Instanton counting and Donaldson invariants

L. Göttsche (joint work w. Nakajima, Yoshioka)

(0) Introduction

X compact 4-manifold $\mapsto$ Donaldson invariants
X proj. surface: computed using moduli spaces of vector bundles on X.

Nekrosov partition function

$$= \text{generating fctn of "D.inv. of } \mathbf{A}^2\text{"}$$

X smooth toric surface $\Longrightarrow$ X pieced together from $\mathbf{A}^2$s.

Aim: Compute Donaldson invariants of X in terms of Nekrasov partition function.

(1) Instanton moduli spaces
$\mathbf{P}^2 = \mathbf{A}^2 \cup l_\infty$
Instanton moduli space

$$M(r,n) := \left\{ \begin{matrix} \text{framed coh. sheaves } E \text{ on } \mathbf{P}^2 \\ rk(E) = r, c_2(E) = n \end{matrix} \right\}$$
$$= \left\{ (E,\phi) \;\middle|\; \phi : E|_{l_\infty} \simeq \mathcal{O}_{l_\infty} \right\} /\text{iso.}$$

$M(r,n)$ is smooth quasiproj, dim $2nr$.

Torus Action: max. torus $T \subset SL(r,\mathbf{C})$ acts by change of framing

$$diag(t_1,\dots,t_r)(E,\phi) = (E, diag(t_1,\dots,t_r) \circ \phi)$$

$\mathbf{C}^* \times \mathbf{C}^*$ acts on $(\mathbf{P}^2, l_\infty)$ by
$(s_1,s_2)\cdot(z_1,z_2) = (s_1z_1, s_2z_2)$
and on $M(r,n)$ by pullback
$\Longrightarrow \widetilde{\mathbf{C}}^* \times \mathbf{C}^* \times T$ acts on $M(r,n)$.

Fact: $M(r,n)^{\widetilde{T}}$ is finite

$$= \left\{ (I_{Z_1},\phi_1) \oplus \dots (I_{Z_r},\phi_r) \;\middle|\; Z_i \in Hilb^{n_i}(\mathbf{A}^2, 0) \right\}$$

(2) Equivariant cohomology and eq. integr
X var./$\mathbf{C}$ w. action by torus $T = \{(t_1, \ldots, t_s)\}$
Equiv. cohomology: $H_T^*(X) = H^*(X \times_{B_T} E_T)$
module over $H_T^*(pt) = H^*(B_T) = \mathbf{C}[w_1, \ldots, w_s]$
E/X vector bdl with lift of T-action
$\Longrightarrow$ equiv. Chern class, e.g. $c_1(\mathcal{O}_X, t_i^k \cdot) = kw_i$

Equiv. integration: $\pi : X \to pt$ projection

$$\int_{X,T} := \pi_* : H_T^*(X) \to H_T^*(pt) = \mathbf{C}[w_1, \ldots, w_s]$$

For $\alpha \in H^*(X)$ let $\widetilde{\alpha}$ equiv. lift $\Longrightarrow$

$$\int_X \alpha = \Big(\int_{X,T} \widetilde{\alpha} \Big)|_{w_1 = \ldots = w_s = 0}$$

X compact, $dim = d$, $X^T = \{p_1, \ldots, p_e\}$,
$i_{p_i} : p_i \hookrightarrow X \Longrightarrow$

$$\int_{X,T} \widetilde{\alpha} = \sum_i \frac{i_{p_i}^*(\widetilde{\alpha})}{c_d(T_{p_i} X))}.$$

(3) Nekrasov Partition function
Take variables $\epsilon_1, \epsilon_2, a_1, \ldots, a_r$ corresponding to $s_1, s_2, t_1, \ldots, t_r$ of $\widetilde{T}$.

$$Z^{inst}(\epsilon_1, \epsilon_2, \vec{a}, \Lambda) = \text{''} \sum_{n \geq 0} \Lambda^{2rn} \int_{M(r,n)} 1\text{''}$$
$$:= \sum_{n \geq 0} \Lambda^{2rn} \sum_{q\ fix} \frac{1}{c_{2rn}(T_q M(r,n))}$$
$$\in \mathbf{C}(\epsilon_1, \epsilon_2, \vec{a})[[\Lambda]]$$

$Z(\epsilon_1, \epsilon_2, \vec{a}, \Lambda) = Z^{inst} \cdot Z^{per}$
perturbation part: explicit function.

