Hilbert schemes: local properties and Hilbert scheme of points

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Abstract. We will first introduce some elementary deformation theory and use this to study the local properties of the Hilbert and Quot scheme, like tangent space, dimension and nonsingularity.

In the second half we will study the Hilbert scheme of points which parametrizes finite subschemes of length n on a smooth quasiprojective variety. We will construct the Hilbert-Chow morphism from the Hilbert scheme of points to the symmetric power that sends a scheme to its support with multiplicities. We study some simple cases for small n and we show that the Hilbert scheme of points on a smooth curve is isomorphic to the symmetric power. We then study a natural stratification of Hilbert scheme of points and use it to sketch the computation of the Betti numbers. Finally we give a brief outline of the Heisenberg algebra action on the cohomology of Hilbert schemes of points.

Contents

CHAPTER 1

Elementary deformation theory

In this section we will give an elementary introduction to the beginnings of deformation theory. Deformation theory is also treated in lecture 5 of the lectures of Illusie [**Ill**] from a somewhat more advanced point of view.

Let k be a field. In this section we will assume all schemes to be schemes over k.

Let Y be a scheme and $p \in Y$ a point. We want to study the local properties of Y near p like tangent space, (non)singularity and dimension. It turns out that this can be done via studying morphisms $f : Spec(A) \to Y$, where A is a local Artin algebra and f maps the closed point to p.

In particular we are interested in the case of the Hilbert scheme. Let X be a projective scheme over a field k and let $Z \subset X$ be a closed subscheme (or more generally X quasiprojective and $Z \subset X$ closed and proper), and we want to study the local properties of $Hilb(X)$ near the point $[Z]$ given by Z. Then the above means that we have to study infinitesimal deformations of Z, i.e. flat families over $Spec(A)$ with the closed fibre equal to Z.

1. Infinitesimal study of schemes

Convention 1.1. For a local ring A we denote by m_A its maximal ideal. Let (Art/k) be the category of local Artinian k-algebras with residue field k. For the rest of this section let D be a covariant functor $(Art/k) \to (Sets)$ such that $D(k)$ is a single point.

The idea is that $D(k)$ is the object that we want to deform (e.g. $D(k)$ consists of a closed subscheme $Z \subset X$). For $A \in (Art/k)$ an element of $D(A)$ should be an infinitesimal deformation over $Spec(A)$ of the element of $D(k)$, (e.g. it is a subscheme $W \subset X \times Spec(A)$ which is flat over $Spec(A)$ and whose fibre over the closed point of $Spec(A)$ is Z).

To understand the functor D it is crucial to understand what happens for surjective morphisms $\sigma : B \to A$ in (Art/k).

(1) What deformations $F \in D(A)$ lift to elements in $D(B)$, i.e. what is the image of $D(\sigma) : D(B) \to D(A)$?
(2) How unique is such a lift, i.e. how big is a fibre of $D(\sigma)$ when it is nonempty?

Let $M := ker(\sigma) \subset B$. By factoring σ into a sequence of morphisms we can assume that at each step $M \cdot m_B = 0$.

DEFINITION 1.2. A *small extension*

$$0 \to M \to B \to A \to 0$$

is a surjection $B \to A$ in (Art/k) with kernel M such that $M \cdot m_B = 0$.

REMARK 1.3. Let $0 \to M \to B \to k \to 0$ be a small extension of k. Then $D(B) \to k$ is surjective (i.e. $D(B)$ is nonempty). In fact $D(B)$ contains a distinguished element 0. As B is a k-algebra we have a homomorphism $k \to B$, such that $k \to B \to k$ is the identity, thus 0 is the image of the unique element of $D(k)$ under $D(k) \to D(B)$.

Our main example of a functor $D : (Art/k) \to Sets$ is the one defined by a complete local ring.

DEFINITION 1.4. Let R be a complete local k-algebra with residue field k and with tangent space of finite dimension (embedding dimension). Then we define $h_R : (Art/k) \to (Sets)$ to be the functor $A \mapsto Hom_k(R, A)$.

REMARK 1.5. The case that we have mostly in mind is that $R = \widehat{\mathcal{O}}_{X,p}$, where X is a scheme and p is a point of X. Let A be a local Artin ring. Then $h_R(A)$ is the same as the set of morphisms $Spec(A) \to X$ sending the closed point to p. This is because such a morphism corresponds to a homomorphism $A \to \mathcal{O}_{X,p}$ and this has to factor through $\mathcal{O}_{X,p}/m_p^k$ for k sufficiently large, thus is factors through the completion.

REMARK 1.6. Let d be the embedding dimension of R. We put $S := k[[t_1, \ldots, t_d]]$ and let n be the maximal ideal of S. Then we can write $R = k[[x_1, \ldots, x_d]]/J$, where J is an ideal contained in n^2. Let $T = (m_R/m_R^2)^\vee = (n/n^2)^\vee$ be the tangent space of R.

We now want to see that the behaviour of h_R under small extensions can be nicely described in terms of T and and J/nJ. This also means that we can find out a lot about R by looking at h_R.

THEOREM 1.7. *For every small extension* $0 \to M \to B \to A \to 0$ *there is a natural exact sequence of sets*

$$0 \to T \otimes_k M \to h_R(B) \to h_R(A) \xrightarrow{ob} (J/nJ)^\vee \otimes_k M.$$

Furthermore this sequence is functorial in small extensions.

REMARK 1.8. (1) The exactness of the sequence at $h_R(A)$ means that an element $a \in h_R(A)$ lifts to B if and only if $ob(a) = 0$, i.e. $ob(a)$ is an obstruction to the lifting. The exactness at $h_R(B)$ means that, if a lifting exists, $T \otimes_k M$ acts transitively on the liftings. Finally that the sequence starts with 0 means that the liftings form an affine space under $T \otimes_k M$.

(2) Functoriality of the sequence means the following: For any morphism of small extensions, i.e. a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & M' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0, \end{array}$$

we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T \otimes_k M & \longrightarrow & h_R(B) & \longrightarrow & h_R(A) & \longrightarrow & (J/nJ)^\vee \otimes_k M \\ & & \downarrow{\scriptstyle id\otimes\varphi} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle id\otimes\varphi} \\ 0 & \longrightarrow & T \otimes_k M' & \longrightarrow & h_R(B') & \longrightarrow & h_R(A') & \longrightarrow & (J/nJ)^\vee \otimes_k M' \end{array}$$

PROOF. Let $0 \to M \to B \to A \to 0$ be a small extension. Assume given a k-algebra homomorphism $\varphi : R \to A$. This induces a homomorphism $\varphi' : S \to A$ (by composing with the natural map $S \to S/J = R$). As S is a power series ring, this lifts to a homomorphism $\widetilde{\varphi} : S \to B$. The lifts $\psi : R \to B$ of φ correspond bijectively to the lifts $\widetilde{\varphi} : S \to B$ with $\widetilde{\varphi}|_J = 0$.

Fix a lifting $\alpha : S \to B$ of φ and let $\beta : S \to B$ be another lifting. Then $h := (\beta - \alpha)$ is a linear map $S \to M$. As $M \cdot m_B = 0$, we see that $\alpha(f)x = \beta(f)x = f(0)x$ for any $f \in S$, $x \in M$, where $f(0)$ is the residue class of f in k. Let $f, g \in S$. Then

$$\begin{aligned} h(fg) &= \alpha(f)\alpha(g) - \beta(f)\alpha(g) + \beta(f)\alpha(g) - \beta(f)\beta(g) \\ &= h(f)g(0) + f(0)h(g). \end{aligned}$$

Thus h is a derivation from S to M, and we get that the set of liftings $\widetilde{\varphi} : S \to B$ of φ is an affine space over the space of derivations from S to M i.e. under $(n/n^2)^\vee \otimes_k M$.

As h is a derivation and $J \subset n^2$, it follows that $h|_J = 0$. Thus $\widetilde{\varphi}|_J$ does not depend on the lifting $\widetilde{\varphi}$, and by $m_B \cdot M = 0$ it has nJ in its kernel. Let $ob(\varphi) : J/nJ \to M$ be the induced map. Then the liftings $\widetilde{\varphi} : S \to B$ of φ give homomorphisms $R \to B$ if and only if $ob(\varphi) = 0$.

The functoriality of the exact sequence is an exercise. □

REMARK 1.9. We see that the tangent space T of R can be read of from h_R: We have a small extension $0 \to M \to k[\epsilon]/\epsilon^2 \to k \to 0$. By Remark 1.3 we see that $h_R(k[\epsilon]/\epsilon^2)$ has a distinguished element 0, thus by Theorem 1.7 it is a vector space isomorphic to T.

2. The case where X represents a functor

Assume a scheme X represents a moduli functor $F : (Schemes) \to (Sets)$. Let $[a] \in X$ corresponding to $a \in F(k)$. Then we can look at the corresponding deformation functor $F_a : (Art/k) \to (Sets)$, which maps A to the the set of all elements in $F(Spec(A))$, whose fibre over the closed point is a. Let $R := \widehat{\mathcal{O}}_{X,[a]}$. Then we get a canonical isomorphism of functors between F_a and h_R.

Let again $D : (Art/k) \to (Sets)$ be a covariant functor with $D(k)$ a point.

DEFINITION 2.1. D is called *pro-representable* if there exists a complete local algebra R of finite embedding dimension with residue field k such that D is isomorphic to h_R. We say then that R pro-represents D.

Assume D is obtained as above from a moduli functor F. Then the infinitesimal study of F often leads in a natural way to a tangent-obstruction theory as follows.

DEFINITION 2.2. Let $D : (Art/k) \to (Sets)$ be a pro-representable functor with $D(k)$ a point. We say D has a *tangent-obstruction theory* if there exist finite-dimensional vector spaces T_1 and T_2 such that the following holds.

(1) For all small extensions $0 \to M \to B \to A \to 0$ there is an exact sequence of sets
$$0 \to T_1 \otimes_k M \to D(B) \to D(A) \xrightarrow{ob} T_2 \otimes_k M.$$
(2) This exact sequence is functorial in small extensions.

We call T_1 the tangent space, T_2 the obstruction space and *ob* the obstruction map.

Thus T_1 and T_2 behave like T and $(J/nJ)^\vee$ in Theorem 1.7. So we can use T_1 and T_2 to study R.

