Introduction to Algebraic topology

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Introduction

The aim of the short course is to give a first introduction to algebraic topology, in particular to singular homology of topological spaces.

The idea of algebraic topology is to assign algebraic data (e.g. groups) to topological spaces. Then problems in topology translate into problems in algebra, which sometimes are much easier. More precisely one wants to associate algebraic data to topological spaces in a functorial way. This means the following: To each topological space X we assign a group $F(X)$ and to each continuous map $f : X \to Y$ a homomorphism $f_* : F(X) \to F(Y)$ such that

(1) $(Id_X)* = id_{F(X)}$,
(2) For $X \xrightarrow{f} Y \xrightarrow{g} Z$ continuous maps

$$(g \circ f)_* = g_* \circ f_*.$$

This will translate problems of topology into problems of algebra.

Example 0.1. A necessary condition for X and Y to be homeomorphic is that $F(X)$ and $F(Y)$ are isomorphic groups. (Let $f : X \to Y$ be a homeopmorphism with inverse f^{-1}. Then $f^{-1} \circ f = id_X$, $f \circ f^{-1} = id_Y$, thus $id_{F(X)} = f_*^{-1} \circ f_*$ and $id_{F(Y)} = f_* \circ f_*^{-1}$. Thus f_* is an isomorphism.) This is usually the easiest way to prove that two spaces with similar topological properties are not homeomorphic.

We will look at two such functors:

(1) The fundamental group, which is a group made of equivalence classes of loops in X.
(2) Homology groups which are made out of maps from simplices into X.

CHAPTER 1

The fundamental group and elements of homotopy theory

1. Homotopy of paths and the fundamental group

Let X be a topological space. Denote by I the closed interval $[0,1]$.

DEFINITION 1.1. A *path* in X from x_0 to x_1 is a continuous map $\gamma : I \to X$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$. X is called *path connected* if for any x_0, $x_1 \in X$ there is a path from x_0 to x_1. A path γ from x_0 to x_0 is called a *loop* at x_0. Let $\sigma, \tau : I \to X$ be two paths from x_0 to x_1. We say that σ and τ are *homotopic with end points fixed*; written

$$\sigma \simeq \tau \ rel\{0,1\},$$

if there is a map $F : I \times I \to X$ s.th.

(1) $F(\bullet, 0) = \sigma$, $F(\bullet, 1) = \tau$
(2) $F(0,t) = x_0$, $F(1,t) = x_1$ $\forall_t$.

We will often drop the $rel\{0,1\}$. F is called a *homotopy* from σ to τ. Note that in this case all $F_t := F(\bullet, t)$ are paths from x_0 to x_1 with $F_0 = \sigma$, $F_1 = \tau$ so that σ gets continuously deformed into τ.

We write $F_t : \sigma \simeq \tau$.

EXERCISE 1.2. $\simeq$ is an equivalence relation, i.e.

(1) $\sigma \simeq \sigma$,
(2) $\sigma \simeq \tau \implies \tau \simeq \sigma$,
(3) $\sigma \simeq \tau$ and $\tau \simeq \gamma \implies \tau \simeq \gamma$

Therefore we can consider the equivalence classes called the *homotopy classes* $[\sigma]$ of paths.

If σ is a path from x_0 to x_1 and τ is a path from x_1 to x_2, we can define the *composition*, a path $\sigma\tau$ from x_0 to x_2 by

$$\sigma\tau(t) = \begin{cases} \sigma(2t) & 0 \leq t \leq \frac{1}{2} \\ \tau(2t-1) & \frac{1}{2} \leq 1 \leq 1 \end{cases}.$$

REMARK 1.3. If $F_t : \sigma \simeq \sigma'$ and $G_t : \tau \simeq \tau'$ Then $F_tG_t : \sigma\tau \simeq \sigma'\tau'$.

