DISTORTED WAKES

A. PRABHU, R. NARASIMHA, AND K. R. SREENIVASAN

Indian Institute of Science, Bangalore 560012, India

1. INTRODUCTION

The spread of pollutants, especially from local, concentrated sources, can be influenced strongly by the distortion resulting from passage between or around buildings, hills, etc. As a first, illustrative step towards analyzing such situations, we consider here the effects of certain types of distortion on nearly plane turbulent wakes. It has recently been shown that such flows can be successfully treated in terms of relaxation towards certain well-defined equilibrium states. The present paper describes a simple integral method for computing such flows and compares the results with measurements on distorted wakes, with particular emphasis on the effects of extremely rapid distortion.

2. THE INTEGRAL METHOD

2.1. *Previous Work*

Various integral methods have been proposed for prediction of turbulent wake development, utilizing the nearly universal similarity in the defect velocity profile that is always found to prevail except in the immediate neighbourhood of the wake generating body. In the work of Hill *et al.* (1963) and of Gartshore (1967), the use of an eddy viscosity eliminates the Reynolds stress as a separate variable, so that (in view of the similarity mentioned above) only a velocity scale w_0 (say the centre-line defect) and length scale δ (the half width of the wake at the $\frac{1}{2}w_0$ point, see Fig. 1), remain to be determined. The momentum integral provides one relation between these quantities. For the second, Hill *et al.* use the moment of momentum equation; for shallow wakes ($w_0 \ll U$, the free stream velocity), their results can be deduced by the more general similarity analysis of Prabhu (1966). Gartshore obtains the second relation by integrating the momentum equation up to $y = \delta$; the stress at this point is determined by a semi-empirical relation

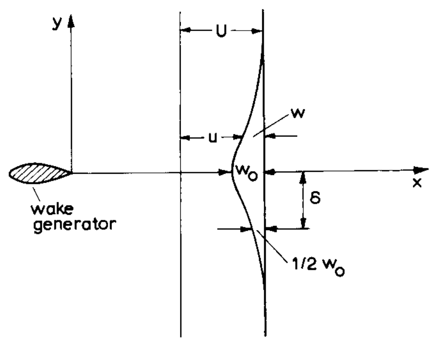

FIG. 1. Sketch defining notation.

between the eddy viscosity and the ratio of longitudinal to transverse strain rate at the point.

Now recent experiments on wakes subjected to various pressure gradients (Narasimha and Prabhu, 1972; Prabhu and Narasimha, 1972; referred to below as I and II, respectively) have shown that the "local" concepts that form the basis of the above techniques are unsatisfactory unless the pressure gradient (or strain) imposed on the flow is small. On the other hand, a relaxation–diffusion model for the stress (Narasimha, 1969), explicitly taking account of the memory of the flow, was found to be remarkably successful in describing flow development under a variety of conditions. We construct here an integral method based on this model.

2.2. *The Equations*

We assume that the Reynolds number is sufficiently large for the boundary layer approximation to be valid and for the viscous stresses to be negligible. We further assume that the normal stresses have no appreciable effect on the flow. The equations governing the development of a two-dimensional incompressible wake are then

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{1}$$

$$u\,\partial u/\partial x + v\,\partial u/\partial y = U\,dU/dx + \partial\tau/\partial y, \tag{2}$$

where u and v are mean velocity components along the x and y axes, $U = U(x)$ is the free stream velocity, and τ is the Reynolds shear stress in kinematic units (i.e., we put density = 1). The additional equation for τ required to have a closed set is assumed here to be (Narasimha, 1969)

$$u\frac{\partial\tau}{\partial x} + v\frac{\partial\tau}{\partial y} = A(\bar{\tau} - \tau) + \frac{\partial}{\partial y}\left(\nu_\tau \frac{\partial\tau}{\partial y}\right), \tag{3}$$

where A is the reciprocal of a relaxation time, ν_τ is a stress diffusivity, and $\bar{\tau}$ is the equilibrium shear stress. A and ν_τ can in general depend on position, velocity, and stress. The boundary conditions on u, v, and τ are (for a symmetric wake)

$$\partial u/\partial y = v = 0 = \tau \quad \text{at} \quad y = 0 \tag{4}$$

and $u \to U$ and $\tau \to 0$ as $y \to \pm\infty$.

