Instanton counting and generalized Donaldson invariants

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(0) Introduction

Donaldson invariants: C^∞-inv. of compact 4-manifolds
For X proj. surface: intersection number $\int_M \mu(L)^d$ on moduli space $M_H^X(c_1, c_2)$ of H-stable rk 2 sheaves on X

Nekrasov partition function $Z(\varepsilon_1, \varepsilon_2, \vec{a}, \Lambda)$:
generating function of "equiv. Donaldson inv. of $\mathbf{A}^2$"
equivariant intersection number of moduli of instantons $M(r, n)$.

X proj. toric surface $\Longrightarrow$ X glued together from $\mathbf{A}^2$'s

Aim A: Compute Donaldson invariants of X in terms of Nekrasov partition function ($r = 2$)

Aim B: Relate $rk = r$ Don. inv. of X to Nekr. part. fct.

K-theory Nekrasov partition function $Z_K(\varepsilon_1, \varepsilon_2, \vec{a}, \Lambda)$:
generating function for equiv. char. of $H^0(M(r,n), \mathcal{O})$
"K-theoretic Donaldson invariants of $\mathbf{A}^2$"

L line bundle on X $\mapsto$ $\overline{L} := \mathcal{O}(\mu(L))$ Donaldson l. b. on $M_H^X(c_1, c_2)$

Aim C: Compute $\chi(M_X^H(c_1, c_2), \overline{L})$ in terms of Z_K

Note: D-inv $\Phi^H_{X,c_1}(c_1(L)^{\dim(M)}) = \int_{M_H^X(c_1,c_2)} c_1(\overline{L})^{\dim(M)}$

Riemann-Roch $\Longrightarrow$ $\chi(M, \overline{L}) = \Phi^H_{X,c_1}(c_1(\overline{L})^d)/d! + l.o.t$

Motivation:

(1) Nekrasov partition function is closely related to relation
Seiberg Witten-invariants $\longleftrightarrow$ Donaldson invariants

(2) Formula can be viewed as analogue of topological vertex formula

(3) $\chi(M, \overline{L})$ should be K-theoretic Donaldson invariants (still not constructed).
Want to understand analogues of all basic properties of Donaldson-invariants

(1) Review of Donaldson invariants for alg. surf.

(X, H) proj. surface

$$M_H^X(c_1, c_2) = \{H\text{-stable rank 2 sheaves}\}$$

$\mathbf{E} \to X \times M$ universal sheaf, $L \in H_2(X)$, $p \in H_0(X)$

$$\mu : H_*(X) \to H^*(M); \mu(\bullet) = \left(c_2(\mathbf{E}) - \frac{1}{4}c_1(\mathbf{E})^2\right)/\bullet$$

$$\Phi^X_{c_1,H}(\exp(Lz + px)) = \sum_n \int_{M_H^X(c_1,n)} \exp(\mu(L)z + \mu(p)x)\Lambda^{dim(M)}$$

- $p_g(X) > 0$, $(b_+ \geq 3)$: independent of H
- $p_g(X) = 0$ $(b_+ = 1)$: depends on H via system of walls and chambers in ample cone $\mathbf{C}_X$

Walls: $\xi \in H^2(X,\mathbf{Z})$ defines wall of type (c_1,c_2) if
$\xi \equiv c_1 \mod 2H^2(X,\mathbf{Z})$ and $4c_2 - c_1^2 + \xi^2 \geq 0$
Wall $W^\xi := \{H \in \mathbf{C}_X \mid H \cdot \xi = 0\}$

Chambers=connected components of $C_X \setminus$ walls
$M_X^H(c_1,c_2)$ and invariants constant on chambers, change when
H crosses wall (i.e. $H_- \to H_+$ with $H_-\xi < 0 < H_+\xi$)

Studied in gauge theory by Kotschick, K.-Morgan, K.-Lisca
Kotschick-Morgan:

$$\Phi^X_{c_1,c_2,H_+} - \Phi^X_{c_1,c_2,H_-} = \sum_\xi \pm\delta^X_{\xi,c_2}$$

Kotschick-Morgan conj.: δ^X_{ξ,c_2} is polynomial in $\langle\xi,L\rangle$, L^2,
coefficients depend only on c_2, ξ^2 and topology of X

Using K-M conjecture [G] determined gen. function for wallcrossing in terms of modular forms

Moore-Witten: Derive same formula for wallcrossing via integration over the u-plane,

Marino-Moore: Generalize this result to r arbitrary. In this case the wallcrossing is subject to higher order wallcrossing

Nekrasov: Rigorous approach to u-plane integral via Z
Predicts: Donaldson invariants obtained by integrating Z over $\vec{a}$.