THEOREM 2.3. *Let $D : (Art/k) \to (Sets)$ be represented by a complete local ring $R = S/J$ with $S = k[[x_1, \dots, x_d]]$, $J \subset n^2$ where n is the maximal ideal of S. Assume that D has a tangent obstruction theory.*

Then $T_1 \simeq (n/n^2)^\vee$ and there is a canonical injective linear map $(J/nJ)^\vee \to T_2$.

PROOF. By Remark 1.3 $D(k[\epsilon]/\epsilon^2) = h_R(k[\epsilon]/\epsilon^2)$ is a vector space isomorphic to T. The same argument shows that it is also a vector space isomorphic to T_1.

So we only need to construct the injective linear map $(J/nJ)^\vee \to T_2$.

Step 1. To make the idea of the proof clear, we first make the (wrong) assumption that T_2 is also an obstruction space for small extensions of complete local rings. In the second step we indicate the necessary changes.

Let $M := J/nJ$, $B := S/nJ$. Then we have a small extension $0 \to M \to B \to R \to 0$. The obstruction to lifting $id : R \to R$ to a map $\varphi : R \to B$ is a canonical element

$$o = ob(id) \in (J/nJ) \otimes_k T_2 = Hom((J/nJ)^\vee, T_2).$$

Assume o is not injective. Then there exists a sub-vectorspace $V \subset J/nJ$ of codimension 1, such that $\pi(o) = 0$ under the projection $\pi : M \to M/V$. Thus there exists a lifting $R \to B/V$ of $id : R \to R$. On the other hand in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & S & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & J/nJ = M & \longrightarrow & B & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & M/V & \longrightarrow & B/V & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

we see that the map $J \to M \to M/V$ is just the canonical quotient map. In particular it is nonzero. Thus by Theorem 1.7 no lift exists. Thus $(J/nJ)^\vee \to T_2$ is injective.

Step 2. We have instead to work with small extensions of Artin algebras. We can still use basically the same argument with some minor changes.

By the theorem of Artin Rees ([**AM**] Proposition 10.9), there exists an $i > 0$, such that $n^i \cap J \subset nJ$. Let $M = (J + n^i)/(nJ + n^i) = J/nJ$, $B := S/(nJ + n^i)$. Thus we have a small extension $0 \to M \to B \to R/n^i \to 0$. The obstruction to lifting the quotient map $p : R \to R/n^i$ to $\varphi : R \to B$ is $o := ob(p) \in Hom((J/nJ)^\vee, T_2)$. If o is not injective, there exists again a subvector space $V \subset M$ of codimension 1 such that there exists a lifting $R \to B/V$ of p. On the other hand the induced map $J \to M/V$ is again as above just the canonical quotient map, and thus nonzero. Thus by Theorem 1.7 no lift exists and o is injective. □

Now we want to show that the tangent-obstruction theory can be used to get information about the structure of R.

Recall that a scheme X is called a *local complete intersection*, at p if there is a smooth variety Y such that $\mathcal{I}_{Y/X}$ is generated at p by $codim_p(X, Y)$ elements. A local ring A is called complete intersection if $A = R/I$ for R a regular local ring and I an ideal generated by $codim(A, R)$ elements.

COROLLARY 2.4. *Under the assumptions of Theorem 2.3, let* $d := dim(T_1)$ *and* $r := dim(T_2)$.

Then $d \geq dim(R) \geq d - r$. *Furthermore if* $dim(R) = d - r$, *then* R *is a complete intersection. If* $r = 0$, *then* $R \simeq k[[x_1, \ldots, x_d]]$.

PROOF. By Theorem 2.3 we get $dim(J/nJ) \leq r$. Thus, by Nakayama's Lemma, J has at most r generators. This implies all the statements. □

REMARK 2.5. If R is the completion $\widehat{\mathcal{O}}_{V,p}$ of the local ring of a scheme at a point, then we get under the assumptions of the Theorem $d \geq dim_p(V) \geq d - r$. Furthermore, if $dim_p(V) = d - r$, then V is a local complete intersection at p and if $r = 0$, then V is nonsingular at p.

PROOF. We know that $dim(R) = dim(\mathcal{O}_{V,p}) = dim_p(V)$ ([**AM**] Corollory 11.19). Furthermore $\mathcal{O}_{V,p}$ is regular if and only if $\widehat{\mathcal{O}}_{V,p}$ is regular ([**AM**] Corollary 11.24). Finally assume that $\mathcal{O}_{V,p} = A/I$ where A is a regular local ring and I is an ideal. Then $R = \widehat{A}/\widehat{I}$ with $\widehat{I}$ the m_A-adic completion of I. Then $I/m_A I = \widehat{I}/m_A\widehat{I}$. By Nakayama's Lemma, if $\widehat{I}$ is generated by s elements, so is I. □

3. Non pro-representable functors

We have assumed that our functor $D : (Art/k) \to (Sets)$ is pro-representable, but many functors occurring in deformation theory are not (basically because the objects that we want to deform might have infinitesimal automorphisms, i.e. the tangent space to the space of automorphisms is nonzero). Also in this case D often has a tangent-obstruction theory, but we have to weaken the axioms a little bit, as one can show that our original axioms imply pro-representability.

DEFINITION 3.1. A functor $D : (Art/k) \to (Sets)$ with $D(k)$ a point is said to have a *tangent-obstruction theory* if there exists finite dimensional k-vector spaces T_1, T_2 such that the following holds.

(1) For all small extensions $0 \to M \to B \to A \to 0$ there exists an exact sequence of sets

$$T_1 \otimes_k M \to D(B) \to D(A) \xrightarrow{ob} T_2 \otimes_k M.$$

(2) In case $A = k$, the sequence becomes
$$0 \to T_1 \otimes_k M \to D(B) \to D(A) \xrightarrow{ob} T_2 \otimes_k M.$$
(3) The exact sequences in (1) and (2) are functorial in small extensions.

DEFINITION 3.2. Let $\alpha : F \to G$ be a morphism of functors: $(Art/k) \to (Sets)$. α is called *smooth* if for all small extensions $0 \to M \to B \to A \to 0$ the canonical map $F(B) \times_{G(A)} G(B)$ is surjective.

Under this condition F is called a *hull* for G, if $F(k[\epsilon]/\epsilon^2) \to G(k[\epsilon]/\epsilon^2)$ is bijective.

REMARK 3.3. (1) Note that by the above $G(k[\epsilon]/\epsilon^2)$ can viewed as the tangent space to the functor. Thus $\alpha : F \to G$ is a hull if and only if it is smooth and an isomorphism of tangent spaces.
(2) The formal criterion for smoothness says that a morphism $f : X \to Y$ of Noetherian schemes is smooth at a point $p \in X$ if and only if the corresponding morphism of functors $h_{\widehat{\mathcal{O}}_{X,p}} \to h_{\widehat{\mathcal{O}}_{y,f(p)}}$ is smooth.

The main result is now that functors $D : (Art/k) \to (Sets)$ that have a tangent-obstruction theory have a pro-representable hull.

THEOREM 3.4. *Let $D : (Art/k) \to (Sets)$ have a tangent-obstruction theory. Then there is a pro-representable functor h_R which is a hull for D.*

Furthermore $R = k[[x_1, \dots, x_d]]/J$, where $d = dim(T_1)$ and J has at most $dim(T_2)$ generators.

The precise story is as follows. Schlessinger [**Schl**] gave axioms for functors $D : (Art/k) \to (Sets)$ and showed that they are equivalent to the existence of a pro-representable hull. It is rather easy to prove that our axioms imply Schlessinger's axioms. Our axioms are stronger but usually easier to check. The last part of the theorem is an easy consequence, as it follows from the definitions that T_1 and T_2 give also a tangent-obstruction theory for the pro-representable hull.

The existence of a hull for D is the infinitesimal counterpart to the Kuranishi family or a miniversal deformation in complex-analytic deformation theory.

For pro-representable functors we used a slightly stronger version of the axioms of a tangent-obstruction theory. We can now see that these imply that the functor is pro-representable.

EXERCISE 3.5. Let $D : (Art/k) \to (Sets)$ be a functor with a tangent-obstruction theory where we replace parts (2) and (3) by part (2) of Definition 2.2

Using Theorem 3.4 show that D is pro-representable.

As said before, most functors appearing in deformation theory are not pro-representable. Roughly speaking pro-representablilty is the infinitesimal version of a fine moduli space and a pro-representable hull that of a chart of an Artin moduli stack. We give without proof some examples of non pro-representable functors with a tangent-obstruction theory. The statements are proved in a slightly different language in the lectures [**Ill**] of Illusie.

EXAMPLE 3.6. (1) Let X be a smooth variety. The functor of deformations of X is given by putting $Def_X(A)$ to be the set of isomorphism classes of flat schemes W over $Spec(A)$ together with an isomorphism of the closed fibre with X.

In this case $T_1 = H^1(X, T_X)$ and $T_2 = H^2(X, T_X)$ (see [**Ill**] Section 5.B).

(2) Let $\mathcal{E}$ be a coherent sheaf on a variety X. The functor of deformations of $\mathcal{E}$ is given by putting $Def_{\mathcal{E}}(A)$ to be the set of isomorphism classes of coherent sheaves $\mathcal{F}$ on $X \times Spec(A)$ flat over $Spec(A)$ together with an isomorphism of the restriction of $\mathcal{F}$ to the closed fibre with $\mathcal{E}$.

In this case $T_1 = \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ and $T_2 = \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ and in case $\mathcal{E}$ is locally free $T_1 = H^1(X, End(\mathcal{E}))$ and $T_2 = H^2(X, End(\mathcal{E}))$ (see [**Ill**] Section 5.A).

As one can see in these examples, usually the T_i are homology groups or Ext groups of some sheaves. It usually happens that if T_1 is the group with index i, then T_2 is the group with index $i+1$. In this case usually the group with index $i-1$ is the group of infinitesimal automorphisms, e.g. $H^0(X, T_X)$ is the tangent space to the automorphisms of X.

4. The local structure of the Hilbert scheme

We want to use the previous results on deformation theory to study the local structure of the Hilbert scheme at a point. The same arguments also give the local structure of the Quot scheme.