To solve these equations, we assume that we can write

$$U - u = w = w_0 f(\eta), \qquad \tau = \tau_0 g(\eta), \qquad \eta = y/\delta(x), \tag{5}$$

where f and g are similarity functions satisfying the boundary conditions

$$f'(0) = f(\pm\infty) = g(0) = g(\pm\infty) = 0, \qquad f(0) = 1;$$

τ_0 is a stress scale. (Primes indicate derivatives with respect to the argument.) Experiments show that the forms (5) always provide a good approximation, in equilibrium as well as nonequilibrium flows, with the additional simplification in the former that τ_0 can be replaced by w_0^2. In general we have therefore three unknowns w_0, τ_0, and δ; we propose to get the required equations by integrating Eqs. (2) and (3), and employing a centre-line condition that has been found advantageous in studying laminar wakes (Vasantha and Narasimha, 1970).

The momentum integral: The integration of Eq. (2) from $-\infty$ to $+\infty$ with respect to y gives the specialization to the wake of the well-known Karman momentum integral equation,

$$d\theta/dx + (H + 2)(\theta/U)\, dU/dx = 0 \tag{6}$$

where θ is the momentum thickness and H is the shape factor. A useful rearrangement of this equation is

$$d \ln(\theta U^3)/dx = (1 - H)U'/U. \tag{7}$$

From their usual definitions θ and H can be written in terms of w_0, δ, and U for given $f(\eta)$:

$$\theta = (w_0\delta/U)b_1(1 - D), \qquad H = 1/(1 - D) \tag{8}$$

where $D \equiv (b_2/b_1)(w_0/U)$,

$$b_1 = \int_{-\infty}^{+\infty} f\, d\eta \quad \text{and} \quad b_2 = \int_{-\infty}^{+\infty} f^2\, d\eta. \tag{9}$$

Using (8) and (9) in (6) we get

$$(w_0'/w_0)(1 - 2D)/(1 - D) + \delta'/\delta + (2U'/U)/(1 - D) = 0. \tag{10}$$

The shear stress integral: Substituting for v from the continuity equation (1), introducing the defect w and integrating the stress equation (3) from $y = 0$ to $y = \infty$, we get

$$(11) \qquad \int_0^\infty (U - w)\frac{\partial \tau}{\partial x}\,dy - \int_0^\infty dy \frac{\partial \tau}{\partial y}\int_0^y \frac{\partial (U - w)}{\partial x}\,dy' = \int_0^\infty A(\bar{\tau} - \tau)\,dy + v_\tau(0)\frac{\partial \tau}{\partial y}\bigg|_{y=0}.$$

Using the similarity assumptions (5) in the above equation and simplifying we get

$$(12) \qquad [\delta(U\tau_0)' + U\tau_0\delta']b_3 - [\delta(w_0\tau_0)' + w_0\tau_0\delta']b_4 = A\delta b_3(\bar{\tau}_0 - \tau_0) - v_\tau(0)b_5\tau_0/\delta$$

where

$$(13) \qquad b_3 = \int_0^\infty g\,d\eta, \qquad b_4 = \int_0^\infty fg\,d\eta, \qquad b_5 = g'(0).$$

The centre-line equation: Taking the momentum equation (2) at the centre line and substituting for w and τ, we get

$$(14) \qquad -\delta(Uw_0)'/\tau_0 + \delta w_0 w_0'/\tau_0 = b_5.$$

Equations (10), (12), and (14) give three first-order ordinary simultaneous differential equations for w_0, τ_0, and δ which can be solved given appropriate initial conditions. If the wake defects are small, higher order terms in w_0/U can be ignored, and the three equations simplify to

$$(15) \qquad w_0\delta U^2 \equiv M = \text{constant},$$

$$(16) \qquad (U\tau_0)' + U\tau_0\delta'/\delta = A(w_0^2 - \tau_0) - v_\tau(0)b_5\tau_0/b_3\delta^2,$$

$$(17) \qquad \delta(Uw_0)'/\tau_0 = -b_5$$

(remembering that $\bar{\tau}_0 = w_0^2$).