Kronheimer: Mathematical definition of higher rank Donaldson invariants

(2) Nekrasov Partition function

Instanton moduli space: $\mathbf{P}^2 = \mathbf{A}^2 \cup l_\infty$,

$$M(r,n) := \left\{ \begin{matrix} \text{framed coh. sheaves } (E,\phi) \text{ on } \mathbf{P}^2 \\ rk(E) = r, c_2(E) = n, \phi : E|_{l_\infty} \simeq \mathcal{O}^{\oplus r} \end{matrix} \right\}$$

smooth quasiproj, dim $2rn$, e.g. $M(1,n) = Hilb^n(\mathbf{A}^2)$

Torus Action:

$\mathbf{C}^* \times \mathbf{C}^*$ acts on $(\mathbf{P}^2, l_\infty)$: $(e^{\varepsilon_1}, e^{\varepsilon_2})(z_1, z_2) = (e^{\varepsilon_1} z_1, e^{\varepsilon_2} z_2)$
$(\mathbf{C}^*)^{r-1} = \left\{ (e^{a_1}, \ldots, e^{a_r}) \,\middle|\, \sum a_i = 0 \right\}$ acts

by change of framing $(e^{a_1}, \ldots, e^{a_r})(E,\phi) = (E, diag(e^{a_1}, \ldots, e^{a_r}) \circ \phi)$
Fixpoint set of $(\mathbf{C}^*)^{2+r-1}$ is finite:

$$\left\{ (I_{Z_1} \oplus \ldots \oplus I_{Z_r}, id) \,\middle|\, Z_i \in Hilb^{n_i}(\mathbf{A}^2, 0), \text{ ideal gen. by monomials} \right\}$$

Let X variety over $\mathbf{C}$ with action of
$T = (\mathbf{C}^*)^k = \{(e^{w_1}, \ldots, e^{w_k})\}$ and $X^T = \{p_1, \ldots, p_e\}$
Equiv. cohom. $H_T^*(X)$: module over $H_T^*(pt) = \mathbf{C}[w_1, \ldots, w_k]$
e.g. E equiv. vector space, eigenval. $e^{b_i} \Longrightarrow c(E) = \prod_i (1 + b_i)$
Equiv. integration: X compact, $\tilde{\alpha}$ equiv. lift of $\alpha \in H^*(X)$

$$\int_X \alpha = \sum_{p_i} \frac{\tilde{\alpha}|_{p_i}}{c_{top}(T_{p_i}X)}\bigg|_{w_1 = \ldots = w_k = 0}$$

Note $\tilde{\alpha}|_{p_i}$, $c_{top}(T_{p_i}X) \in \mathbf{C}[w_1, \ldots, w_k]$

Nekrasov Partition function
$M(r,n)$ with action of $(\mathbf{C}^*)^{2+r-1} = \{(e^{\varepsilon_1}, e^{\varepsilon_2}, e^{\vec{a}})\}$

$$Z^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}, \Lambda) = \text{''} \sum_{n \geq 0} \Lambda^{2rn} \int_{M(r,n)} 1\text{''} \in \mathbf{C}(\varepsilon_1, \varepsilon_2, \vec{a})[[\Lambda]]$$

$Z(\varepsilon_1, \varepsilon_2, \vec{a}, \Lambda) = Z^{inst} \cdot Z^{per}$

Nekrasov conjecture (Nekrasov-Okounkov, Yoshioka-Nakajima, Braverman-Etinghof)

(1) $Z = \exp(\frac{F(\varepsilon_1,\varepsilon_2,\vec{a},\Lambda)}{\varepsilon_1\varepsilon_2})$, F regular at $\varepsilon_1 = \varepsilon_2 = 0$

(2) $F_0 = F|_{\varepsilon_1=\varepsilon_2=0}$ is Seiberg-Witten prepotential (periods of SW-curve, a family of hyperelliptic curves).