Let X be a scheme over k and let $Y \subset X$ be a closed subscheme. Let T be a scheme such that $T_{red} \simeq Spec(k)$ (hence $\mathcal{O}_T(T)$ is a local Artin ring with residue field k). Let $S \subset T$ be a closed subscheme with ideal M such that $M \cdot m_T = 0$. Thus we have a small extension $0 \to M \to \mathcal{O}_T \to \mathcal{O}_S \to 0$. Let $\pi : \mathcal{O}_{X\times T} \to \mathcal{O}_{X \times S}$ be the natural projection. Let $Y_S \subset X \times S$ be a closed subscheme, flat over S with fibre Y over the closed point.

Let $Y_T \subset X \times T$ be a closed subscheme such that $Y_T \cap (X \times S) = Y_S$. We first want to find a criterion for Y_T to be flat over T

PROPOSITION 4.1. *We have a commutative diagram of coherent sheaves on $X \times T$.*

$$(*)\quad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_Y \otimes_k M & \xrightarrow{\gamma} & \mathcal{O}_X \otimes_k M & \longrightarrow & \mathcal{O}_Y \otimes_k M \ \ (a) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Y_T} \ \ (b) & \longrightarrow & \mathcal{O}_{X\times T} & \longrightarrow & \mathcal{O}_{Y_T} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\pi} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Y_S} & \longrightarrow & \mathcal{O}_{X\times S} & \longrightarrow & \mathcal{O}_{Y_S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows. The columns are exact except possibly in (a) and (b). They are exact in (a) if and only if they are exact in (b).

PROOF. Rows (2) and (3) are exact by definition. Row (1) is $\otimes_k M$ applied to $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ and M is k-flat.

Column (2) is $\otimes_k \mathcal{O}_X$ applied to $0 \to M \to \mathcal{O}_T \to \mathcal{O}_S \to 0$ and $\mathcal{O}_X$ is k-flat. Column (3) is $\otimes_{\mathcal{O}_T} \mathcal{O}_{Y_T}$ applied to $0 \to M \to \mathcal{O}_T \to \mathcal{O}_S \to 0$ and thus it is right exact.

α and β are defined by easy diagram chasing. α is injective since γ and δ are. β is injective by diagram chasing. That the exactness of the columns at (a) and (b) is equivalent is also shown by diagram chasing. The diagram chasing is left as an easy exercise. □

Now we show our criterion for flatness.

PROPOSITION 4.2. *The following are equivalent.*

(1) *Y_T is flat over T.*
(2)
$$\mathcal{I}_Y \otimes_k M \subset \mathcal{I}_{Y_T} \subset \pi^{-1}(\mathcal{I}_{Y_S})$$
and the map $\overline{\pi} : \mathcal{I}_{Y_T}/(\mathcal{I}_Y \otimes_k M) \to \mathcal{I}_{Y_S}$ induced by π is an isomorphism.

LEMMA 4.3. *Let $\mathcal{F}$ be an $\mathcal{O}_T$ module. Then $\mathcal{F}$ is T-flat if and only if*

(1) *$\mathcal{F} \otimes_{\mathcal{O}_T} M \to \mathcal{F}$ is injective.*
(2) *$\mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{O}_S$ is $\mathcal{O}_S$-flat.*

PROOF. It is a standard fact that $\mathcal{F}$ is T-flat if and only if for all coherent ideals $\mathcal{A} \subset \mathcal{O}_T$ the map $\mathcal{F} \otimes_{\mathcal{O}_T} \to \mathcal{F}$ is injective ([**Hartshorne**] III,9.1A(a)).

"⇒" (1) is the special case $\mathcal{A} = M$. (2) follows because flatness is compatible with base extension. ([**Hartshorne**], III,9.1A(b)).

"⇐" Let $\mathcal{A} \subset \mathcal{O}_T$ be an ideal. We have an exact sequence

$$0 \to M \cap \mathcal{A} \to \mathcal{A} \xrightarrow{\varphi} \mathcal{O}_S$$

Let $\mathcal{B} \subset \mathcal{O}_S$ be the image of φ. We apply $\otimes_{\mathcal{O}_T} \mathcal{F}$ to get

$$\begin{tikzcd}
\mathcal{F} \otimes_{\mathcal{O}_T} (\mathcal{A} \cap M) \arrow[r] \arrow[dr, "\alpha"] & \mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{A} \arrow[r] \arrow[d, "\beta"] & (\mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{B} \arrow[r] \arrow[d, "\gamma"] & 0 \\
& \mathcal{F} \arrow[r] & \mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{O}_S &
\end{tikzcd}$$

γ is injective, because $\mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{O}_S$ is S-flat. Thus α injective implies β injective. Because $m_T \cdot M = 0$, we see that as an $\mathcal{O}_T$-module M is a vector space and so is $\mathcal{A} \cap M$. Therefore there is a sub-vector space C so that $M = (\mathcal{A} \cap M) \oplus C$. Hence $\mathcal{F} \otimes_{\mathcal{O}_T} (A \cap M) \to \mathcal{F} \otimes_{\mathcal{O}_T} M$ is injective and $\mathcal{F} \otimes M \to \mathcal{F}$ is injective. □

PROOF. (of Proposition 4.2). "(1)⇒(2) " By (1) there exists a diagram (*). Thus $\mathcal{I}_Y \otimes_k M \subset \mathcal{I}_{Y_T} \subset \pi^{-1}(\mathcal{I}_{Y_S})$. The diagram is exact in (a) because $\mathcal{O}_{Y_T}$ is T-flat, hence it is exact in (b) and (2) follows.

"(2)⇒(1)" (2) implies the diagram (4.1) is exact in (b) and hence in (a). Therefore $\mathcal{O}_{Y_T}$ is T-flat by Lemma 4.3. □

We want to use Proposition 4.2 to determine tangent and obstruction space for the Hilbert scheme in terms of extension classes.

REMARK 4.4. There is an exact sequence

$$0 \to \mathcal{O}_Y \otimes_k M \to \pi^{-1}(\mathcal{I}_{Y_S})/(\mathcal{I}_Y \otimes M) \xrightarrow{\overline{\pi}} \mathcal{I}_{Y_S} \to 0$$

of coherent sheaves on $X \times T$.

PROOF. This is by putting together standard exact sequences: By

$$0 \to \mathcal{O}_X \otimes_k M \to \mathcal{O}_{X \times T} \xrightarrow{\pi} \mathcal{O}_{X \times S} \to 0$$

we get

$$0 \to \mathcal{O}_X \otimes_k M \to \pi^{-1}(\mathcal{I}_{Y_S}) \to \mathcal{I}_{Y_S} \to 0,$$

and thus

$$0 \to \mathcal{O}_Y \otimes_k M = \frac{\mathcal{O}_X \otimes_k M}{\mathcal{I}_Y \otimes_k M} \to \frac{\pi^{-1}(\mathcal{I}_{Y_S})}{\mathcal{I}_Y \otimes_k M} \to \mathcal{I}_{Y_S} \to 0.$$

□

COROLLARY 4.5. *To give $\mathcal{I}_{Y_T}$ such that Y_T is T-flat and $Y_T \cap (X \times S) = Y_S$ is equivalent to finding a section of*

$$\overline{\pi} : \pi^{-1}(\mathcal{I}_{Y_S})/(\mathcal{I}_Y \otimes_k M) \to \mathcal{I}_{Y_S}.$$

PROOF. The existence of a section σ of $\overline{\pi}$ is equivalent to the existence of a coherent subsheaf $\mathcal{G} \subset \pi^{-1}(\mathcal{I}_{Y_S})/(\mathcal{I}_Y \otimes_k M)$, such that $\overline{\pi}|_G$ is an isomorphism. Given $\mathcal{I}_{Y_T}$ we can define $\mathcal{G} := \mathcal{I}_{Y_T}/(\mathcal{I}_Y \otimes_k M)$. Given $\mathcal{G}$, $\mathcal{I}_{Y_T}$ is its inverse image via $\pi^{-1}(\mathcal{I}_{Y_S}) \to \pi^{-1}(\mathcal{I}_{Y_S})/(\mathcal{I}_Y \otimes_k M)$. $\square$

We recall a standard lemma about extensions (see e.g. [**Hartshorne**] Ex. III.6.1).

LEMMA 4.6. *Let $0 \to \mathcal{E} \to \mathcal{F} \xrightarrow{\overline{\pi}} \mathcal{G} \to 0$ be an exact sequence of coherent sheaves. $\overline{\pi}$ has a section if and only if the extension class of the sequence in $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F})$ vanishes. In this case the sections of $\overline{\pi}$ are and affine space over $\mathrm{Hom}(\mathcal{G}, \mathcal{F})$.*

Finally we can put the results together.

THEOREM 4.7.
(1) *There is an obstruction to the existence of Y_T flat over T in $\mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{I}_Y, \mathcal{O}_Y) \otimes_k M$.*
(2) *If Y_T exists, the set of all possible Y_T is an affine space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{O}_Y) \otimes_k M$.*

PROOF. By the Corollary 4.5 and Lemma 4.6 this holds for the $\mathrm{Ext}^i_{\mathcal{O}_{X \times T}}(\mathcal{I}_{Y_S}, \mathcal{O}_Y \otimes_k M)$. Since $\mathcal{I}_{Y_S}$ is S-flat and $\mathcal{O}_Y \otimes_k M$ is annihilated by m_S there is a natural isomorphism $\mathrm{Ext}^i_{\mathcal{O}_{X \times T}}(\mathcal{I}_{Y_S}, \mathcal{O}_Y \otimes_k M) \simeq \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{O}_Y) \otimes_k M$. $\square$

It is also easy to see that the corresponding exact sequence of sets is functorial in small extensions. Applying Theorem 4.7 and Remark 2.5 now gives the local structure of the Hilbert scheme.

THEOREM 4.8. *Let X be quasiprojective and $[Z] \in Hilb(X)$. Let $d := dim(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z))$ and $r := dim(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z))$.*

Then $d \geq dim_{[Z]} Hilb(X) \geq d - r$. Furthermore if $dim_{[Z]} Hilb(X) = d$, then $Hilb(X)$ is nonsingular at $[Z]$. If $dim_{[Z]} Hilb(X) = d - r$, then $Hilb(X)$ is is a local complete intersecion at $[Z]$.

REMARK 4.9. In the proof we dealt with the Hilbert scheme instead of the Quot scheme. But we can deal in the same way with the Quot scheme. Let X be projective and let $\mathcal{G}$ be a coherent sheaf on X. Let $\mathcal{G} \to \mathcal{F}$ be a coherent quotient with kernel $\mathcal{K}$. Then we can replace in our arguments $\mathcal{O}_X$ by $\mathcal{G}$, $\mathcal{O}_Y$ by $\mathcal{F}$ and $\mathcal{I}_Y$ by $\mathcal{K}$ to get that

Theorem 4.8 applies to the local structure of $Quot(\mathcal{G}/X)$ at $[\mathcal{G} \to \mathcal{F}]$, if we put $d := dim(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}))$ and $r := dim(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{K}, \mathcal{F}))$.