Therefore we get a well-defined product of homotopy classes

$$[\sigma][\tau] := [\sigma\tau].$$

THEOREM 1.4. *Let $\pi_1(X, x_0)$ be the set of homotopy classes of loops in X at x_0, with the product $[\sigma][\tau] := [\sigma\tau]$. Then $\pi_1(X, x_0)$ is a group. The neutral element is the constant loop x_0 and the inverse of $[\sigma]$ is the class of σ^{-1} defined by $\sigma^{-1}(t) = \sigma(1-t)$.*

$\pi_1(X, x_0)$ is called the *first homotopy group* with respect to x_0.

PROOF. We show $\sigma\sigma^{-1} \simeq x_0$. The homotopy is given by the diagram

i.e. we put

$$F(s,t) := \begin{cases} \sigma(2s) & 0 \leq s \leq t \\ \sigma(t) & t \leq 2s \leq 2-t \\ \sigma^{-1}(2s-t) & 2-t \leq 2s \leq 2. \end{cases}$$

The functions are continuous on each triangle and agree on the intersections, thus F is continuous. The associativity is proven by

(note that this also proves the associativity for products of paths and not just for loops) and that x_0 is the neutral element is proven by

$\square$

EXERCISE 1.5. Write this down in formulas.

By definition $\pi_1(X,x)$ depends on the choice of the base point x. Is there a relation between $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$? Obviously this can only be the case if x_0 and x_1 lie in the same path-component.

Proposition 1.6. *Let α be a path from x_0 to x_1. Then the map*

$$\alpha_* : \pi_1(X, x_0) \to \pi_1(X, x_1);\ [\sigma] \to [\alpha^{-1}\sigma\alpha]$$

is an isomorphism.

Proof. α_* is clearly a group homomorphism, and α_*^{-1} is the inverse, by the diagram

$\square$

Corollary 1.7. *If X is path-connected, then $\pi_1(X, x)$ is independent (up to isomorphism) of x.*

In this case we will often write $\pi_1(X)$ for $\pi_1(X, x_0)$ and call it the *fundamental group* of X.

Definition 1.8. A topological space is called *simply connected* if is path-connected and $\pi_1(X) = \{1\}$.

The simplest example of a simply connected topological space is one point $\{*\}$.

We want to see that the fundamental group is functorial.

Definition 1.9. Let X, Y be topological spaces and let $x_0 \in X$, $y_0 \in Y$. Let $f : (X, x_0) \to (Y, y_0)$ (this just means $f(x_0) = y_0$)be continuous. Then we define

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0);\ [\sigma] \to [f \circ \sigma].$$

It is easy to see that this is well-defined and a homomorphism.

REMARK 1.10. The following are obvious:

(1) $(id_X)_* = id_{\pi_1(X,x_0)}$,
(2) for $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ continuous, we have

$$(g \circ f)_* = g_* \circ f_*.$$

Thus the fundamental group is functorial.

2. Homotopy of maps

DEFINITION 2.1. Let X, Y be topological spaces and $f, g : X \to Y$ continuous maps. A *homotopy* between f and g is a continuous map $F : X \times I \to Y$ with $F(\bullet, 0) = f$ and $F(\bullet, 1) = g$. (Writing again $F_t := F(\bullet, t)$ we get a continuous family of maps with $F_0 = f$, $F_1 = g$).

In this case f and g are called *homotopic* and we write $f \simeq g$ of $F_t : f \simeq g$.

If $A \subset X$ is a subset and $g|_A = f|_A$, F is called a a *homotopy rel* A if $F(a, t) = f(a) = g(a)$ for all t and all $a \in A$. Note that two paths σ, τ are homotopic with fixed endpoints if they are homotopic *rel* $\{0, 1\}$.

DEFINITION 2.2. Two topological spaces X, Y are called *homotopy equivalent* if there are continuous maps $f : X \to Y$, $g : Y \to X$ with

$$g \circ f \simeq id_X, \qquad f \circ g \simeq id_Y.$$

In this case f (or g) is called a *homotopy equivalence.*

X is called *contractible* if it is homotopy equivalent to a point.

