=0.564$; o data points.

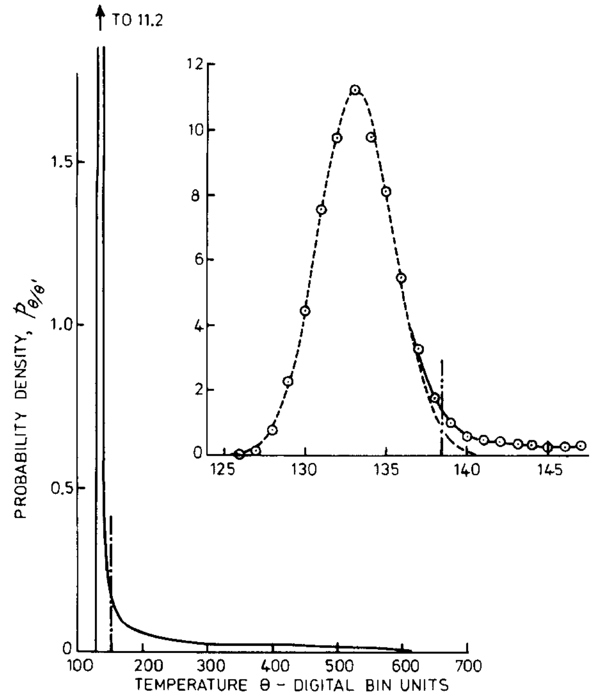

FIG. 2. Probability density function of temperature in wake of heated cylinder showing detail of spike structure. Data of LaRue and Libby.[8] $x/d = 400$, $y/l_c = 0.431$. One digital bin corresponds to $\Delta\theta/\theta' = 1.192\times 10^{-2}$; --------fitted Gaussian; —·—·— threshold for $\gamma = 0.175$, —··—··—··— threshold for $\gamma = 0.232$; ○ data points.

intermittency are shown together with the values used by LaRue and Libby. Experience with different people carrying out the curve fitting indicates that the area under the Gaussian (and hence $1-\gamma$) can be determined to better than ± 0.02 when there are about 10 points to fit.

Since the turbulent and nonturbulent parts of the flow are mutually exclusive, we may write

$$p_\theta(\theta) = (1-\gamma)\, p_\theta^f(\theta) + \gamma p_\theta^t(\theta)\ , \qquad (1)$$

where $p_\theta(\theta)$, $p_\theta^f(\theta)$, $p_\theta^t(\theta)$ are the actual probability density functions for the temperature θ as a whole, in the free stream, and in the turbulent fluid, respectively. In the presence of noise in the measurement system the measured probability density function $p_\theta^m(\theta)$ will be, by convolution:

$$p_\theta^m(\theta) = \int_{-\infty}^{\infty} p_\theta^n(\theta-\phi)\left[(1-\gamma)\, p_\theta^f(\phi) + \gamma p_\theta^t(\phi)\right] d\phi\ . \qquad (2)$$

Here, $p_\theta^n(\theta)$ is the probability density function of the noise expressed in terms of temperature. If $p_\theta^f(\theta) = \delta(\theta - \overline{\theta}_f)$, a Dirac delta function, and $p_\theta^n(\theta)$ is Gaussian with standard deviation θ_n', we have

$$p_\theta^m(\theta) = \frac{1-\gamma}{\theta_n'\sqrt{2\pi}} \exp\left[-\frac{\frac{1}{2}(\theta-\overline{\theta}_f)^2}{\theta_n'^2}\right] + \gamma \int_{-\infty}^{\infty} \frac{1}{\theta_n'\sqrt{2\pi}} \exp\left[-\frac{\frac{1}{2}(\theta-\phi)^2}{\theta_n'^2}\right] p_\theta^t(\phi)\, d\phi\ . \qquad (3)$$

If the free stream has temperature fluctuations which are Gaussian with a standard deviation θ_f', then the first term on the right-hand side is modified by replacing θ_n' by $(\theta_n'^2 + \theta_f'^2)^{1/2}$. If we have $\theta_t'^2 \gg \theta_n'^2$, where θ_t' is the standard deviation of the temperature in the turbulent fluid only, then the noise will have no significant effect on the second term: for the LaRue and Libby data the signal-to-noise ratio exceeds 30 dB and this is well satisfied. With these assumptions

$$p_\theta^m(\theta) = \frac{1-\gamma}{[2\pi(\theta_n'^2 + \theta_f'^2)]^{1/2}} \exp\left[-\frac{\frac{1}{2}(\theta-\overline{\theta}_f)^2}{(\theta_n'^2+\theta_f'^2)}\right] + \gamma p_\theta^t(\theta)\ . \qquad (4)$$