We now assume, following I, that

$$(18) \qquad A = aU/\theta, \qquad v_\tau(0) = k_1 U\theta = k_0 w_0 \delta;$$

eliminating δ'/δ from Eq. (16) we get the following equations for w_0 and τ_0:

$$(19) \qquad w_0' = -b_5\tau_0 w_0 U/M - w_0 U'/U,$$

and

$$(20) \qquad \tau_0' = \frac{aU^3}{Mb_1}(w_0^2 - \tau_0) - b_5\frac{U\tau_0}{M}\left(\frac{k_0}{b_3}w_0^2 + \tau_0\right).$$

The constants b_i, $i = 1$ to 5, depend on the velocity and stress profiles adopted, and have been evaluated here by assuming $f(\eta) = \exp(-\eta^2 \ln 2)$ and $g(\eta) = 2k_0(\ln 2)\eta f$; use of the available measured similarity profiles would affect the results only by a few per cent. Thus we get

$$b_1 = (\pi/\ln 2)^{1/2} = 2b_2\,, \qquad b_3 = k_0\,, \tag{21}$$
$$b_4 = k_0/2, \qquad b_5 = 2k_0 \ln 2.$$

Unless otherwise stated, values of a and k_0 are taken as[1]

$$a = 8.2 \times 10^{-4}, \qquad k_0 = 0.066. \tag{22}$$

In the experiments it is usually necessary to apply convergence corrections if the flow is not strictly two dimensional; applying them on the lines suggested in II, the only modification required in Eqs. (15), (19), and (20) is that M, instead of being a constant, should be now taken as a function of x as given by the experiments.

Equation (19) can in fact be integrated directly to obtain w_0 in terms of τ_0 (see II). However, it is just as convenient to solve both Eqs. (19) and (20) numerically, starting with given initial values (at $x = x_1$, say) of w_0 and τ_0 and the distributions of $U(x)$ and $M(x)$. We have used for this purpose a Runge–Kutta–Gill procedure with a step size of $\Delta x = \delta(x_1)$. $\delta(x)$ is then computed from Eq. (15).

It should be noted here that if the wake is in equilibrium (i.e., $U' \simeq 0$ and $\tau_0 = w_0^2$), Eqs. (19) and (20) are identical and their solution gives the usual equilibrium development ($w_0 \sim x^{-1/2}$) of the wake.

2.3. Results

To illustrate the accuracy of the proposed method, we show in Fig. 2 the results of calculations for what is perhaps the most severe test case available, namely flow M1 of II. Measurements as well as the exact numerical solution of Eqs. (2) and (3) are also shown in Fig. 2; there is excellent agreement among the data and the predictions.

Keffer (1965, 1967) has reported results of experiments conducted on nearly plane wakes subjected to two different types of distortion, involving respectively lateral stretching with normal compression and vice versa. In this situation Eq. (15) is of course not obeyed; however, using the measured

[1] There is a misprint in the value quoted in I for the parameter $\bar{w}_0^2 x/U^2\theta$. Applying certain small corrections to obtain true asymptotic values and utilizing data from more recent experiments, the following slightly revised values are now suggested for the characteristic mean flow parameters of an equilibrium wake:

$$\bar{\delta}/(x\theta)^{1/2} \simeq 0.30, \qquad \bar{w}_0/U(x/\theta)^{1/2} \simeq 1.64, \qquad k_0 \simeq 0.066.$$