Homeomorphic topological spaces are in particular homotopy equivalent.

EXERCISE 2.3. Show that $\mathbb{R}^n$ is contractible.

Now we want to see that homotopy equivalences induce isomorphisms of fundamental groups. There is a small problem, because if f and g are as above, then we usually have $g(f(x_0)) \neq x_0$.

LEMMA 2.4. *Let $F : I \times I \to X$ be continuous. Put $\alpha = F(0, \bullet)$, $\beta = F(1, \bullet)$, $\gamma = F(\bullet, 0)$, $\delta = F(\bullet, 1)$.*

Then $\delta \simeq \alpha^{-1}\gamma\beta \; rel\{0,1\}$.

PROOF. Let $x_0 = \delta(0)$, $x_1 = \delta(1)$. Then the proof is by putting together the squares

That is

$$E(s,t) = \begin{cases} x_0 & s \leq t \\ \alpha(1+t-s) & s \geq t, \end{cases}$$

$$E(s,t) = \begin{cases} \beta(t+s) & 1-s \geq t \\ x_1 & 1-s \leq t. \end{cases}$$

Then

$$E_t F_t G_t : \delta \simeq \alpha^{-1}\gamma\beta \; rel\{0,1\}.$$

□

This implies that homotopic maps induce almost the same map of homotopy groups.

COROLLARY 2.5. *Let* $f, g : X \to Y$ *be homotopic via* $F : X \times I \to Y$. *Let* $x_0 \in X$. *Let* $\alpha = F(x_0, \bullet) : I \to Y$ *(a path from* $f(x_0)$ *to* $g(x_0)$*). Then we have*

$$g_* = \alpha_* \circ f_* : \pi_1(X, x_0) \to \pi_1(X, g(x_0)).$$

(Note that $\alpha_* : \pi_1(X, f(x_0)) \to \pi_1(X, g(x_0))$ *is an isomorphism, thus* f_* *is an isomorphism if and only* g_* *is).*

Proof. For any loop σ at x_0 we have a diagram

□

Theorem 2.6. *Let $f : X \to Y$ be a homotopy equivalence, $x_0 \in X$. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. We have $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$. Thus, by the corollary, $g_* f_*$ and $f_* g_*$ are isomorphisms. Thus f_* is an isomorphims. □

Corollary 2.7. *If X is contractible, then X is simply connected.*

Example 2.8. In particular $\mathbb{R}^n$ is simply connected.

Note that however simply connected manifolds are not necessarily contractable, e.g. the n-sphere S^n for $n > 1$ is simply connected, but not contractable.

3. Fundamental group of the circle

We will compute the fundamental group of the circle S^1, using its cover $\mathbb{R}$. It turns out that the homotopy class of a loop is determined by how many times it winds around the circle. The methods used here generalize: all "nice" topological spaces X have a so called *universal covering* $\rho : Y \to X$, such that ρ is a local homeomorphism and the fibres of ρ are discrete and Y is simply connected. Then $\pi_1(X)$ can be identified with the group of homeomorphisms $f : Y \to Y$ with $\rho = \rho \circ f$. The arguments are very similar to the ones we use below for the special case of S^1. We will not pursue this here.

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The exponential map $e : \mathbb{R} \to S^1, t \to e^{2\pi i t}$ is a local homeomorphism. In particular

$$e|_{(-1/2,1/2)} : (-1/2, 1/2) \to S^1 \setminus \{-1\}$$

is a homeomorphism, let *log* be its inverse.

We want to show that paths in S^1 and homotopies between them can be lifted uniquely to $\mathbb{R}$.