A subscheme Z of a scheme X is called a *local complete intersection in* X, if at all points $p \in Z$, $\mathcal{I}_{Z/X}$ is locally generated by $codim_p(Z,X)$ elements. In the case of local complete intersections in X we can replace the Ext groups by cohomology groups. Let $Z \subset X$ is a local complete intersection in X. Let $N_{Z/X} = (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee$ be the normal bundle of Z in X, which is locally free.

COROLLARY 4.10. *Theorem 4.8 holds for* Z *with* $d = H^0(Z, N_{Z/X})$, $r = H^1(Z, N_{Z/X})$.

PROOF. By definition $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = \mathrm{Ext}^i_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z)$. As $\mathcal{I}_Z/\mathcal{I}_Z^2$ is locally free, we get $\mathrm{Ext}^i_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) = H^i(Z, (I_Z/\mathcal{I}_Z^2)^\vee)$. $\square$

Finally we want to study the local structure of the space of morphisms from a projective variety to another. Let X, Y be projective and assume Y is nonsingular. Let $Mor(X,Y) \subset Hilb(X \times Y)$ be the Hilbert scheme of morphisms. Let $f : X \to Y$ be a morphism.

COROLLARY 4.11. *Theorem 4.8 holds for* $Mor(X,Y)$ *at* f, *if we put* $d := dim(H^0(X, f^*T_Y))$ *and* $r := dim(H^1(X, f^*T_Y))$.

PROOF. Recall that $Mor(X,Y)$ is identified with an open subscheme of $Hilb(X \times Y)$ via sending a map to its graph. Let $\Delta_Y \subset Y \times Y$ be the diagonal. Then Δ is a local complete intersection in $Y \times Y$ because Δ and $Y \times Y$ are nonsingular. The ideal of $\Gamma_f := (f \times id_Y)^{-1}(\Delta_Y)$ in $X \times Y$ is locally generated by the preimages of the generators of $\mathcal{I}_{\Delta_Y}$. Thus Γ_f is a local complete intersection in $X \times Y$, and we see that $\mathcal{I}_{\Gamma_f}/\mathcal{I}^2_{\Gamma_f} = (f \times id_Y)^* N^\vee_{\Delta_Y/Y \times Y}$. Thus $N_{\Gamma_f/X \times Y} = (f \times id_Y)^* N_{\Delta_Y/Y \times Y}$. Under the isomorphism $\Gamma_f \to X$ (by projection to the first factor) $(f \times id_Y)^* N_{\Delta_Y/Y \times Y}$ becomes f^*T_Y. $\square$

CHAPTER 2

Hilbert schemes of points

1. Introduction

Let X be a quasiprojective scheme over a field k with an ample line bundle $\mathcal{O}(1)$. The Hilbert scheme $Hilb(X)$ of X parametrizes all closed, proper subschemes of X. We know that $Hilb(X)$ can be written as a disjoint union

$$Hilb(X) = \coprod_P Hilb^P(X),$$

of quasiprojective schemes where $Hilb^P(X)$ parametrizes the subschemes with Hilbert polynomial P [**Nit**] (i.e. it represents the contravariant functor sending a scheme T to the set of all closed subschemes $Z \subset X \times T$, which are flat over T and the Hilbert polynomial of the fibres is P)

We want to deal with the simplest case that P is the constant polynomial n. As the degree of the Hilbert polynomial is the dimension of the subscheme, we see that $Hilb^n(X)$ parametrizes 0-dimensional subschemes of length n of X. In other words this means that

$$dim H^0(Z, \mathcal{O}_Z) = \sum_{p \in supp(Z)} dim_k(\mathcal{O}_{Z,p}) = n.$$

$len(Z) := dim(H^0(Z, \mathcal{O}_Z)$ is the length of Z as module over itself. In future we will also write $X^{[n]}$ for $Hilb^n(X)$ The simplest example of an element in $X^{[n]}$ is just a set $\{p_1, \ldots, p_n\}$ of n distinct points on X. It is easy to see that these form an open subset of $X^{[n]}$. $X^{[n]}$ parametrizes sets of n not necessarily distinct points on X, with additional non-reduced structure when some of these points come together. Another space that parametrizes in a different way sets of n points on X is the symmetric power $X^{(n)}$ of X, the quotient of X^n by the action of the symmetric group $G(n)$ in n-letters by permuting the factors. $X^{(n)}$ parametrizes effective 0-cycles of degree n on X, i.e. formal sums $\sum_i n_i[p_i]$ with $p_i \in X$, $n_i \in \mathbb{Z}_{>0}$ and $\sum n_i = n$. There is an obvious set-theoretic map

$$\rho : X^{[n]} \to X^{(n)}, Z \mapsto \sum_p len(\mathcal{O}_{Z,p})[p],$$

Z is sent to its support with multiplicities. We shall see below that this is indeed a morphism of schemes.

In case X is a nonsingular curve, we will show that ρ is an isomorphism for all n, and if X is a nonsingular surface, ρ is a birational resolution of singularities, in particular $X^{[n]}$ is nonsingular. This is not true for X of dimension at least 3.

The Hilbert scheme of points on a surface has recently received a lot of interest both in mathematics and in theoretical physics. Partially this is because it is a canonical resultion of singularities of the symmetric power, but also because of its relations to moduli of vector bundles and to infinite dimensional Lie algebras. See [**G2**] for an overview of some of these relations. A nice and very readable introduction to Hilbert schemes of points and some of the newer results is [**Nakajima**].

2. The symmetric power and the Hilbert-Chow morphism

As said in the introduction, the Hilbert scheme $X^{[n]}$ of n points on X is closely related to the symmetric power $X^{(n)} = X^n/G(n)$. We first need to know that the symmetric power exists as an algebraic variety.

Definition 2.1. Let X be a quasiprojective variety over k, and let X be a group acting (by automorphisms) on X. A variety Y together with a surjective morphism $\pi : X \to Y$ is called a *quotient* of X by G if and only if the following holds.

(1) The fibres of π are the orbits of G.
(2) Any G-invariant morphism $\varphi : X \to Z$ to a scheme Z factors through π.

It follows that the quotient is unique up to isomorphism, if it exists. We denote it by X/G.

In general it is a difficult question whether a quotient exists. However if G is finite and G quasiprojective, the problem is easy.

Theorem 2.2. *Let X be a quasiprojective variety with an action of a finite group G. Then the quotient X/G exists as a quasiprojective variety.*

Proof. (Sketch) First assume that X is affine. Let $k[X]$ be the affine coordinate ring. Condition (2) in the definition of the quotient implies that $k[X/G]$ should be the ring of invariants $k[X]^G \subset k[X]$. It is easy to see that $k[X]^G$ is a finitely generated k-algebra, so we define $X/G := Spec(k[X]^G)$, which is an affine variety. Let $\pi : X \to X/G$ be the morphism induced by the inclusion $k[X]^G \subset k[X]$. It is not difficult to show that π is surjective and the fibres are the G-orbits.

If X is not affine, it has an open cover (U_i) by affines, and as G is finite, we can choose the affine sets in such a way that each orbit is contained in one of the U_i. Replacing the U_i with $W_i = \bigcap_{g\in G} g(U_i)$ we get an open cover of X by G-invariant affine open subsets (as X is a variety the intersection of affine open sets is affine). Then it is not difficult to show that the W_i/G glue to give the quotient X/G. $\square$

In particular if X is a quasiprojective variety, then the symmetric power $X^{(n)} := X^n/G(n)$ (where the symmetric group $G(n)$ acts by permutation of the factors) exists as a quasiprojective variety.

Example 2.3.
(1) By the fundamental theorem on symmetric functions $k[x_1, \dots, x_n]^{G(n)} = k[s_1, \dots, s_n]$ where the s_i are the elementary symmetric functions in the x_i. Thus $(\mathbb{A}^1)^{(n)} = \mathbb{A}^n$. Similarly one shows $(\mathbb{P}^1)^{(n)} = \mathbb{P}^n$.
(2) Show as exercise that $(\mathbb{A}^2)^{(2)} \simeq \mathbb{A}^2 \times Spec(k[u,v,w]/(uw-v^2))$. Thus $(\mathbb{A}^2)^{(2)}$ is singular.

By definition the points of the symmetric power $X^{(n)}$ are the orbits of the n-tuples of points on X under permutation, i.e. they are the effective 0-cycles $\sum n_i[x_i]$ with $x_i \in X$, $n_i > 0$ and $\sum n_i = n$. This allows to give a different description of $X^{(n)}$ as a Chow variety of 0-cycles.

Let $X \subset \mathbb{P}^d$ be locally closed. We see that $X^{(n)}$ is a locally closed subvariety of $(\mathbb{P}^d)^{(n)}$. Let $\check{\mathbb{P}}^d$ by the dual projective space of hyperplanes in $\mathbb{P}^d$. Let $Div^n(\check{\mathbb{P}}^d) \simeq \mathbb{P}^{\binom{n+d}{d}-1}$ be the space of effective divisors of degree n on $\check{\mathbb{P}}^d$. For any $p \in \mathbb{P}^d$ let

$$H_p := \{ l \in \check{\mathbb{P}}^d \mid p \in l \}.$$

Then $p \mapsto H_p$ defines an isomorphism $\mathbb{P}^d \simeq Div^1(\check{\mathbb{P}}^d)$. For $(x_1, \dots, x_n) \in (\mathbb{P}^d)^n$ let

$$ch(x_1, \dots, x_n) := \sum_i H_{x_i} \in Div^n(\check{\mathbb{P}}^d).$$

Then $ch : (\mathbb{P}^d)^n \to Div^n(\check{\mathbb{P}}^d)$ is a $G(n)$-invariant morphism, and thus gives a morphism $ch : (\mathbb{P}^d)^{(n)} \to Div^n(\check{\mathbb{P}}^d)$. One checks that the image is closed (exercise) and that ch is an isomorphism onto its image. In particular we can also identify $X^{(n)}$ with its image in $Div^n(\check{\mathbb{P}}^d)$.