Nekrasov conjecture:

1. $Z = \exp(\frac{F(\epsilon_1, \epsilon_2, \vec{a}, \Lambda)}{\epsilon_1 \epsilon_2})$, F reg. at $\epsilon_1 = \epsilon_2 = 0$.

2. $F_0 = F|_{\epsilon_1 = \epsilon_2 = 0}$ is Seiberg-Witten prepotential.

Shown by Nekrasov-Okounkov, Yoshioka-Nakajima and next terms determined

SW-prepot.: $\vec{u} = (u_2, \ldots, u_r) \in \mathbf{C}^{r-1}$ (u-plane)
$C_{\vec{u}} = [y^2 = (P(z) - 2\Lambda^r)(P(z) + 2\Lambda^r)]$,
$P(z) = z^r + u_2 z^{r-2} + \ldots + u_r$
hyperell. curves

In physics important for Donaldson$\simeq$SW.

Periods: $a_i = \int_{A_i} dS, a_i^D = -2\pi i \int_{B_i} dS$
$\exists F_0(\vec{a}) : a_i^D = -\frac{\partial F_0}{\partial a_i}$ (SW-prepotential).

$r = 2$: expressed in terms of modular forms and elliptic functions.

(4) Donaldson inv. vs. Nekrasov part. fctn
For simplicity rank $r = 2$
Donaldson invariants, C^∞-inv. of X
X algebraic surface, computed using $M_H^X(c_1, c_2)$
(moduli space of coh. sheaves on X)
For $\alpha \in H^*(X)$ Donaldson invariants:

$$\phi^X_{c_1,H}(\exp(\alpha)) := \sum_{c_2} \int_{M_H^X(c_1,c_2)} \exp((c_2(E) - c_1(E)^2/4)/\alpha)$$

$p_g(X) > 0$ independent of H
$p_g(X) = 0$ depends on H via
system of chambers in ample cone
changes only when H crosses a wall
Kotschick-Morgan conjecture:
wallcrossing depends only on topology
Using K-M conjecture determined wallcrossing
in terms of modular forms.
(same as in Nekrasov part. function)
Related work Moore-Witten, Marino-Moore.

X smooth toric surface, has $T = \mathbf{C}^* \times \mathbf{C}^*$-action
fixpoints $p_1, \ldots, p_e$
x_i, y_i local coord. at p_i, $t_i \cdot x_i = t^{u_i} x_i$, $t_i \cdot y_i = t^{v_i} y_i$
$w(x_i) := \epsilon_1 u_1 + \epsilon_2 u_2$, $w(y_i) := \epsilon_1 v_1 + \epsilon_2 v_2$ weights

Conjecture (Nekrasov) (arbitrary rank)

$$\phi^X_{c_1,H}(\exp(\alpha)) := \lim_{\epsilon_1,\epsilon_2 \to 0} \oint d\vec{a} \sum_{\vec{L}} \prod_{i=1}^{e} Z\Big(w(x_i), w(y_i), i^*_{p_i}(\vec{L}) + \vec{a}, \exp(-i^*_{p_i}(\alpha)/2r)\Big)$$

$\vec{L}$ runs through $(L_1, \ldots, L_r)$ with $\sum L_j = c_1$

Thm: Fix H_0 with $M^X_{H_0}(c_1, c_2) = \emptyset \ \forall c_2$.