LEMMA 3.1. (1) *(lifting paths) Let $\sigma : I \to S^1$ be a path with $\sigma(0) = 1$. Then there is a unique path $\sigma' : I \to \mathbb{R}$ with $\sigma(0) = 0$ and $e \circ \sigma' = \sigma$.*

(2) *(lifting homotopies) Let $\tau : I \to S^1$ be another path and let $F_t : \sigma \simeq \tau\ rel\{0,1\}$. Then there is a unique $F' : I \times I \to \mathbb{R}$, with $F'_t : \sigma' \simeq \tau'\ rel\{0,1\}$ and $e \circ F' = F$.*

PROOF. (1) and (2) are very similar and have the same proof. Let either $J = I$ or $J = I \times I$. Let $f : J \to S^1$ be either σ or F.

(Existence) Since J is compact, F is uniformly continuous, i.e. there exists $\varepsilon > 0$ s.th. $|t - t'| < \varepsilon \implies |f(t) - f(t')| < 1$. In particular $f(t') \neq -f(t)$ so that $log(f(t)/f(t'))$ is defined. Choose n so that $|t| < n\varepsilon$ for all $t \in J$. Define

$$f'(t) := \sum_{i=1}^{n} log\Big(f(ti/n)/f(t(i-1)/n)\Big).$$

Then $f' : J \to \mathbb{R}$ is continuous, $f'(0) = 0$, $e \circ f' = f$.

(Uniqueness) Let f'' be another lift. Then $g := f' - f''$ is continuous and has image in $\mathbb{Z}$. As J is connected, g is constant and thus $f' = f''$. $\square$

THEOREM 3.2. $\pi_1(S^1) = \mathbb{Z}$.

PROOF. By the Lemma above the end point of the lift σ' depends only on the homotopy class $[\sigma]$. Define a map

$$w : \pi_1(S,1) \to \mathbb{Z},\ [\sigma] \mapsto \sigma'(1)$$

w is a homomorphism: Let $[\sigma], [\tau] \in \pi_1(S^1, 1)$. Let $m = \sigma'(1)$, $n = \tau'(1)$. Let $\tau'' : I \to \mathbb{R}$ be the path $t \mapsto \tau'(s) + m$. Then $\sigma'\tau''$ is a lifting of $\sigma\tau$ with endpoint $n + m$.

w is surjective: Given $n \in \mathbb{Z}$ define $\sigma'(s) = ns$. Then $w([e \circ \sigma']) = n$.

w is injective: Assume $w([\sigma]) = 0$, i.e. σ' is a loop in $\mathbb{R}$. As $\mathbb{R}$ is contractable, we get $F_t : \sigma' \simeq 0\ rel\{0,1\}$. Thus $e \circ F_t : \sigma \simeq 0\ rel\{0,1\}$. $\square$

So we have computed the fundament group in at least one nontrivial case.

EXERCISE 3.3. $\pi_1(X \times Y, (x,y)) = \pi_1(X,x) \times \pi_1(Y,y)$.

Sketch: The map

$$\pi_1(X \times Y, (x,y)) \to \pi_1(X,x) \times \pi_1(Y,y);\ [(\sigma,\tau)] \to ([\sigma],[\tau])$$

is an isomorphism.

So we see that a torus $S^1 \times S^1$ has fundamental group $\mathbb{Z} \times \mathbb{Z}$.

As an application of the Theorem we want to prove the case $n = 2$ of the Brouwer fixed point theorem.

DEFINITION 3.4. Let X be a topological space and $A \subset X$. A is called a retract of X if there is a continuous map $f : X \to A$ with $f|_A = id_A$.

Let $E^n = \left\{ z \in \mathbb{R}^n \;\middle|\; ||z|| \leq 1 \right\}$ be the closed unit n disc. Note that E^n is contractable.

PROPOSITION 3.5. *S^1 is not a retract of E^2.*

PROOF. Supposed $f : E^2 \to S^1$ is a retraction. Let $i : S^1 \to E^2$ be the inclusion. Then $f \circ i = id_{S^1}$. Thus $f_* \circ i_* = id$, which is impossible because $\pi_1(E^2) = 0$ and $\pi_1(S^1) = \mathbb{Z}$. $\square$

COROLLARY 3.6. *(Case $n = 2$ of Brouwer fixed point theorem). Any continuous map $f : E \to E$ has a fixed point.*

PROOF. Let $f : E^2 \to E^2$ have no fixed point. Define $r : E^2 \to S^1$ by letting $r(x)$ be the intersection point (on the side of x) of the line through x and $f(x)$ with S^1.