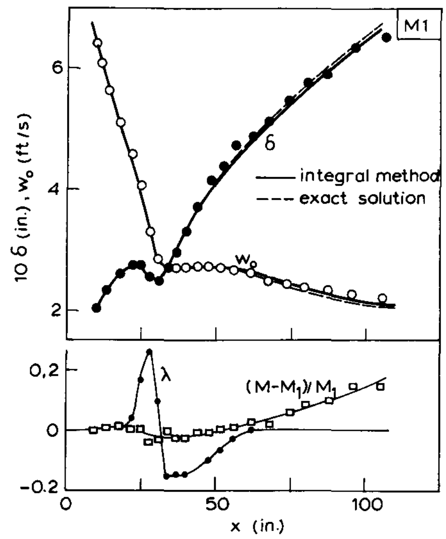

FIG. 2. Comparison of wake development calculated by the integral method with the exact solution as well as the measurements in flow M1 of II. $\lambda \equiv U'\delta/w_0$, $M_1 \equiv M(x_1)$.

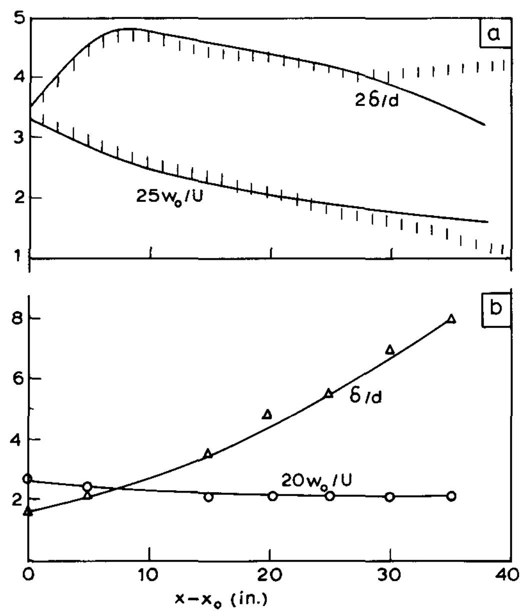

FIG. 3. Comparison between the measured (hatched areas, points) and calculated (curves) flow development in the distorted wakes studied by Keffer (1965, 1967). d is the diameter of the wake generating cylinder; distortion begins at x_0. (a) Lateral stretching normal compression (Keffer, 1965): $d = 3/16$ in., $x_0 = 10$ in. (b) Lateral compression, normal stretching (Keffer, 1967): $d = \frac{1}{2}$ in., $x_0 = 20$ in.

values of $M(x)$, Eqs. (19) and (20) may be solved as described above for an effectively two-dimensional wake with a slight convergence. The results of such calculations are again in excellent agreement with experiment (Fig. 3); the discrepancy that may be noticed towards the last few stations in one of the experiments (Keffer, 1965) is very likely due to the existence of a pressure gradient (of unknown magnitude) near the downstream end of the distorting duct.

3. Effects of Severe Distortion

When the pressure gradients are sufficiently large certain interesting features of the flow development become apparent; it is our intention here to discuss these, in the light of the results of Section 2.

3.1. Mean Velocity Field

The ratio of the second term on the right-hand side of Eq. (19) to the first gives us an effective pressure gradient or distortion parameter; taking τ_0 to be of order w_0^2, this parameter is seen to be the strain ratio $U'\delta/w_0$, called λ in I. If λ is large, the stress term containing τ_0 in Eq. (19) can be ignored, and we get the "ideal fluid" solution requiring Uw_0 to be independent of x (this result can be seen also as a consequence of Bernoulli's law). It follows from Eq. (15) (or from vorticity conservation) that $U\delta$ must also be independent of x.

Figure 4 shows measurements from flow F4 of II, in comparison with the above ideal fluid solutions as well as the more elaborate calculations of Section 2. The latter agree with experiment throughout the flow, and converge towards the former beyond $\lambda = 1.0$. It would appear therefore that, as the longitudinal strain rate becomes appreciably larger than the shear (it can be shown that this does not necessarily invalidate the boundary layer approximation), the Reynolds stresses do not strongly affect the mean flow development.