$$\Phi^X_{c_1,H}(\exp(\alpha)) =$$
$$- \lim_{\epsilon_1,\epsilon_2 \to 0} res_{a=0}\Big(\sum_{L_1,L_2} \prod_{i=1}^{e}$$
$$Z\Big(w(x_i), w(y_i), i^*_{p_i}(L_1) - a/2, i^*_{p_i}(L_2) + a/2,$$
$$\exp(-i^*_{p_i}(\alpha)/4)\Big)\Big)$$

Sum is over L_1, L_2 with $L_1 + L_2 = c_1$ and $(L_1 - L_2)H > 0 > (L_1 - L_2)H_0$

Gives new proof of wallcrossing formula for toric surfaces without KM-conj.

(5) Method of proof

Relate wallcrossing to partition function

Wallcrossing: $\xi \in H^2(X, \mathbf{Z}) + c_1/2$ defines wall $\xi^\perp = \{H \mid H \cdot \xi = 0\}$

When H crosses wall sheaves in

$$0 \to I_Z(\xi + c_1/2) \to E \to I_W(-\xi + c_1/2) \to 0$$

are replaced by extensions the other way round.

Geometrically $M_H^X(c_1, c_2)$ is blown up along bundle w. fibre $\mathbf{P}(\mathrm{Ext}^1(I_W, I_Z(2\xi)))$ over

$$Hilb^n(X \sqcup X) = \bigcup_{l+m=n} Hilb^l(X) \times Hilb^m(X),$$

then blown down to $\mathbf{P}(\mathrm{Ext}^1(I_Z, I_W(-2\xi)))$.

With Ellingsrud computed wallcrossing in terms of integral over $Hilb^n(X \sqcup X)$

Note that the fixpoints on $M(2,n)$
correspond to $((Z_1,\phi_1),(Z_2,\phi_2))$
$(Z_1,Z_2)\in Hilb^n(\mathbf{A}^2\sqcup \mathbf{A}^2)$
Now use localization on $Hilb^n(X\sqcup X)$
to compute the integral.
Gives expression in Nekrasov part. fct.

Why does it work?
(1) The fixpoints on $Hilb^n(X\sqcup X)$ are

$$((Z_1,W_1),\ldots,(Z_e,W_e)),\ \text{with}$$

(Z_i,W_i) supported at p_i, the same fixpoints as in $\prod_{i=1}^e M(2,l_i)$.
(2) The tangent space of
$Hilb^n(X\sqcup X)$ at (Z_1,Z_2) is

$$\mathrm{Ext}^1(Z_1,Z_1)\oplus \mathrm{Ext}^1(Z_2,Z_2)$$

The tangent space of $M(2,n)$
at $((Z_1,\phi_1),(Z_2,\phi_2))$ is

$$\text{this}\oplus \mathrm{Ext}^1(Z_1,Z_2)*t^2\oplus \mathrm{Ext}^1(Z_2,Z_1)*t^{-2}$$

Summing over the p_i and multiplying by the perturbation gives the same weights as

$$\mathrm{Ext}^1(I_W, I_Z(2\xi)) \oplus \mathrm{Ext}^1(I_Z, I_W(-2\xi))$$

Integral on Hilbert scheme $Hilb^n(X \sqcup X)$ involves the Segre class of

$$\mathrm{Ext}^1(I_W, I_Z(2\xi)) \oplus \mathrm{Ext}^1(I_Z, I_W(-2\xi))$$

Taking the residue wrt a replaces $\frac{1}{c_{top}(E)}$ by $\frac{1}{c(E)} = s(E)$.

(6) Generalizations

1. equivariant version: proof gives equivariant D. inv. in terms of the Nekrasov partition function.

2. higher rank: still working, seems to work, wallcrossing in higher rank is much more difficult

3. K-theory version: compute χ of determinant bundles on moduli space in terms of K-theory version of partition function. Easy modification of the arguments.

4. With matter, integrate against c_{top} of some natural vector bundle on moduli space. Relate this to version of Nekrasov partition function with matter.