(Exercise: write the formula for this.) Then r is a retraction, a contradiction. $\square$

EXERCISE 3.7. Assume X is the union of two simply connected open sets U, V with $U \cap V$ nonempty and path-connected. Then X is simply connected.

EXERCISE 3.8. Show that S^n is simply connected for $n \geq 2$.

$\pi_1(X)$ can be very complicated and does not have to be commutative. For instance let X be a bouquet of k circles:

Then $\pi_1(X)$ is the free group on k generators (i.e. a group with k generators and no relations).

4. Higher Homotopy groups

Let S^n be the n-sphere, and let $s \in S^n$ be a point. It is easy to see from the definition that $\pi_1(X, x_0)$ can be interpreted as the set of homotopy classes of maps from (S^1, s) to (X, x_0). Higher homotopy groups $\pi_n(X, x_0)$ can be defined as the set of homotopy classes of maps (S^n, s) to (X, x_0). One can show that $\pi_n(X, x_0)$ is a commutative group for $n > 1$. Higher homotopy groups are in general difficult to understand. A lot of work has for instance gone (and still is going) into the computation of the higher homotopy groups of spheres. It is not difficult to show that $\pi_k(S^n) = 0$ for $k < n$ and that $\pi_n(S^n) = \mathbb{Z}$. For $k > n$ the answer is very complicated and not completely known.

CHAPTER 2

Homology

The main part of the course is about homology. The definition of homology is a bit more complicated and less intuitive than that of the fundamental group and of the higher homotopy groups. Homology is however in many ways more powerful and easier to handle. There are many ways of defining homology groups, which at least for nice spaces are all equivalent. The most elementary and most intuitive (which also was the first historically) is simplicial homology. For this a topological space is triangulated, i.e. divided into pieces homeomorphic to simplices, and these simplices are then used to define the homology groups.

This has the advantage of being intuitive. There are however a number of technical disadvantages. In particular it takes a lot of effort to show that the homology groups are independent of the triangulation.

We will use singular homology, which is defined in terms of maps from simplices into X. It has the advantage that it obviously is a topological invariant of X. In fact it is not difficult to show that the homology groups depend only on the homotopy type of X and furthermore that they are functorial.

1. Singular homology

The singular homology groups $H^*(X, \mathbb{Z})$ of a topological space X are made out of continuous maps from simplices (points, lines, triangles, tetraeders) into X. So first we need to talk about simplices.

Definition 1.1. Let $p_0, \dots, p_k \in \mathbb{R}^n$. The *convex hull* of $p_0, \dots, p_k$ is

$$\Big\{ \sum_{i=0}^{k} t_i p_i \;\Big|\; 0 \le t_i,\ \sum t_i = 1 \Big\}.$$

In $\mathbb{R}^n$ let

$$\begin{aligned} e_0 &= (0, 0, \dots, 0) \\ e_1 &= (1, 0, \dots, 0) \\ e_2 &= (1, 0, \dots, 0) \\ &\dots \\ e_n &= (0, \dots, 0, 1) \end{aligned}$$

The *standard n-simplex* $\Delta_n \subset \mathbb{R}^n$ is the convex hull of $e_0, \dots, e_n$.

Definition 1.2. Let X be a topological space. A *singular n-simplex* in X is a continuous map $\sigma : \Delta_n \to X$.

Example 1.3. Given $p_0, \dots, p_n$ in $\mathbb{R}^k$, they define a singular n-simplex

$$(p_0, \dots, p_n) : \Delta_n \to \mathbb{R}^k;\ \sum t_i e_i \mapsto \sum t_i p_i.$$

$(p_0, \dots, p_n)$ is called the singular simplex generated by $p_0, \dots, p_n$.