3.2. The Reynolds Stresses

It has earlier been suggested (I) that the changes that occur in the turbulence quantities on the imposition of an impulsive pressure gradient are similar in nature to what might be expected in the limit of extremely rapid distortion, when both inertia and viscous forces may be neglected. To obtain a quantitative appreciation of this similarity, we examine here the change in various turbulence quantities observed in flow F2 of I between a station just before the commencement of pressure gradient ($x = 19$ in., denoted by subscript a), and another just after its end ($x = 37$ in., subscript b), along the centre-line of the wake. Table I lists ratios of the r.m.s. values $\tilde{u}$, $\tilde{v}$ of the

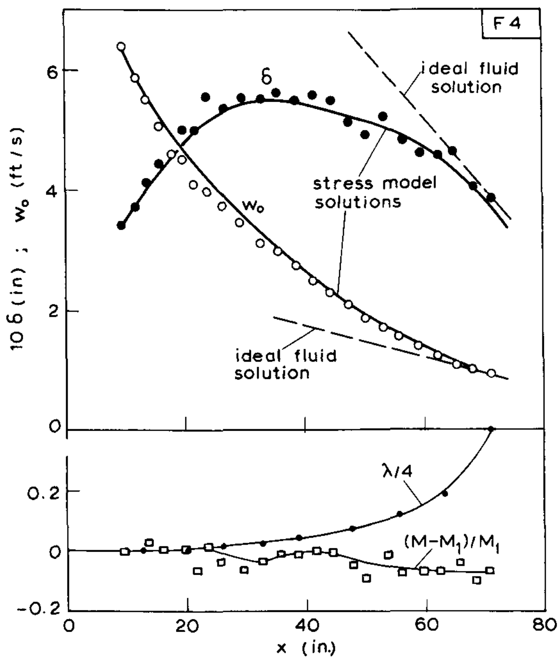

FIG. 4. Comparison of the calculated wake development using integral method and inviscid solution with the measurements in flow F4 of II. Values of Uw_0 and $U\delta$ at the last station are used for calculating the inviscid solution.

fluctuating velocity components along x and y, and compares them first with the values that would be expected in a normal equilibrium wake. It is clear that the favourable pressure gradient results in an appreciable decrease in $\tilde{u}$, a slight increase in $\tilde{v}$, and an appreciable increase in $\tilde{v}/\tilde{u}$.

These are generally the kind of changes that may be expected in the rapid distortion limit. Although a complete theory valid for shear flows is not yet available, the results of Batchelor and Proudman (1954) for homogeneous, initially isotropic turbulence provide a valuable pointer. This is particularly

TABLE 1. Changes in turbulence intensities in impulsively accelerated wake

	Measured	Equilibrium value	Rapid distortion theory	Corrected
$\tilde{u}_b/\tilde{u}_a$	0.69	0.75	0.935	0.70
$\tilde{v}_b/\tilde{v}_a$	0.77	0.75	1.072	0.80
$\dfrac{\tilde{v}_b/\tilde{u}_b}{\tilde{v}_a/\tilde{u}_a}$	1.12	1.0	1.15	1.15

so as a recent extension to a simple but nontrivial case of axisymmetric turbulence (Sreenivasan, 1972) shows that, unless the total distortion imposed or the departure from isotropy is extremely large, the isotropic results provide a good first approximation to the changes in the component energies.