Now we want to define the Hilbert-Chow morphism $\rho : X^{[n]} \to X^{(n)}$ as a morphism $X^{[n]} \to Div^n(\check{\mathbb{P}}^d)$. $Div^n(\check{\mathbb{P}}^d)$ represents the contravariant functor associating to each scheme T the effective relative Cartier divisors $D \subset \check{\mathbb{P}}^d \times T$. Relative means equivalently, either that D is flat over T or that the restriction to each fibre over a point $t \in T$ is a Cartier divisor. Thus in order to construct ρ we need a way to obtain effective Cartier divisors.

For this we first review a construction of Mumford which associates under suitable conditions to a coherent sheaf $\mathcal{F}$ on a scheme Y an effective Cartier divisor $div(\mathcal{F})$ on Y. We will not carry out the construction in full detail or in full generality but only give a sketch in the case of interest to us. The general construction can be found in [**GIT**] Chap. 5 Sec. 3.

We first deal with a special case. Let X be a smooth connected variety over k and let $\mathcal{F}$ be a coherent sheaf on X with $supp(\mathcal{F}) \neq X$. For an irreducible hypersurface $V \subset X$ let $[V]$ be its generic point, and put $R := \mathcal{O}_{X,[V]}$. Let M be the stalk of $\mathcal{F}$ at $[V]$. Then R is a discrete valuation ring, and by [**Hartshorne**] III.6.11A, III.6.12A, M has homological dimension 1, i.e. there exists a free resolution

$$0 \to R^n \xrightarrow{\varphi} R^n \to M \to 0.$$

Note that the two modules on the left have the same rank, because $supp(\mathcal{F}) \neq X$. Let $m_V \in \mathbb{Z}_{\geq 0}$ be the valuation of $\det(\varphi) \in R$. (More geometrically we can describe this as follows: There is an open subset $U \subset X$ whose intersection with V is open, on which we have a resolution

$$0 \to \mathcal{O}^n \xrightarrow{\varphi} \mathcal{O}^n \to \mathcal{F} \to 0.$$

Let m_V be the order of vanishing of $det(\varphi)$ on an open subset of V.)

We need to see that m_V is independent of the choice of the resolution. Fixing the map $R^n \xrightarrow{\psi} M \to 0$ the possible $\varphi' : R^n \to R^n$ are obtained from φ by composing with an automorphism α of R^n. Thus $det(\varphi') = det(\alpha)det(\varphi)$ and $det(\alpha) \in R^*$. Thus the valuation does not change. The map $\psi : R^n \to M$ corresponds to the choice of a set of generators of M as R-module. We obtain any other choice by successively adding and removing generators. Thus let $\psi : R^n \to M$ be given by $m_1, \ldots, m_n \in M$ and $\psi' : R^{n+1} \to M$ given by $m_1, \ldots, m_n, x \in M$. Then in M we have a relation

$$\sum_{i=1}^{n} a_i m_i + x = 0, \qquad a_i \in R,$$

and given a resolution $0 \to R^n \xrightarrow{\varphi} R^n \xrightarrow{\psi} M \to 0$ we get a resolution

$$0 \to R^{n+1} \xrightarrow{\varphi'} R^{n+1} \xrightarrow{\psi'} M \to 0, \qquad \varphi' = \begin{pmatrix} \varphi & 0 \\ a_1 \dots a_n & 1 \end{pmatrix}$$

and $det(\varphi') = det(\varphi)$. Thus m_V is well-defined.

Definition 2.4. Let $div(\mathcal{F}) := \sum_V m_V V$. This is by definition an effective Cartier divisor on X. The sum is finite because m_V can only be nonzero if $V \subset supp(\mathcal{F})$.

We need this divisor $div(\mathcal{F})$ in a relative situation.

Theorem 2.5. *Let X be a smooth irreducible variety. Let S be a scheme and let $\mathcal{F}$ be a coherent sheaf on $X \times S$, flat over S. Assume that $supp(\mathcal{F}_s) \neq X$ for all $s \in S$. Then there exists an effective Cartier divisor $div(\mathcal{F})$ on $X \times S$ such that*

(1) *The formation of $div(\mathcal{F})$ commutes with base change.*
(2) *If S is a point, then $div(\mathcal{F})$ is the same as in Definition 2.4.*

Proof. (Sketch). First note the following: if $0 \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0 \to 0$ is a short exact sequence of locally free sheaves with $rk(\mathcal{E}_i) = r_i$, then there exists a canonical isomorphism

$$\bigwedge^{r_2} \mathcal{E}_2 \otimes \bigwedge^{r_0} \mathcal{E}_0 \simeq \bigwedge^{r_1} \mathcal{E}_1.$$

(Locally on an open set choose a basis $f_1, \dots, f_{r_2}, g_1, \dots, g_{r_0}$ of $\mathcal{E}_1$ such that the f_i are the image of a basis of $\mathcal{E}_2$ and the g_i map to a basis of $\mathcal{E}_0$ (both denoted by the same letters). Then the isomorphism is given by

$$(f_1 \wedge \dots \wedge f_{r_2}) \otimes (g_1 \wedge \dots \wedge g_{r_0}) \mapsto f_1 \wedge \dots \wedge f_{r_2} \wedge g_1 \wedge \dots \wedge g_{r_0}.$$

This is independent of the choice of the g_i and thus glues to a global isomorphism.)

If $0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \dots \to \mathcal{E}_0 \to 0$ is an exact sequence of locally free sheaves we can split it up into two exact sequences of locally free sheaves

$$0 \to \widetilde{\mathcal{E}} \to \mathcal{E}_1 \to \mathcal{E}_0 \to 0, \quad 0 \to \mathcal{E}_n \to \dots \to \mathcal{E}_2 \to \widetilde{\mathcal{E}} \to 0$$

and obtain by induction a canonical isomorphism $\bigotimes_{i=0}^n \left(\bigwedge^{r_i} \mathcal{E}_i\right)^{(-1)^i} \simeq \mathcal{O}_X$.

Now let

$$0 \to \mathcal{E}_n \to \dots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

be a locally free resolution of $\mathcal{F}$ on $X \times S$ (this exists because of flatness). Let $U \subset X \times S$ be an open subset such that the $\mathcal{E}_i$ are free on

U. Let $V := U \setminus supp(\mathcal{F})$. Then by the above we have a canonical isomorphism

$$\mathcal{O}_{X\times S} \simeq \bigotimes_{i=0}^{n} \big(\bigwedge^{r_i} \mathcal{E}_i\big)^{(-1)^i} \qquad \text{on } V.$$

On the other hand, as the $\mathcal{E}_i$ are free, there is an isomorphism

$$\bigotimes_{i=0}^{n} \big(\bigwedge^{r_i} \mathcal{E}_i\big)^{(-1)^i} \simeq \mathcal{O}_{X\times S} \qquad \text{on } U,$$

unique up to a unit. The composition defines a nonzero section $f \in \Gamma(V, \mathcal{O}_V)$, giving a rational function (f) on U.

Let $[V] \in X \times S$ be a point of depth 1. Let $0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0$ be a free resolution in a neighbourhood of $[V]$. Let $\widetilde{\mathcal{E}}$ be the kernel of $\mathcal{E}_0 \to \mathcal{F} \to 0$. Then $\widetilde{\mathcal{E}}$ is free in a smaller neighbourhood U_0 of $[V]$. Thus we have exact sequences

$$0 \to \widetilde{\mathcal{E}} \to \mathcal{E}_0 \to \mathcal{F} \to 0, \qquad 0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_1 \to \widetilde{\mathcal{E}} \to 0.$$

Therefore on U_0 we have a canonical isomorphism

$$\bigwedge^{r_0} \widetilde{\mathcal{E}} \simeq \bigotimes_{i=1}^{n} \big(\bigwedge^{r_i} \mathcal{E}_i\big)^{(-1)^{i-1}},$$

and $div(\mathcal{F})$ is on U_0 defined by $0 \to \widetilde{\mathcal{E}} \to \mathcal{E}_0 \to \mathcal{F} \to 0$.

As two Cartier divisors are equal if they coincide at points of depth 1, this shows first that (f) is independent of the choice of the resolution and thus glues to give an effective Cartier divisor $Div(\mathcal{F})$ on $X \times T$.

Secondly, as the points of depth 1 on a smooth variety are precisely the generic points of prime divisors, it also shows that in case S is a point we get the same definition as in Definition 2.4.

Let $h : T \to S$ be a morphism. For a sheaf $\mathcal{E}$ on $X \times S$ we denote by $\mathcal{E}_T$ its pullback via $id_X \times h$. Using flatness of $\mathcal{F}$, one checks that if $0 \to \mathcal{E}_n \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0$ is a resolution of $\mathcal{F}$ on $X \times S$, then

$$0 \to (\mathcal{E}_n)_T \to \ldots \to (\mathcal{E}_0)_T \to (\mathcal{F})_T \to 0$$

is a resolution of $(\mathcal{F})_T$. It follows that the pullback of $div(\mathcal{F})$ is $div(\mathcal{F}_T)$, thus $div(\mathcal{F})$ is compatible with base change. □

Finally we can construct the Hilbert-Chow morphism. Let $\mathbb{H} \subset \mathbb{P}^d \times \check{\mathbb{P}}^d$ be the incidence correspondence. $\mathbb{H}$ is a fibre bundle over $\mathbb{P}^d$ with fibre $\mathbb{P}^{d-1}$. Let S be a scheme, and let $Z \subset \mathbb{P}^d \times S$ be a closed subscheme, flat of degree n over S. We denote by $p, \check{p}$ the projections to $\mathbb{P}^d$ and $\check{\mathbb{P}}^d$ respectively. Let $p_S := p \times id_S$, $\check{p}_S := \check{p} \times id_S$. Let

$Z^* := p_{\check{S}}^{-1}(Z) \subset \mathbb{H} \times S$. Let $\mathcal{F} := (\check{p}_S)_*(\mathcal{O}_{Z^*})$. Then $\mathcal{F}$ is a coherent sheaf on $\check{\mathbb{P}}^d \times S$, flat over S. Let Z_s be the fibre of Z over $s \in S$. Then

$$supp(\mathcal{F}_s) = \{ l \in \check{\mathbb{P}}^d \mid l \cap Z_s \neq 0 \},$$

in particular $supp(\mathcal{F}_s) \neq \check{\mathbb{P}}^d$. Thus $div(\mathcal{F})$ is a relative Cartier divisor on $\check{\mathbb{P}}^d \times S$. Thus we have constructed a morphism $\rho : X^{[n]} \to Div^n(\check{\mathbb{P}}^d)$. Finally we can check from the definitions that $div(\mathcal{F}_s) = \sum_{p \in supp(Z_s)} len(\mathcal{O}_{Z,p}) H_p$. Thus we see that the support of the image of $X^{[n]}$ is $X^{(n)}$, so if we give $X^{[n]}$ the reduced structure, the morphism will factor through $X^{(n)}$. Thus we have shown the following theorem.