The approximation involved in Eq. (4) will be more significant in the neighborhood of $\overline{\theta}_f$ where the contribution of $p_\theta^t(\theta)$ may, due to the influence of the noise, substantially overlap and cloud the behavior of $p_\theta^f(\theta)$ to which the Gaussian is being fitted. This clouding should be small if

$$\epsilon \equiv \int_{\overline{\theta}_f}^{\overline{\theta}_f + \theta_n'} p_\theta^t(\theta)\, d\theta \ll 1-\gamma\ . \qquad (5)$$

The behavior of $p_\theta^t(\theta)$ is shown in Fig. 3. It appears that when γ is low, $p_\theta^t(\theta)$ becomes very large near $\overline{\theta}_f$. Accordingly, the value of the integral of $p_\theta^t(\theta)$ is shown in Fig. 4. (In these figures θ_0' is the standard deviation of θ on the centerline so that all curves are for temperature normalized by the same factors.) For the experiments of LaRue and Libby,[8] $\theta_n'/\theta_0' \approx 0.025$, i.e., centerline signal-to-free-stream noise $S/N \approx 32$ dB (from here onward we shall usually take θ_n' to include the contribution of θ_f'). The uncertainty in $1-\gamma$ (and hence γ) will also be of the order of ϵ. From Fig. 4 it appears that this uncertainty will be about 0.01 which agrees with experience in curve fitting in which the accuracy has already been quoted as being better than ± 0.02. This level of uncertainty is well within that obtained by subjective examination of the signal traces; LaRue[4] implies that this is of the order of ± 0.05 in the range $0.15 < \gamma$

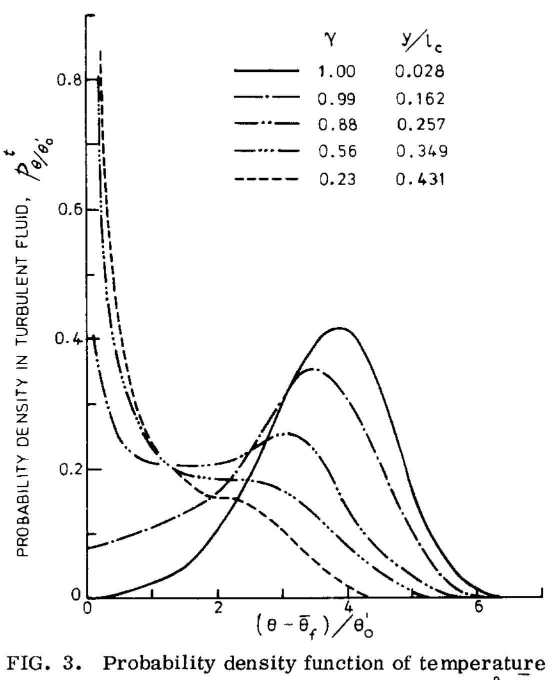

FIG. 3. Probability density function of temperature within the turbulent fluid only. Data of LaRue and Libby.[8] $\overline{\theta}_f$ is the mean temperature of the free stream, and θ_0' the standard deviation of temperature for measurement on the centerline.

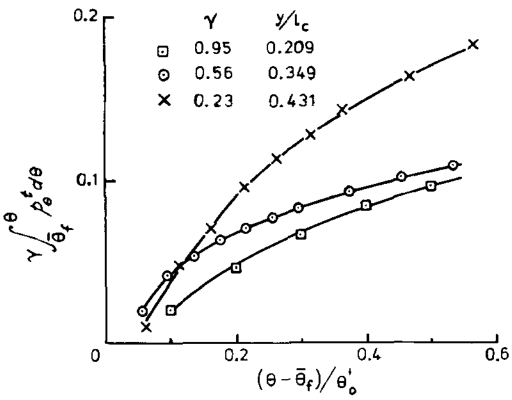

FIG. 4. Integrated probability within the turbulent fluid only. Data of LaRue and Libby[8]. $\bar{\theta}_f$ and θ_0' defined as for Fig. 3.

< 0.85.