However, the values of the various parameters obtained by an application of the Batchelor–Proudman results, also listed in Table I, show that while the change in the ratio $\tilde{v}/\tilde{u}$ is very closely predicted, the individual intensities are not. This is not unexpected because of the appreciable changes that occur in an undistorted wake in the normal course of decay. A simple correction for this can be obtained by superposing the two effects, either additively (Townsend, 1956) or multiplicatively (Ribner and Tucker, 1953); in the present case there is no great difference between the two procedures. Thus we may put

$$\tilde{u}_b/\tilde{u}_a = (1 - \Delta_1 \tilde{u}/\tilde{u}_a)(1 - \Delta_2 \tilde{u}/\tilde{u}_a),$$

where the changes on the right-hand side are respectively those due to distortion and normal decay. The last column in Table I shows these corrected numbers, which are in satisfactory agreement with measurements.

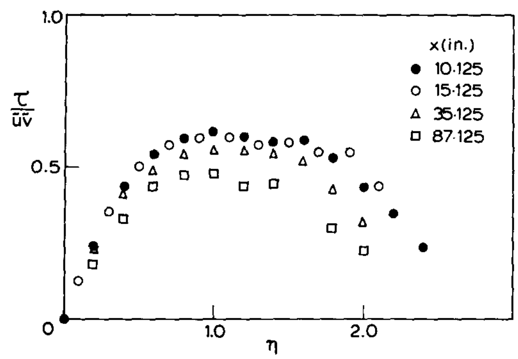

FIG. 5. Distribution of $\tau/\tilde{u}\tilde{v}$ across the wake in flow F2b of I.

In these rapid distortion theories the Reynolds shear stress τ is assumed to be zero, but a rough idea of the likely change in τ can nevertheless be obtained by examining the product $\tilde{u}\tilde{v}$, for measurements show that the correlation $\tau/\tilde{u}\tilde{v}$ changes much less from the equilibrium value (Fig. 5) than either $\tilde{u}$ or $\tilde{v}$ separately. Figure 6 shows the variation in $\tilde{u}\tilde{v}$ in a homogeneous flow subjected to a two-dimensional contraction $U(x)/U(x_a)$, for various values of the parameter

$$R = \tilde{u}_a^2/\tilde{v}_a^2,$$

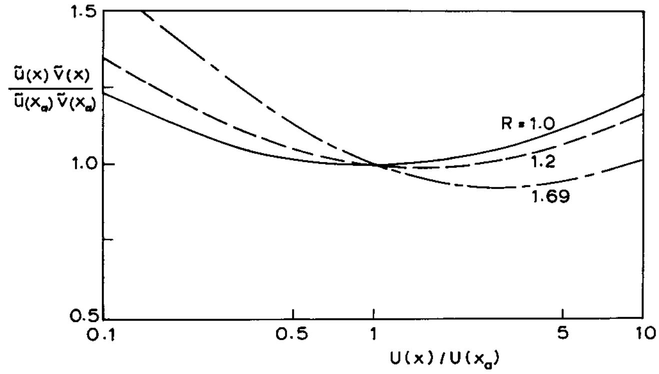

FIG. 6. Variation of the ratio of the product of r.m.s. values of the turbulence intensities before and after a two-dimensional contraction of ratio $U(x)/U(x_a)$, in homogeneous initially axisymmetric turbulence characterized by the parameter $R = \tilde{u}_a^2/\tilde{v}_a^2$.

which in isotropic turbulence is unity; the details of this calculation will be published elsewhere. It is seen that for $R < 1.69$, a fivefold increase in free stream velocity will not change the product $\tilde{u}\tilde{v}$ by more than 10 %. This appears to be the most likely explanation for the kind of "stress freezing" that has been observed in accelerated wakes (I).

In the model equation (3), stress freezing can be accounted for by putting $A = 0$ [more precisely $a = 0$ in (18)]. Figure 7 compares the measured values of τ_{max} with calculations using the integral method mentioned earlier with three different assumptions on the relaxation number a. It is seen from the figure that calculated values with $a = 0$ for $x < 38.5$ in. and

