Let $(X, \mathcal{O}_X)$ be a smooth C^∞ manifold, $\pi : E \to X$ a rank r vector C^∞ complex vector bundle on X.

Fue every open subset U of X we define $\mathcal{E}(U)$ to be the vector space of sections of $E|_U$. Note that $\mathcal{E}$ has the following properties:

1) it is a sheaf of vector spces on X;

2) it is a sheaf of $\mathcal{O}_X$ modules (that is, given a section s and a regular function f, $f \cdot s$ is also a section);

3) it is locally free of rank r. That is, locally on r, we can find an isomorphism $\mathcal{E} = \mathcal{O}_X^{\oplus r}$ as sheaves of $\mathcal{O}_X$-modules.

The latter corresponds to the fact that E is locally trivial.

Conversely, assume we are given a sheaf $\mathcal{E}$ on X satisfying 1)-3); we can define a vector bundle on X by glueing such that $\mathcal{E}$ is the sheaf of sections, as follows. Let U_i be an open cover of X such that $\mathcal{E}|_{U_i}$ is trivial; fix an isomorphism, i.e. a basis s_i^a $(a = 1, \dots, r)$ of $\mathcal{E}(U)$ as a free $\mathcal{O}_X(U)$-module.

Then on $U_{ij} = U_i \cap U_j$ we have two different basis for $\mathcal{E}(U_{ij})$ as $\mathcal{O}_X(U_{ij})$-module: s_i^a and s_j^b restricted to U_{ij}. What we can do is give automorphism of the trivial vector bundle $U_{ij} \times \mathbf{C}^k$ as follows: write $s_i^a = \sum_b f_b^a(ij) s_j^b$. The $f_b^a(ij)$ are C^∞ functions on U_{ij} and the matrix they form has nonzero determinant; in fact $f(ij)f(jk) = f(ik)$. So we get a cocylce in $Gl(r, \mathbf{C})$, which defines a bundle. One can show that E is unique up to canonical isomorphism and that $\mathcal{E}$ is its sheaf of sections.

When we take $(X, \mathcal{O}_X)$ to be a scheme, we can do pretty much the same; except that now we view $\mathbf{C}^k$ as a scheme and not as a manifold. It is not difficult to prove that the product of two schemes is a scheme and that the automoprihsm $f(ij)$ defined above are automorphisms of schemes. So the whole construction still works, and we call such an $\mathcal{E}$ a locally free sheaf on X and the associated scheme $\pi : E \to X$ a vector bundle on X. Usually algebraic geometers do not distinguish between a vector bundle and the associated locally free sheaf.

If $f : Y \to X$ is a morphism of schemes, we can pullback vector bundles; if Y is a closed subscheme of X, we write restriction instead of pullback.

That's all very easy, and if our scheme happens to be a manifold we get a tangent, a cotangent bundle of the rank you expect, and we can define p forms and vector fields.

Here: explain the cotangent (and tangent) sheaf of $\mathbf{P}^n$ and of a complete intersection in $\mathbf{P}^n$.

When X is a general quasiprojective scheme, we can still use the exact sequence above to define a sheaf Ω^1_X on X. That is, we let $\Omega_X(U) = \Omega_{\mathbf{P}^n}|_X(U)/\equiv$, where a 1-forms ω is equivalent to zero iff locally on $\mathbf{P}^n$ there is a function vanishing on X such that $\alpha - df|_X$ is zero. (Note that this is how you make the cokernel of a map of sheaves, if you do it naively you don't get a sheaf).

The sheaf we obtain this way has the following properties:

1) it is a sheaf of vector spaces on X;

2) it is a sheaf of $\mathcal{O}_X$ modules in the sense above;

3) X can be covered by affine open sets U such that $U = Spec\, A$ and there is an exact sequence

$$\mathcal{O}_U^r \to \mathcal{O}_U^g \to \mathcal{E}|_U \to 0.$$

In this case we say that $\mathcal{E}$ is a coherent sheaf. Note that locally on every such U $\mathcal{E}$ can be described as the cokernel of an $r \times g$ matrix of regular functions, so at every point we get a vector space of dimension between g and $g - r$. However, in general this vector spaces will not fit together to define a scheme E with a morphism $\pi : E \to X$ such that the sections of π give back $\mathcal{E}$. When this happens, the sheaf $\mathcal{E}$ is said to be reflexive (in technical terms, it means that it is equal to its double dual).

The beautiful thing about coherent sheaves is that you can define cohomology for them. I will not tell you the definition used by the algebraic geometers because it's frightening. Cohomology has the usual long exact sequences (this is actually how you compute it, most of the time) but it's important to also know the Cech interpretation. Let X be any scheme, and U_i an open cover such that the intersection of any finite number of U_i is affine. Then we can form the usual Cech complex and its cohomology **is** the cohomology of the sheaf. No need to go to the inverse limit over all covers!

Among the beautiful facts of life is that if the scheme is Hausdorff (with its euclidean topology) then the intersection of two affines is affine (give trivial counterexample otherwise). In particular this is true for quasiprojective schemes.

So from this it follows that:

1) A coherent sheaf on an affine scheme has no higher cohomology; in fact, it can be globally written as a cokernel of locally free's.

2) A coherent sheaf on a quasiprojective scheme of dimension n has zero k-th cohomology whenever $k > n$.

Moreover, there are some vanishing theorems that give you some vanishing for free.