The behavior of $p_\theta^t(\theta)$ near $\bar{\theta}_f$ shown in Fig. 3 is of some interest. As depicted in Fig. 3 it appears that there may be an asymptotic approach $p_\theta^t(\theta) \to \infty$ as $\theta \to \bar{\theta}_f$. This would be consistent with the notion that the temperature profile through the interface is asymptotic to the free-stream temperature, a result commonly found in steady-state solutions of conduction and diffusion problems. Such asymptotic behavior would mean that even in the absence of noise, an arbitrary threshold temperature must be chosen to define the boundary between "heated" and "unheated" fluid. The "delta function associated with the free stream" would become broadened on the heated side and convolution with a Gaussian would no longer yield a Gaussian. The actual data for $p_\theta^t(\theta)$ do not in fact show an asymptote near $\theta = \bar{\theta}_f$ as can be seen in Figs. 1 and 2. Preliminary experience gained with other flows indicates that a spike in $p_\theta^t(\theta)$ near $\bar{\theta}_f$ is not always found, even at low γ. It may be that the temperature profile through the interface changes sufficiently abruptly to asymptote to the free stream temperature, so that the boundary between "unheated" and "heated" fluid is effectively defined at a threshold which is well below θ_n'.

THE PROPOSED METHOD

In its simplest form the method for determining the intermittency factor γ is as has already been indicated. A Gaussian curve is fitted to the spike corresponding to the free stream temperature and the area under the Gaussian gives $1-\gamma$. This is most conveniently carried out with digitized data. A useful check on the fit is obtained by determining the threshold Th such that

$$\int_{-\infty}^{\mathrm{Th}} p_\theta(\theta)\, d\theta = \int_{-\infty}^{\infty} G(\theta)\, d\theta \quad , \tag{6}$$

where $G(\theta)$ is the fitted Gaussian. Equation (6) may be written

$$\int_{-\infty}^{\mathrm{Th}} [p_\theta(\theta) - G(\theta)]\, d\theta = \int_{\mathrm{Th}}^{\infty} G(\theta)\, d\theta \quad . \tag{7}$$

When this threshold is drawn on the plot, as in Figs. 1 and 2, the areas indicated by Eq. (7) should appear equal. Of course, $p_\theta(\theta)$ is really a histogram when digitized data are used, and this fact can be important if the number of points defining the spike is small.

Experience indicates that the points on the left-hand side of the spike up to and including the peak are usually a good fit to the Gaussian over the whole range of γ. A least squares routine could be used to obtain the fit with the computer. This should be done fitting the exponential to p and not a parabola to $\log_e p$ since in the latter case, data for low p are over-emphasized. At least one point on the right-hand side of the peak should always be included to insure correct centering. There are some indications that the noise is somewhat peakier than Gaussian (i.e., kurtosis > 3) so that there are dangers in trying to fit the Gaussian from one side alone.

Figure 3 and the discussion in the previous section indicate that the attainable absolute accuracy is of the order of $\frac{1}{2}\theta_n'/\theta_0'$ and so very high signal-to-noise ratios are desirable. This will require that a small bin size be used to generate the probability density function so that adequate resolution of the spike is obtained. If we take the range of the full probability density function to be $8\theta_0'$ and we required a bin size of $0.4\,\theta_n'$, then the number of bins required is $20\,\theta_0'/\theta_n'$. The 1024 steps in the usual 10 bit digitizer soon becomes the limiting factor.

It is a corollary of Eq. (2) that if the measurement system gain remains unaltered, then θ_n' (including the contribution from θ_f') will remain constant across the flow as will the probability density function of the "noise spike":

$$p_\theta^s(\theta) \equiv \int_{-\infty}^{\infty} p_\theta^n(\theta - \phi)\, p_\theta^f(\phi)\, d\phi \quad . \tag{8}$$

This can be determined unequivocally in the free stream, and in the intermittent region only multiplication by $1-\gamma$ should be necessary to fit $p_\theta^s(\theta)$ to the spike. In practice, variations in the direct current level of the measurement system will often require that θ_f be determined by the fit as well as $1-\gamma$. The point is that

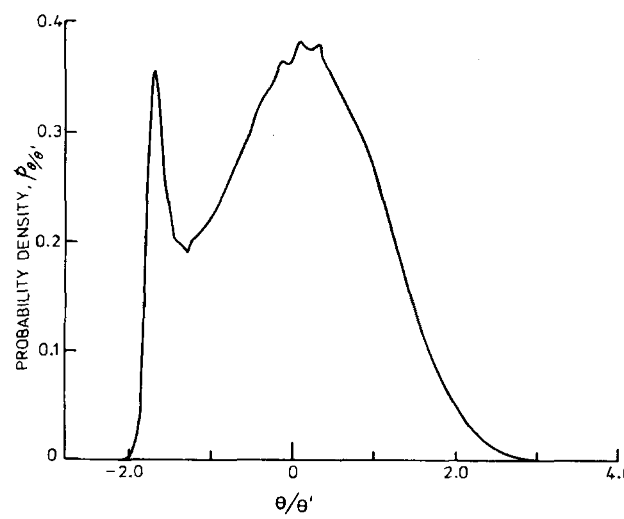

FIG. 5. Probability density function of temperature in the axisymmetric heated turbulent jet in a co-flowing stream investigated by Antonia *et al.*[6] $\eta = 0.89$, $\gamma = 0.91$.