THEOREM 2.6. *Let X be a smooth quasiprojective variety. There is a surjective morphism $\rho : X^{[n]}_{red} \to X^{(n)}$, given on the level of points by $Z \mapsto \sum_{p \in supp(Z)} len(\mathcal{O}_{Z,p})[p]$.*

3. Irreducibility and nonsingularity

We will show that, if X is a nonsingular quasiprojective curve or surface over k, then $X^{[n]}$ is irreducible and nonsingular.

LEMMA 3.1. *Let X be a connected variety over k, then $X^{[n]}$ is connected for all $n \geq 0$.*

PROOF. First we recall that the Quot scheme allows us to define the projectivization of any coherent sheaf on a X.

For a coherent sheaf $\mathcal{F}$ on X let $\mathbb{P}(\mathcal{F}) := Quot^1(\mathcal{F})$, thus $\mathbb{P}(\mathcal{F})$ parametrizes 1-dimensional quotients of the fibres of $\mathcal{F}$. $\mathbb{P}(\mathcal{F})$ is a quasiprojective scheme with a morphism to X, and the fibre of $\mathbb{P}(\mathcal{F})$ over $x \in X$ is

$$\mathbb{P}(\{\lambda : \mathcal{F}(x) \to k(x) \text{ surjection}\}) \simeq \mathbb{P}(F(x)).$$

Now we want to show the claim by induction on n. $X^{[0]}$ is one point corresponding to the empty set. Assume we have shown that $X^{[n]}$ is connected. Then $X \times X^{[n]}$ is connected. Let $Z_n(X) \subset X \times X^{[n]}$ be the universal family with ideal sheaf $\mathcal{I}_{Z_n(X)}$. Let $\mathbb{P} := \mathbb{P}(\mathcal{I}_{Z_n(X)})$, with the projection $\pi : \mathbb{P} \to X \times X^{[n]}$. The fibre of $\mathbb{P}$ over (x, Z) is a projective space and thus connected. Thus $\mathbb{P}$ is connected.

On $X \times \mathbb{P}$ we have a universal exact sequence

$$0 \to \mathcal{I} \to (\mathcal{I}_{\pi^{-1}(Z_n(X))}) \to Q \to 0.$$

In particular $\mathcal{I}$ is an ideal sheaf on $X \times \mathbb{P}$ defining a subscheme $\mathcal{Z} \subset X \times \mathbb{P}$. The above exact sequence gives rise to an exact sequence

$$0 \to Q \to \mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\pi^{-1} Z_n(X)} \to 0.$$

As $\mathcal{O}_{\pi^{-1}Z_n(X)}$ is flat of degree n over $\mathbb{P}$ and Q is flat of degree 1, we see that $\mathcal{Z}$ is flat of degree $(n+1)$ over $\mathbb{P}$. Thus we have a morphism $\psi : \mathbb{P} \to X^{[n+1]}$, which on points is given by sending $(\lambda : \mathcal{I}_Z \to k(x))$ to the subscheme of X with ideal $ker(\lambda)$. We want to see that ψ is surjective. Let $W \in X^{[n+1]}$. Let $p \in supp(W)$. Choose $f \in \mathcal{O}_W$ an element in the kernel of the multiplication by the maximal ideal m at p. Let $Z \subset W$ be the subscheme with ideal (f). Then $Z \in X^{[n]}$. Let $f = g_0, g_1, \dots, g_k$ be a basis of $\mathcal{I}_Z/m\mathcal{I}_Z$ and define $\lambda : \mathcal{I}_Z \to k(p)$ by $\sum a_i g_i \mapsto a_0$. Then $\mathcal{I}_W = ker(\lambda)$. Thus ψ is surjective, and thus $X^{[n+1]}$ is connected. $\square$

Let X be a nonsingular quasiprojective variety of dimension d. Let $X_0^n \subset X^n$ be the dense open set of $(p_1, \dots, p_n)$ with the p_i distinct. Let $X_0^{(n)}$ be its image in $X^{(n)}$, which parametrizes effective zero cycles $\sum_i [p_i]$ with the p_i distinct. This is also open and dense. As $G(n)$ acts freely on $(X^n)_0$ we see that $X_0^{(n)}$ is nonsingular of dimension nd. Let $X_0^{[n]}$ be the preimage in $X^{[n]}$. One checks that at any point of $X_0^{[n]}$ the dimension of the tangent space is nd and that $\rho|_{X_0^{[n]}}$ is an isomorphism. Thus $X^{[n]}$ contains a nonsingular open subset which is isomorphic to an open subset of $X^{(n)}$. In the case that X is a curve or a surface one can use this to show that $X^{[n]}$ is nonsingular and irreducible.

THEOREM 3.2. (1) *Let C be an irreducible nonsingular quasiprojective curve and $n \geq 0$. Then $C^{[n]}$ is nonsingular and irreducible of dimension n.*
(2) *(Fogarty* [**F**]*) Let S be an irreducible nonsingular quasiprojective surface and $n \geq 0$. Then $S^{[n]}$ is nonsingular and irreducible of dimension $2n$.*

PROOF. Let $X = C$ or $X = S$ and let $d = dim(X)$. As $X^{[n]}$ is connected and contains a nonsingular open subset of dimension nd, it is enough to show that the dimension of the tangent space $T_{[Z]}X^{[n]}$ is nd for all $[Z] \in X^{[n]}$. This will first show the nonsingularity of $X^{[n]}$ at any point in the closure $\overline{X_0^{[n]}}$. If $X^{[n]}$ was reducible, then by connectedness there would be another irreducible component intersecting $\overline{X_0^{[n]}}$, and the intersection point would be a singular point of $X^{[n]}$.

We know $T_{[Z]}X^{[n]} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$. Applying $\mathrm{Hom}(\bullet, \mathcal{O}_Z)$ to $0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ we obtain

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Z) \to &\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) \\ &\to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Z).\end{aligned}$$

The first map is an isomorphism $k^n \to k^n$. Thus $\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \subset \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$, and it is enough to show that $ext^1(\mathcal{O}_Z, \mathcal{O}_Z) \leq nd$.

In the case of a curve C we have $Hom(\mathcal{O}_Z, \mathcal{O}_Z) = H^0(\mathcal{O}_Z) = k^n$, and by Serre duality $Ext^1(\mathcal{O}_Z, \mathcal{O}_Z) = H^0(\mathcal{O}_Z \otimes K_C)^\vee = k^n$.

In the case of a surface S we have $Hom(\mathcal{O}_Z, \mathcal{O}_Z) = H^0(\mathcal{O}_Z) = k^n$ and by Serre duality $Ext^2(\mathcal{O}_Z, \mathcal{O}_Z) = H^0(\mathcal{O}_Z \otimes K_S)^\vee = k^n$. Thus it suffices to show that

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum_{i=0}^{2} ext^i(\mathcal{O}_Z, \mathcal{O}_Z) = 0.$$

Let $0 \to \mathcal{E}_l \to \ldots \to \mathcal{E}_0 \to \mathcal{O}_Z \to 0$ be a locally free resolution of $\mathcal{O}_Z$ on S. Then $\sum_i (-1)^i rk(\mathcal{E}_i) = 0$ and

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = \sum_{i=0}^{n} (-1)^i \chi(\mathcal{E}_i, \mathcal{O}_Z) = \sum_{i=0}^{n} (-1)^i n \cdot rk(\mathcal{E}_i) = 0.$$

□

REMARK 3.3. Note that in this proof we do not show that the obstruction space $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$ vanishes; in fact it usually will not.

REMARK 3.4. Let X be a nonsingular variety. Then $X^{[n]}$ is nonsingular for $n \leq 3$.

PROOF. Let $d = dim(X)$. It is enough to show that $hom_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) \leq dn$ for all $[Z] \in X^{[n]}$. Obviously

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = \bigoplus_{p \in supp(Z)} \mathrm{Hom}_{\mathcal{O}_{X,p}}(\mathcal{I}_{Z,p}, \mathcal{O}_{Z,p}).$$

Thus we can reduce to the case that $supp(Z)$ is a point p. Let m be the maximal ideal at p. Is is easy to show that there are local parameters $x_1, \ldots, x_d$ at p such that $\mathcal{O}_{Z,p}$ is of the form.

$$\begin{cases} k[x_1, \ldots, x_d]/m & n = 1, \\ k[x_1, \ldots, x_d]/(m^2 + (x_2, \ldots, x_d)) & n = 2, \\ & \\ k[x_1, \ldots, x_d]/(m^3 + (x_2, \ldots, x_d)) & n = 3. \\ k[x_1, \ldots, x_d]/(m^2 + (x_3, \ldots, x_d)) & \end{cases}$$

In all cases one easily checks that $\mathrm{hom}_{\mathcal{O}_{X,p}}(\mathcal{I}_{Z,p}, \mathcal{O}_{Z,p}) = d\, len(\mathcal{O}_{Z,p})$. □

REMARK 3.5. Let X be nonsingular of dimension 3. Then $X^{[4]}$ is singular.

Proof. Let $[Z] \in X^{[4]}$ be the point $\mathcal{O}_Z = \mathcal{O}_p/m^2$. Then

$$Hom_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = Hom_k(m^2/m^3, m/m^2) = k^{18}.$$

So the dimension is bigger then $dn = 12$. □

An example of Iarrobino shows that if X is nonsingular of dimension $d \geq 3$ and n is sufficiently large, then $X^{[n]}$ is reducible.

4. Examples of Hilbert schemes

Let X be a nonsingular projective variety of dimension d. We give some examples of $X^{[n]}$ for small values of n.