Seiberg-Witten curve: parametrized by $\vec{u} = (u_2, \ldots, u_r)$.

$$C_{\vec{u}} : y^2 = P(z)^2 - 4\Lambda^{2r}, \quad P(z) = z^r + u_2 z^{r-2} + \ldots + u_r$$

hyperelliptic curves via $(y, z) \mapsto z \in \mathbf{P}^1$

Seiberg-Witten differential $dS = -\frac{1}{2\pi}\frac{zP'(z)dz}{y}$

Put $a_\alpha := \int_{A_\alpha} dS, \quad a_\beta^D = 2\pi\sqrt{-1}\int_{B_\beta} dS$

$(\alpha = 1, \ldots, r,\ \beta = 2, \ldots, r)$. F_0 characterized by $a_\beta^D = -\frac{\partial F_0}{\partial a_\beta}$

K-theory Nekrasov Partition function

$$Z_K^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}, \Lambda) = \sum_{n \geq 0} \Lambda^{2rn} e^{-(\varepsilon_1+\varepsilon_2)rn/2} \mathrm{ch}(H^0(M(r,n), \mathcal{O})) \in \mathbf{C}(e^{\varepsilon_1}, e^{\varepsilon_2}, e^{\vec{a}})[[\Lambda]]$$

Character = formal sum of (finite-dim) weight spaces

e.g. $r = 1$: $Z_K^{inst}(\varepsilon_1, \varepsilon_2, \Lambda) = 1 + \frac{e^{-(\varepsilon_1+\varepsilon_2)/2}\Lambda}{(1-e^{\varepsilon_1})(1-e^{\varepsilon_2})} + O(\Lambda^2)$

$Z_K = Z_K^{inst} \cdot Z_K^{per}$

(Yoshioka-Nakajima): Similar result for Z_K

Same statement, different family of hyperelliptic curves

Know also next two orders in $\varepsilon_1, \varepsilon_2$ of F and F_K in case $r = 2$

(3) Donaldson inv. and $\chi(M,\overline{L})$ versus Nekrasov part. fctn

Fix $r=2$. X sm. toric surface, $\mathbf{C}^*\times\mathbf{C}^*=\{(e^{\varepsilon_1},e^{\varepsilon_2})\}$-action
fixpoints $\{p_1,\ldots,p_e\}$, $w(x_i),w(y_i)$ weights of action on $T_{p_i}X$
Fix F with $M_F^X(c_1,c_2)=\emptyset$ (exists after blowing up X)
Donaldson invariants:

$$\Phi^X_{c_1,H}(\exp(Lz+px))=\sum_\xi \frac{1}{\Lambda}\operatorname*{res}_{t=\infty}\lim_{\varepsilon_1,\varepsilon_2\to 0}\left(\prod_{i=1}^e Z\Big(w(x_i),w(y_i),\frac{t-\xi|_{p_i}}{2},\Lambda e^{(Lz+px)|_{p_i}/4}\Big)\right)dt$$

Here $\xi\in H^2(X,\mathbf{Z})$ with $\xi\equiv c_1 \mod 2$ and $\xi H>0>\xi F$.

Holomorphic Euler Characteristic: $L\in H_2(X)$ with $\langle c_1,L\rangle$ even
$\overline{L}$ line bundle on $M_H^X(c_1,n)$ with $c_1(\overline{L})=\mu(L)$.

$$\sum_n \chi(M_H^X(c_1,n),\overline{L})\Lambda^{dim M}$$

$$=\sum_\xi\frac{1}{\Lambda}\Big(\operatorname*{res}_{e^t=0}-\operatorname*{res}_{e^t=\infty}\Big)\lim_{\varepsilon_1,\varepsilon_2\to 0}\left(\prod_{i=1}^e Z_K\Big(w(x_i),w(y_i),\frac{t-\xi|_{p_i}}{2},\Lambda e^{(L-K_X)|_{p_i}/4}\Big)\right)\frac{d(e^t)}{e^t}$$

(4) Method of proof

X toric surface $\Longrightarrow$ $M_H^X(c_1, c_2)$ and Don. inv. depend on chamber of H. In one chamber moduli space is empty $\Longrightarrow$ everything determined by wallcrossing

Aim: Relate wallcrossing to partition function.
Let ξ define wall W^ξ. When H crosses wall, sheaves E in

$$0 \to I_Z((\xi + c_1)/2) \to E \to I_W((-\xi + c_1)/2) \to 0, \quad (Z, W) \in X^{[l]} \times X^{[m]}$$

($l + m = 4c_2 - c_1^2 + \xi^2$) are replaced by extensions the other way round

Geometrically: Start with $M_{H_-}^X(c_1, c_2)$.