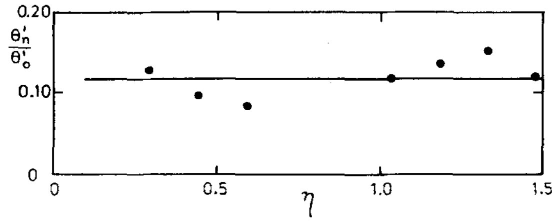

FIG. 6. Variation across the jet of the standard deviation of nonturbulent fluid fluctuations normalized by the standard deviation of the whole signal on the center line. Data of Antonia *et al.*[6]

θ_n' is known, and it is not necessary to assume a Gaussian. The cumulative probability distribution $P(\theta)$ can also be fitted in this way. The normal discriminating circuit with hold time set to zero and varying the threshold will yield an intermittency function whose average is $1-P(\theta)$. By varying the threshold in millivolt steps near the free stream voltage it may be possible to generate $P(\theta)$ over the range of the spike and so determine the value of $1-\gamma$ needed to fit the $P(\theta)$ distribution found in the free stream.

PRELIMINARY RESULTS FOR A HEATED JET

Only two of the probability density functions of LaRue and Libby[8] were available to us in detailed digital form (those shown in Figs. 1 and 2). Furthermore, the system gain was altered in obtaining them. A preliminary examination of the *FM* tape records of Antonia *et al.*[6] has been made. These were digitized with a constant system gain and probability density functions generated with 128 equal bins each comprising 8 digital steps. No record of the free stream noise was made. Examination of the probability density functions generated indicated that the signal-to-noise ratio at 18 dB was considerably poorer than that of LaRue and Libby[8] (≈ 32 dB). It appears that a large part of this noise may have arisen in digitizing the data. The data will be re-analyzed and presented more fully elsewhere. However, some aspects of the "noisily" analyzed data are of particular interest and are presented here.

Figure 5 shows a typical probability density function obtained in the heated jet in a co-flowing stream. The left-hand spike corresponds to the free stream. This spike is shorter and wider than would have been obtained with less noise. Table I shows the values of γ obtained by fitting a Gaussian curve to this spike in the manner outlined here. The numbers in parentheses are obtained by a second person fitting the data. They are compared with the values obtained by Antonia *et al.*[6] who set their value of the threshold by visual comparison of oscilloscope traces of the temperature signal and intermittency function generated by the discriminator. These settings were confirmed by examining the influence of the threshold on γ and f_γ; the chosen threshold lay on a plateau for f_γ and on a not too steeply

TABLE I. Comparison of intermittency values determined in a heated jet.

	γ	
η[a]	Present work	Antonia *et al.*[6]
0.296	0.990	0.993
0.444	0.988	0.986
0.592	0.983	0.971
0.889	0.913	0.902
1.037	0.808(0.792)	0.813
1.185	0.655	0.610
1.333	0.385(0.410)	0.452
1.481	0.277	0.280
1.555	0.176	0.187
1.629	0.129	0.117

[a]η is equal to r/L_0, where L_0 is radius to the point where the mean temperature is halfway between that at the center line and that in the free stream.

varying region for γ. The differences are most marked at the two points near $\gamma = 0.5$ where visual discrimination is most difficult and the present method is probably most accurate. Even here, the agreement is well within the precision (±0.05) mentioned earlier for the visual method in this range. The agreement of the two different workers using the present method is within the accuracy of $\frac{1}{2}\theta_n'/\theta_0'$ which here works out to be ±0.06. Figure 6 shows the variation of θ_n' across the flow; it is seen to be essentially constant in agreement with the discussion in the last section.

CONCLUSIONS

We have shown that the spike in the probability density function for temperature associated with the free-stream fluid can be used to obtain an effective measure of the intermittency factor γ. The method is much less subjective than other methods, and its precision is limited only by the signal-to-noise ratio of the temperature signal.

ACKNOWLEDGMENTS

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