Example 4.1.
(1) $X^{[0]}$ is one reduced point, corresponding to the empty subscheme of X.
(2) Subschemes of length 1 of X are just points of X and $X^{[1]} = X$. The universal family is just the diagonal $\Delta \subset X \times X$.
(3) Points in $X^{[2]}$ are either a set $\{p_1, p_2\}$ of distinct points on X or a subscheme Z of length 2 concentrated in one point p. Let m be the ideal of p. Then $m \supset \mathcal{I}_Z \supset m^2$. Thus $\mathcal{I}_Z$ is given by a one-codimensional subspace of m/m^2, i.e. by a point in $\mathbf{P}(T_{X,p}) = \mathbb{P}^{n-1}$. In other words a point in $X^{[2]}$ is either a set of two points in X or a point p and a tangent direction at p.

    This allows us to describe $X^{[2]}$ globally as follows: Let $\widehat{X^2}$ be the blowup of $X \times X$ along the diagonal. Let E be the exceptional divisor. The action of $G(2)$ on X^2 extends to $\widehat{X^2}$. Let Y be the quotient. E is the fixlocus of the nontrivial element of $G(2)$. As this is a divisor, we see that Y is nonsingular. It is easy to see that $\pi : \widehat{X^2} \to Y$ is flat of degree 2. This gives a morphism $Y \to X^{[2]}$, which is birational and an bijective. Thus it is an isomorphism.

For general n we can say something about $X^{[n]}$ if X has dimension 1 or 2. In the case of a nonsingular curve we see that the Hilbert scheme of points is just the symmetric power.

Proposition 4.2. *Let C be a nonsingular quasiprojective curve. Then $\rho : C^{[n]} \to C^{(n)}$ is an isomorphism.*

Proof. As the local ring of C at a point p is a discrete valuation ring, all ideals in $\mathcal{O}_{C,p}$ are powers of the maximal ideal m_p. Thus for all $[Z] \in C^{[n]}$ we have

$$\mathcal{O}_Z = \bigoplus_i \mathcal{O}_{C,p_i}/m_{p_i}^{n_i}, \qquad \sum_i n_i = n.$$

Then the ρ sends Z to $\sum_i n_i[p_i]$. Thus ρ is bijective. As it is also birational, it is an isomorphism by Zariskis Main Theorem.

Alternatively one can see that

$$\pi : C \times C^{(n-1)} \to C^{(n)}, (p, \sum n_i[p_i]) \mapsto \sum n_i[p_i] + [p].$$

is flat of degree n over $C^{(n)}$, defining an inverse to ρ. □

THEOREM 4.3. *(Fogarty* [**F**]*) Let S be a nonsingular quasiprojective surface. Then $\rho : S^{[n]} \to S^{(n)}$ is a resolution of singularities.*

PROOF. $S^{[n]}$ is nonsingular and irreducible, and ρ is an isomorphism over the open subset $S_0^{(n)}$. Thus it is a resolution of singularities. □

REMARK 4.4. Example 4.1 can easily be generalized to show that $S^{(n)}$ is singular for all $n \geq 2$. Thus ρ is not an isomorphism.

One of the reasons for the interest in the Hilbert scheme of points on a surface is that it gives a canonical resolution of the singularities of the symmetric power.

5. A stratification of the Hilbert scheme

For the rest of these lectures let S be a smooth projective surface over the complex numbers. We want to study a natural stratification of $S^{(n)}$ and $S^{[n]}$.

DEFINITION 5.1. For any partition $\nu = (n_1, \ldots, n_r)$ of n (i.e. $n_1 \geq n_2 \geq \ldots \geq n_r > 0$ and $\sum n_i = n$), we define a locally closed subset

$$S_\nu^{(n)} := \big\{ \sum n_i[x_i] \in S^{(n)} \;\big|\; x_i \in S \text{ distinct points} \big\}$$

of $S^{(n)}$. We have thus a stratification

$$S^{(n)} = \coprod_\nu S_\nu^{(n)}$$

into locally closed strata. Putting $S_\nu^{[n]} := \rho^{-1}(S_\nu^{(n)})$ we obtain a stratification of $S^{[n]}$ into locally closed strata.

Now we want to study these stratifications. First we note that the strata $S_\nu^{(n)}$ are nonsingular. Let $\nu = (n_1, \ldots, n_r)$. Write $\nu = (1^{\alpha_1}, 2^{\alpha_2}, \ldots, s^{\alpha_s})$ where α_i is the number of times i occurs in $(n_1, \ldots, n_r)$. Let $(S^{\alpha_1} \times \ldots \times S^{\alpha_r})_*$ be the open subset where all components $p_i \in S$ are distinct. Then $S_\nu^{(n)}$ is the quotient of $(S^{\alpha_1} \times \ldots \times S^{\alpha_r})_*$ by the action of $G(\alpha_1) \times \ldots \times G(\alpha_s)$, where each $G(\alpha_i)$ permutes the factors of S^{α_i}. As this action is free, $S_\nu^{(n)}$ is nonsingular. We also see that $S_\nu^{(n)}$ is an open subset of $S^{(\alpha_1)} \times \ldots \times S^{(\alpha_r)}$. One can show that $S^{(\alpha_1)} \times \ldots \times S^{(\alpha_r)}$ is the normalization of $S^{(n)}_\nu$.

Now we want to see that over any stratum $S^{(n)}_\nu$, the Hilbert-Chow morphism ρ is a locally trivial fibre bundle (in the strong topology).

First we look at the worst stratum $\rho : S^{[n]}_{(n)} \to S^{(n)}_{(n)} \simeq S$. For a point p in S the fibre $\rho^{-1}(n[p])$ is the set of subschemes Z of length in S with support p. The ideal of any such scheme is contained in m_p^n. We denote

$$H_n := Hilb^n(\mathbb{C}[x,y]/(x,y)^n).$$

Then the choice of holomorphic local coordinates x, y in a neighbourhood U of p determines an isomorphism

$$\rho^{-1}(U) \simeq U \times H_n.$$

Thus $\rho : S^{(n)}_{(n)} \to S$ is a locally trivial fibre bundle in the strong topology with fibre H_n. In fact being a bit more careful, we can replace in this argument local coordinates by local parameters in a Zariski open neighbourhood of p (i.e. x, y, s.th dx, dy span the cotangent space at every point of U). Thus this is even a Zariski locally trivial fibre bundle.

Now let $\nu = (n_1, \ldots, n_r)$ be a partition of n. Let $\xi := n_1[p_1] + \ldots + n_r[p_r] \in S^{(n)}_\nu$. Then the fibre $\rho^{-1}(\xi)$ is just $H_{n_1} \times \ldots \times H_{n_r}$. In fact we can choose (in the strong topology) disjoint open neighbourhoods U_i of the p_i in S, and these give rise to an open neighbourhood U of ξ in $S^{(n)}_\nu$ such that

$$\rho^{-1}(U) \simeq U \times H_{n_1} \times \ldots \times H_{n_r}.$$

Thus $\rho : S^{[n]}_\nu \to S^{(n)}_\nu$ is a locally trivial fibre bundle in the strong topology with fibre $H_{n_1} \times \ldots \times H_{n_r}$. Again being more careful one can prove slightly more: the bundle is locally trivial in the étale topology.

Now we want to have a look at the fibres of ρ. By the above, we only need to look at

$$H_n = \rho^{-1}(n[p]) \simeq Hilb^n(k[x,y]/(x,y)^n)$$

H_n called the *punctual Hilbert scheme*. It has been studied quite extensively (see for instance [**Ia**], [**Br**]).

It is clear that $H_1 = p$ and we have seen that $H_2 = \mathbb{P}_1$.

For $n \geq 3$ we have to distinguish two cases depending on the embedding dimension $dim(T_pZ)$ of the scheme Z. A scheme $Z \in H_n$ is called *curvilinear* if its embedding dimension is 1. This means that it locally lies on a smooth curve in S. We denote $H_n^c \subset H_n$ the open subscheme of curvilinear subschemes.

If $Z \in H_n$ is curvilinear, then in suitable local coordinates we can write

$$I_Z = (y + a_1 x + \ldots + a_{n-1} x^{n-1}, x^n).$$

We can see that these schemes form a locally trivial $\mathbb{A}^{n-2}$-bundle over $\mathbb{P}^1$. In the case $n = 3$ we see that the only subscheme which is not curvilinear is the scheme with ideal m_p^2 and H_3^c is dense in H_3. In fact this is true in general.

THEOREM 5.2. [**Br**] *H_n^c is open and dense in H_n.*

6. The Betti numbers of the Hilbert scheme of points

Now I want to summarize what we have shown so far and put it into context.

DEFINITION 6.1. Let $f : X \to Y$ be a projective morphism of varieties over $\mathbb{C}$. Suppose that Y has a stratification

$$Y = \coprod_{\alpha} Y_\alpha$$

into locally closed subvarieties. Write $X_\alpha := f^{-1}(Y_\alpha)$. Assume that for all α the restriction $f : X_\alpha \to Y_\alpha$ is a locally trivial fibre bundle with fibre F_α in the strong topology.

Then f is called *strictly semismall* (with respect to the stratification) if for all α

$$2dim(F_\alpha) = codim(Y_\alpha)$$

Thus we have shown:

PROPOSITION 6.2. *$\rho : S^{[n]} \to S^{(n)}$ is strictly semismall with respect to the stratification by the $S_\nu^{(n)}$. Furthermore the fibres of ρ are irreducible.*

This can be used to compute the Betti numbers of the Hilbert schemes: For proper morphisms of complex varieties, there is the decomposition theorem of [**BBD**] for the intersection homology complex. This becomes much simpler simple for semismall morphisms and computes the intersection homology of X in terms of the intersection homologies of the closures $\overline{Y}_\alpha$ of the strata of Y. If X is nonsingular and projective, then its intersection homology groups coincide with the usual homology groups of X (with $\mathbb{Q}$ coefficients). Thus we can compute the cohomology groups of X in terms of the intersection homology of the $\overline{Y}_\alpha$. In the case of the Hilbert scheme of points on a surface this has been carried out in [**GS**]. The proof is explained in section 8 of [**EG**]. If $\nu = (1^{\alpha_1}, \ldots, s^{\alpha_s})$ the fact that $S^{(\alpha_1)} \times \ldots \times S^{(\alpha_s)}$

is the normalization of $S_\nu^{(n)}$ will imply that the the intersection homology groups of $S_\nu^{(n)}$ are equal to the usual cohomology groups of $S^{(\alpha_1)} \times \ldots \times S^{(\alpha_s)}$ which are know by Macdonalds formula [**Md**].