DEFINITION 1.4. The *face* $F_i : \Delta_{n-1} \to \Delta_n$ of the standard simplex is

$$(e_0, \dots , \widehat{e_i}, \dots , e_n) : \Delta_{n-1} \to \Delta_n.$$

For a singular simplex $\sigma : \Delta_n \to X$ the composition $\sigma^{(i)} := \sigma \circ F_i : \Delta_{n-1} \to X$ is called the i-th face of σ.

REMARK 1.5. Note that the image $F_i(\Delta_n)$ is indeed an $(n-1)$-dimensional face of Δ_n; in fact it is the face oposite to e_i.

Now we want to define singular chains as formal linear combinations of singular simplices.

DEFINITION 1.6. Let $S_k(X)$ be the free abelian group generated by the singular q-simplices, i.e.

$$S_k(X) = \Big\{\text{formal sums} \sum_\sigma a_\sigma \sigma \;\Big|\; \sigma \text{ singular } k\text{-simplex} \;,\; a_\sigma \in \mathbb{Z}\Big\}.$$

That is such a formal sum is 0 if all coefficients are. Elements of $S_k(X)$ are called singular k-simplices.

The set of singular k-chains is very big, so we cannot use it directly.

DEFINITION 1.7. The *boundary* $\partial\sigma$ of a singular k-simplex $\sigma : \Delta_k \to X$ is

$$\partial\sigma := \sum_{i=0}^{k} (-1)^i \sigma^{(i)} \in S_{k-1}(X).$$

Thus

We extend ∂ by linearity to a group homomorphism

$$\partial : S_k(X) \to S_{k-1}(X);\ \sum a_\sigma \sigma \mapsto \sum a_\sigma \partial\sigma.$$

The boundary of a 0 chain is defined to be 0.

The starting point for the definition of homology is the fundamental identity $\partial \circ \partial = 0$.

LEMMA 1.8. $\partial \circ \partial = 0$.

PROOF. It is enough to show $\partial\partial\sigma = 0$ for all singular k-simplices $\sigma : \Delta_k \to X$.

$$\begin{aligned}
\partial\partial\sigma &= \partial \sum_{i=0}^{k} (-1)^i \sigma \circ (e_0, \dots, \widehat{e_i}, \dots, e_k) \\
&= \partial \sum_{i=0}^{k} (-1)^i \Bigg(\sum_{j=1}^{i-1} (-1)^j \sigma \circ (e_0, \dots, \widehat{e_j}, \dots, \widehat{e_i}, \dots, e_k) \\
&\qquad - \sum_{j=i+1}^{k} (-1)^j \sigma \circ (e_0, \dots, \widehat{e_i}, \dots, \widehat{e_j}, \dots, e_k) \Bigg) \\
&= \sum_{j<i} (-1)^{i+j} \sigma \circ (e_0, \dots, \widehat{e_j}, \dots, \widehat{e_i}, \dots, e_k) \\
&\qquad - \sum_{i<j} (-1)^{i+j} \sigma \circ (e_0, \dots, \widehat{e_i}, \dots, \widehat{e_j}, \dots, e_k) = 0
\end{aligned}$$

□

DEFINITION 1.9. A singular k-chain z with $\partial z = 0$ is called a *k-cycle*. If $z = \partial c$ for a $(k+1)$-chain c, then z is called a *boundary*. We denote $Z_k(X) \subset S_k(X)$ the subgroup of k-cycles and $B_k(X)$ the subgroup of k-boundaries. By $\partial \circ \partial = 0$ we have $B_k(X) \subset Z_k(X)$. The quotient group $H_k(X, \mathbb{Z}) = Z_k(X)/B_k(X)$ is called the k-th singular homology group of X.