Successively for $l = 0, 1 \ldots, 4c_2 - c_1^2 + \xi^2$:
- blowup bundle $\mathbf{P}(\mathrm{Ext}^1(I_W, I_Z(\xi)))$ over $X^{[l]} \times X^{[m]}$,
- blow down exceptional divisor D to $\mathbf{P}(\mathrm{Ext}^1(I_Z, I_W(-\xi)))$

Finally arrive at $M_{H_+}^X(c_1, c_2)$.

Fix l, m, let $p : D \to X^{[l]} \times X^{[m]}$ projection

$$\left[\text{wallcrossing for D-inv}\right] = \int_D (Ap^*(B)) = \int_{X^{[l]} \times X^{[m]}} (p_* A) B$$

$$\left[\text{wallcrossing for } \chi(\overline{L})\right] = \chi(D, A' \otimes p^*(B')) = \chi(X^{[l]} \times X^{[m]}, p_*(A') \otimes B')$$

Now apply Bott formula on $X^{[l]} \times X^{[m]}$:

Action of $\mathbf{C}^* \times \mathbf{C}^*$ on X lifts to $X^{[l]} \times X^{[m]}$

$$\bigcup_{l,m} (X^{[l]} \times X^{[m]})^{(\mathbf{C}^*)^2} = \bigcup_{n_1,\ldots,n_e} M(n_1)^{(\mathbf{C}^*)^3} \times \ldots \times M(n_e)^{(\mathbf{C}^*)^3}$$

Show: contribution for both sides is the same at every fixpoint

$T_{(Z,W)} X^{[n]} \times X^{[m]} = \text{Ext}^1(I_Z, I_Z) \oplus \text{Ext}^1(I_W, I_W)$

$T_{(Z,W)} M(n) = \text{Ext}^1(I_Z, I_Z) \oplus \text{Ext}^1(I_W, I_W) \oplus \text{Ext}^1(I_Z, I_W) e^{2a} \oplus \text{Ext}^1(I_W, I_Z) e^{-2a}$

(5) Explicit formulas in modular forms and elliptic functions

Develop $F = \varepsilon_1\varepsilon_2 \log Z$, $F_K = \varepsilon_1\varepsilon_2 \log Z_K$:

$$F(\varepsilon_1,\varepsilon_2,a,\Lambda) = F_0 + (\varepsilon_1+\varepsilon_2)H + \varepsilon_1\varepsilon_2 F_1 + (\varepsilon_1+\varepsilon_2)^2 G + h.o.t$$

Similarly for F_K.

$$\begin{aligned}
&\prod_{i=1}^{e} Z\Big(w(x_i), w(y_i), \frac{t-\xi|_{p_i}}{2}, \Lambda e^{(Lz+px)|_{p_i}/4}\Big)\\
&\quad = \exp\left(\sum_{i=1}^{e} \frac{1}{w(x_i)w(y_i)} F\Big(w(x_i), w(y_i), \frac{t-\xi|_{p_i}}{2}, \Lambda e^{(Lz+px)|_{p_i}/4}\Big)\right)\\
&\quad = \exp\left(\sum_{i} \frac{1}{w(x_i)w(y_i)} \frac{\partial^2 F_0}{(\partial a)^2}(t/2,\Lambda)\frac{(\xi|_{p_i})^2}{8} + \dots\right)\\
&\quad = \exp\Big(\frac{\partial^2 F_0}{(\partial a)^2}\frac{\xi^2}{8} - \frac{\partial^2 F_0}{\partial \log\Lambda \partial a}\frac{\langle \xi, L\rangle z}{8} + \frac{\partial^2 F_0}{(\log\Lambda)^2}\frac{\langle L, L\rangle z^2}{32} + \frac{\partial F_0}{(\log\Lambda)}\frac{x}{4} + \dots\Big)
\end{aligned}$$

by Bott formula on X. Similarly for Z_K.

Donaldson invariants

$$\Phi^X_{c_1,H}(\exp(Lz+px)) = \sum_\xi \pm \operatorname*{Coeff}_{q^0}\left(q^{-(\xi/2)^2}\exp\left(h\langle\xi/2,L\rangle z + T\langle L^2\rangle z^2 - ux\right)\theta_{01}(\tau)^{\sigma(X)}B\right)$$