The final result is best stated in terms of generating functions. We write $b_i(X) := dim(H^i(X,\mathbb{Q}))$ for the Betti numbers, and $p(X) := \sum b_i(X)z^i$ for the Poincaré polynomial. Then the result is the following.

$$\sum_{n\geq 0} p(S^{[n]})t^n = \prod_{k=1}^{\infty} \frac{(1+z^{2k-1}t^k)(1+z^{2k+1}t^k)}{(1-z^{2k-2}t^k)(1-z^{2k}t^k)(1-z^{2k+2}t^k)}.$$

This result can also proved in other ways. For instance in [**G1**] the fact that $S^{[n]}_{(n)} \to S$ is Zariski locally trivial and $S^{[n]}_{(\nu)} \to S^{(n)}_{(\nu)}$ is étale locally trivial is used to compute the numbers of points of $S^{[n]}$ over finite fields and then compute the Betti numbers of the $S^{[n]}$ using the Weil conjectures.

7. The Heisenberg algebra

In this section all the cohomology that we consider is with $\mathbb{Q}$-coefficients. The last formula suggests that somehow all the cohomology groups of $S^{[n]}$ for different n are tied together, and that one should try to look at all of them at the same time. So we denote $\mathbb{H}_n := H^*(S^{[n]})$ and consider the direct sum of all these cohomologies

$$\mathbb{H} := \bigoplus_{n\geq 0} \mathbb{H}_n.$$

We want to see that this carries an additional structure: $\mathbb{H}$ carries an irreducible representation of the Heisenberg algebra modelled on the cohomology of $S^{[n]}$. This was conjectured by Vafa and Witten [**VW**] and proven by Nakajima and Groijnowski [**N**], [**Gr**]. The main purpose of the lectures [**EG**] was to explain this result and its proof. In this lecture I will just try to briefly explain the result. For simplicity of exposition we will assume that $H^1(S) = H^3(S) = 0$.

We want to relate the Hilbert schemes $S^{[n]}$ for different n. Thus we need to find a way to go from $S^{[n]}$ to $S^{[n+m]}$. To relate $S^{[n]}$ and $S^{[n+1]}$ the obvious thing is to add to any subscheme $Z \in S^{[n]}$ a point in S. This can be done by looking at the incidence correspondence, i.e. the closed subscheme

$$S^{[n,n+1]} := \left\{(Z,W) \in S^{[n]} \times S^{[n+1]} \;\middle|\; Z \subset W\right\},$$

where we mean by $Z \subset W$ that Z is a subscheme of W. Then $S^{[n,n+1]}$ parametrizes all ways to obtain a subscheme of length $n+1$ by adding a point to a subscheme of length n. We used $S^{[n,n+1]}$ before to show that $S^{[n]}$ is connected: one can show that $S^{[n,n+1]}=\mathbb{P}(\mathcal{I}_{Z_n(S)})$.

Thus to relate $S^{[n]}$ to $S^{[n+m]}$ we also want to use an incidence correspondence. The obvious generalization would be to use just the incidence variety of $S^{[n]}$ and $S^{[n+m]}$, which parametrizes all ways to obtain a subscheme of length $n+m$ by adding a subscheme of length m to a subscheme of length n, but it turns out that we want the difference to be supported at a point of S. Thus we put

$$Z_{n,m} := \big\{(Z,p,W)\in S^{[n]}\times S\times S^{[n+m]} \bigm| \rho(W)-\rho(Z)=n[p]\big\},$$

with the projections pr_1, pr_2, pr_3 to $S^{[n]}$, S, $S^{[n+m]}$ respectively.

This correspondence defines for each $\alpha\in H^*(S,\mathbb{Q})$ a map

$$p_m(\alpha): H^*(S^{[n]})\to H^*(S^{[n+m]});\ y\mapsto PD(pr_{3*}(pr_2^*(\alpha)\cup pr_1^*(y)\cap[Z_{n,m}])).$$

where $[Z_{n,m}]$ is the fundamental class of $Z_{n,m}$ and PD denotes Poincaré duality. We call the $p_m(\alpha)$ the *creation operators*. Intuitively this map can be described as follows: Assume that α and y can be represented as the cohomology classes Poincaré dual to the fundamental class of submanifolds $A\subset S$ and $Y\subset S^{[n]}$. Then $p_m(\alpha)y$ will be the class of the closure of

$$\big\{Z\sqcup P \bigm| [Z]\in Y, P\in S^{[n]}_{(n)} \text{ with } supp(P)\in A\big\}.$$

Thus $p_m(\alpha)$ is the operation of adding a fat point in A. Thus for all $m>0$ and all $\alpha\in H^*(S)$ we obtain operators $p_m(\alpha):\mathbb{H}\to\mathbb{H}$, sending $\mathbb{H}_n$ to $\mathbb{H}_{n+m}$. Note that $S^{[0]}$ is a point, and thus $\mathbb{H}_0=\mathbb{Q}$. Let $\mathbf{1}$ be its unit element.

A weak version of the result of Nakajima and Groijnowski says that all the cohomology of the Hilbert schemes can be obtained by just applying the creation operators to $\mathbf{1}$. In fact given a basis of $H^*(S)$ this gives a canonical basis of $H^*(S^{[n]})$.

THEOREM 7.1. *The $p_m(\alpha)$, with $m>0$ and $\alpha\in H^*(S)$ commute. Let $\{a_i\}_{i\in L}$ be a basis of $H^*(S)$, then the set all monomials*

$$p_{n_1}(\alpha_{i_1})\dots p_{n_k}(\alpha_{i_k})\mathbf{1},\quad ,k\geq 0,\ i_j\in L,\ \sum n_j=n$$

is a basis of $\mathbb{H}_n$.

This means that, given the intuitive description of the $p_m(\alpha)$ above, we get at least intuitively a very explicit description of the cohomologies of the Hilbert scheme: Assume that the α_i are represented by submanifolds $A_i\subset S$. Then a basis of $H^*(S^{[n]})$ is given by the closures of the classes of subsets of $S^{[n]}$ of the form

$$\big\{P_1\sqcup P_2\dots\sqcup P_l \bigm| P_j\in S^{[n_j]}_{(n_j)} \text{ with support in } A_{i_j}\big\}.$$

In order to get the Heisenberg algebra, we also need to consider *annihilation operators*, $p_{-m}(\alpha) : \mathbb{H}_{n+m} \to \mathbb{H}_n$. We define $p_0(\alpha) = 0$ and let $p_{-m}(\alpha)$ be the adjoint operator of $p_m(\alpha)$ with respect to the intersection pairing on the cohomology of $S^{[n]}$ and $S^{[n+m]}$.

Again one can get an analoguous intuitive interpretation. If y is the class of $Y \subset S^{[n+m]}$, and a the class of a submanifold $A \subset S$, then $p_{-m}(a)y$ should be the class of the closure of

$$\big\{Z \in S^{[n]} \bigm| \exists_{W\in Y} W \text{ differs from } Z \text{ in only one point of } A\big\}.$$

Thus $p_{-m}(a)$ is obtained by subtracting a fat point in A.

We denote by $\langle\ ,\ \rangle$ the intersection pairing on S. Denote by $[p_n(\alpha), p_m(\beta)] := p_n(\alpha)p_m(\beta) - p_m(\beta)p_n(\alpha)$ the commutator. Then the main result of [**N**],[**Gr**] is:

THEOREM 7.2. $[p_n(\alpha), p_m(\beta)] = n\delta_{n,-m}\langle\alpha,\beta\rangle id_{\mathbb{H}}$

Thus all the creation operators $p_n(\alpha)$ commute with each other, and also all annihilation operators $p_{-m}(\beta)$ commute. Furthermore $p_n(\alpha)$ and $p_{-m}(\beta)$ commute unless $n = m$, when we just get a multiple of the identity. From our intuitive description this is quite plausible: Adding a fat point of length n in A and a fat point of length m in B should commute and similarly for subtracting fat points. Also adding a fat point of length n in A and subtracting a fat point of length m in B should commute unless $n = m$. However if $n = m$ this will no longer be true, because we get an extra term from subtracting the point that we have just added, and this extra term will just be a multiple of what we started with. Obviously this is not a proof, but still it gives the basic idea.

How does Theorem 7.1 imply Theorem 7.2, and what does it have to do with the Heisenberg algebra? Let V be a $\mathbb{Q}$-vector space with a nondegenerate bilinear form $\langle\ ,\ \rangle$. Let T be the tensor algebra on $V[t, t^{-1}]$. Elements of T are of the form

$$v_1t^{i_1} \otimes \ldots \otimes v_kt^{i_k}, \quad v_j \in V,\ i_j \in \mathbb{Z}, \quad j \geq 0.$$

Let $\mathbf{e}$ be the neutral element of the tensor algebra corresponding to the empty tensor product. We have $T = \bigoplus_{n\in\mathbb{Z}} T^i$, where the grading is determined by giving t^i the degree i. The *Heisenberg algebra* $H(V)$ modelled on V is obtained from T by imposing the relations

$$[ut^i, vt^j] = i\delta_{i,-j}\langle u, v\rangle\mathbf{e}.$$

The *Fock space* $F(V)$ is the subalgebra of $H(V)$ obtained by replacing $V[t, t^{-1}]$ by $tV[t]$. $F(V)$ becomes an $H(V)$-module, by putting

$ut^0 \cdot w := 0$ for all $w \in F(V)$ and $ut^{-i} \cdot \mathbf{e} := 0$ for all $i > 0$. Then one can show that $F(V)$ is an irreducible module for $H(V)$ and

$$\sum_{n\geq 0} dim(F(V)^d)t^d = \prod_{k\geq 1} \frac{1}{1-t^k}.$$

Now let $V = H^*(S)$ with the intersection pairing. Then Theorem 7.2 says that there is an $H(V)$-module homomorphism

$$F(V) \to \mathbb{H}, ut^i \mapsto p_i(u)\mathbf{1}.$$

As $F(V)$ is irreducible and both have the same Poincaré series, this is an isomorphism. This implies in particular that $H^*(S^{[n]})$ has the basis given in Theorem 7.1. Note that the fact that $p_i(u)$ and $p_{-i}(u)$ are adjoint operators for the intersection pairings on the Hilbert schemes, also makes it easy to determine the intersection pairing in this basis.

This Heisenberg algebra action has been further used to study the ring structure of the $H^*(S^{[n]})$, see e.g. [**L**],[**L-S**].

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