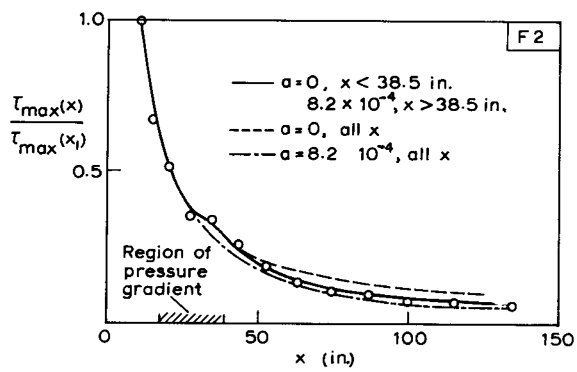

FIG. 7. Comparison of measured and calculated value of τ_{max} in flow F2b of I with three different assumptions on the number a.

$a = 8.2 \times 10^{-4}$ for $x > 38.5$ in. agree with the experiments well,[2] even in the region of peak pressure gradient. This indicates that the relaxation number might depend on the pressure gradient or strain imposed on the flow and rapidly goes to zero as the strain increases. It may be mentioned that similar considerations have been found to give meaningful results for the development of the outer part of a reverting turbulent boundary during rapid acceleration (Narasimha and Sreenivasan, 1973), and that further experiments are currently being carried out in wakes subjected to larger pressure gradients. On the whole it appears that inviscid development of the turbulent quantities is likely to occur earlier in the pressure gradient scale than does inviscid mean flow development.

4. Conclusions

The simple integral method described here using the relaxation diffusion model for the stress is adequate to describe wake flow subjected to various pressure gradients as well as to constant pressure distortions. In the case of large pressure gradients or severe distortion, it is likely that the development of turbulent stresses can be understood in terms of rapid distortion analysis; eventually even the mean velocity field becomes amenable to an ideal fluid treatment.

Note added in proof: A detailed, quantitative analysis of stress freezing has been recently given by K. R. Sreenivasan and R. Narasimha [Rapid distortion of shear flows, *Aeronaut. Soc. India Silver Jubilee Tech. Conf., Bangalore, 1974*, Pap. No. 2.3]. For difficulties in using the experimental data of Hill *et al.* (1963), see A. Prabhu [Nonequilibrium wake flows. Ph.D. Thesis, Dept. Aeronaut. Eng., Indian Institute of Science, 1971].

Acknowledgment

This work has been supported in part through a fellowship granted to K. R. Sreenivasan by the University Grants Commission.

References

Batchelor, G. K., and Proudman, I. (1954). *Quart. J. Mech. Appl. Math.* **7**, 83–103.
Gartshore, I. S. (1967). *J. Fluid Mech.* **30**, 547–560.
Hill, P. G., Schaub, U. W., and Senoo, Y. (1963). *J. Appl. Mech.* **30**, 518–524.
Keffer, J. F. (1965). *J. Fluid Mech.* **22**, 135–159.
Keffer, J. F. (1967). *J. Fluid Mech.* **28**, 183–193.
Narasimha, R. (1969). *Curr. Sci.* **38**, 47–48.
Narasimha, R., and Prabhu, A. (1972). *J. Fluid Mech.* **54**, 1–18.
Narasimha, R., and Sreenivasan, K. R. (1973). *J. Fluid Mech.* **61**, 417–447.

[2] For $x < x_a \simeq 17$ in., the wake is in equilibrium and $\tau = \bar{\tau}$, so the precise value used for a is irrelevant.

Prabhu, A. (1966). *AIAA J.* **4**, 925–926.
Prabhu, A., and Narasimha, R. (1972). *J. Fluid Mech.* **54**, 19–38.
Ribner, H. S., and Tucker, M. (1953). *NACA (Nat. Adv. Comm. Aeronaut.) Rep.* No. 1113.
Sreenivasan, K. R. (1972). *I. I. Sci. Aeronaut. Rep.* No. 72 FM7.
Townsend, A. A. (1956). "The Structure of Turbulent Shear Flow." Cambridge Univ. Press, London and New York.
Vasantha, S., and Narasimha, R. (1970). *J. Eng. Math.* **4**, 51–63.