Holomorphic Euler characteristic

$$\sum_{n\in\mathbf{Z}}\chi(M^X_H(c_1,n),\overline{L})\Lambda^{dim(M)}$$

$$= \sum_\xi \pm \operatorname*{Coeff}_{q^0}\left(q^{-(\xi/2)^2}\exp\left(\langle\xi/2, L-K_X\rangle h_K + (L-K_X)^2 T_K\right)\theta_{01}(\tau)^{\sigma(X)}B_K\right)$$

$$q := e^{\pi i\tau},\ \tau := -\frac{1}{2\pi i}\frac{\partial^2 F_0}{(\partial a)^2}, \quad u := -\frac{1}{4}\frac{\partial F_0}{\partial \log\Lambda} = -\frac{\theta_{00}^4+\theta_{10}^4}{\theta_{00}^2\theta_{10}^2}\Lambda^2,$$

$$h := \frac{\partial u}{\partial a} = \frac{2\sqrt{-1}}{\theta_{00}\theta_{10}}\Lambda,\ T := \frac{1}{32}\frac{\partial^2 F_0}{(\partial\log\Lambda)^2} = -h^2G_2 - \frac{u}{6};$$

$$h_K(\Lambda) = 2\pi i\left(i\frac{\theta_{11}(\bullet,\tau)}{\theta_{01}(\bullet,\tau)}\right)^{-1} = h\Lambda + O(\Lambda^3),\ T_K = \log\left(\frac{\theta_{01}(\frac{h_K}{2\pi i},\tau)}{\theta_{01}(\tau)}\right) = T\Lambda^2 + O(\Lambda^4),$$

$$B_K = B + O(\Lambda^2)$$

$$\theta_{\mu\nu}(z,\tau) := \sum_{n\in Z}(-1)^{(n+\frac{\mu}{2})\nu}q^{(n+\frac{\mu}{2})^2}e^{2\pi i(n+\frac{\mu}{2})z},\ \theta_{\mu\nu}(\tau) := \theta_{\mu\nu}(0,\tau)$$

Same formula holds for δ_ξ^X for X any simply conn. alg. surfaces with $b_+ = 1$

Proof: Show many intersect. numbers on $X^{[n]}$ are polynom. in inters. numbers on X, thus formula determined by case of toric surfaces.

Cor. Kotschick-Morgan conj. true for alg. surfaces.

(6) Higher rank case:

$\mathbf{E} \to X \times M_H^X(r, c_1, c_2)$ universal sheaf, $L \in H_2(X)$, $p \in H_0(X)$

$$\mu : H_*(X) \to H^*(M); \mu(\bullet) = \left(c_2(\mathbf{E}) - \frac{r-1}{2r} c_1(\mathbf{E})^2\right)/\bullet$$

$$\Phi^X_{c_1,H}(\exp(Lz + px)) = \sum_n \int_{M_H^X(r,c_1,n)} \exp(\mu(L)z + \mu(p)x)\Lambda^{dim(M)}$$

$r = 3$: 2nd order wallcrossing:

$\xi_1, \xi_2 \in H^2(X)$ with W^{ξ_1}, W^{ξ_2} intersecting transv.

$H_{++}, H_{+-}, H_{-+}, H_{--}$ ample with $H_{+-}\xi_1 > 0$, $H_{+-}\xi_2 < 0$, etc.

$\delta_{\xi_1,+} = \Phi_{H_{++}} - \Phi_{H_{-+}}$, $\delta_{\xi_1,-} = \Phi_{H_{+-}} - \Phi_{H_{--}}$, etc.

$D_{\xi_1,\xi_2} = \delta_{\xi_1,+} - \delta_{\xi_1,-} = \delta_{\xi_2,+} - \delta_{\xi_2,-}$

$D_{\xi_1,\xi_2}(\exp(Lz+px)) = E_{\xi_1,\xi_2} + E_{-\xi_1,-\xi_2}$ with

$$E_{\xi_1,\xi_2} = \frac{1}{\Lambda^2} \operatorname*{res}_{t_2=\infty} \operatorname*{res}_{t_1=\infty} \lim_{\varepsilon_1,\varepsilon_2\to 0} \prod_{i=1}^{e} Z\Big(w(x_i), w(y_i), \tfrac{t_1-\xi_1|_{p_i}}{3} - \tfrac{t_2-\xi_2|_{p_i}}{2}, \tfrac{t_1-\xi_1|_{p_i}}{3} + \tfrac{t_2-\xi_2|_{p_i}}{2}; \Lambda e^{(Lz+px)|_{p_i}/6}\Big)$$

Similar formula for $(r-1)$-th order wallcrossing for general r.

Still preliminary (not carefully checked)
Based on work of Mochizuki