REMARK 1.10. One can instead define homology $H_k(X, R)$ with coefficients in an arbitrary commutative ring R with 1. The most important ones are $R = \mathbb{Z}$ and $R = \mathbb{Q}$. One takes

$$S_k(X) := \Big\{ \sum_\sigma a_\sigma \sigma \;\Big|\; \sigma \text{ singular } k \text{ simplex } a_\sigma \in R \Big\},$$

and proceeds as before. Almost everything we do works also in this more general context.

We will however stick to $R = \mathbb{Z}$ and most of the time drop the $\mathbb{Z}$ from the notation.

Directly from the definition it is very hard to compute the homology groups, except in the most simple case, which is however important.

Proposition 1.11. *Let $X = \{p\}$ be a point. Then $H_i(X) = 0$ for $i > 0$ and $H_0(X) = \mathbb{Z}$.*

Proof. For each $n \geq 0$ there is a unique singular n-simplex, the constant map $c_n : \Delta_n \to \{p\}$. We have

$$\partial c_n = \sum_{i=0}^{n} (-1)^i c_{n-1} = \begin{cases} c_{n-1} & n \text{ even } > 0 \\ 0 & n \text{ odd or } 0 \end{cases}$$

Thus $B_n(X) = Z_n(X)$ for $n > 0$. However $B_0(X) = 0$ and $H_0(X) = Z_0(X) = \big\{nc_0 \bigm| n \in \mathbb{Z}\big\}$. $\square$

The following proposition relates homology to path components. It is the key to all results in algebraic topology about connectedness.

Proposition 1.12. (1) *Let $(X_k)_k$ be the set of path-components of X. Then $H_n(X) = \bigoplus_k H_n(X_k)$ for all $n \geq 0$.*
(2) *If X is path-connected, then $H_0(X) = \mathbb{Z}$.*

Proof. (1) As Δ_n is path-connected, $\sigma(\Delta_n)$ lies in a path component of X for any singular simplex σ. Thus $S_n(X) = \bigoplus_k S_n(X_k)$. Furthermore ∂ is just the direct sum of the ∂ on the X_k. Thus $Z_n(X) = \bigoplus_k Z_n(X_k)$, $B_n(X) = \bigoplus_k B_n(X_k)$, $H_n(X) = \bigoplus_k H_n(X_k)$.

(2) For $p \in X$ write also p for the singular simplex $\Delta_0 \to \{p\}$. Choose $x_0 \in X$. For any $x \in X$, let $\sigma_x : \Delta_1 = I \to X$ be a path from x_0 to x. Thus $\partial\sigma_x = x - x_0$.

Given a 0-chain $c = \sum a_x x$, we claim $c \in B_0(X) \iff \sum a_x = 0$.

If $\sum a_x = 0$, then

$$c = \sum a_x x - \sum a_x x_0 = \partial\Big(\sum a_x \sigma_x\Big).$$

The converse is clear.

By definition every 0-chain is a cycle. Thus the map sending c to the sum of the coefficients is a surjective homomorphism $S_0(X) = Z_0(X) \to \mathbb{Z}$ with kernel $B_0(X)$. Thus $H_0(X) = \mathbb{Z}$. $\square$

Finally we will show that homology is functorial, i.e. if $f : X \to Y$ is continuous, then there exists homomorphism $f_* : H_n(X) \to H_n(Y)$ with $(id_X)_* = id_{H_n(X)}$, $(g \circ f)_* = g_* \circ f_*$. We will use this occasion to introduce some basic concepts of homological algebra.

Definition 1.13. A sequence

$$S_* = S_0 \xleftarrow{\partial} S_1 \xleftarrow{\partial} S_2 \xleftarrow{\partial} \dots$$

of abelian groups and group homomorphism is called a *chain complex* if $\partial \circ \partial = 0$.

Given a chain complex S_*, its i-th *homology group* is

$$H_i(S_*) := Ker(\partial|S_i)/Im(\partial|_{S_{i-1}}).$$

A *morphism* $F : S_* \to T_*$ of chain complexes is a sequence $F : S_i \to T_i$ of homomorphism such that $\partial \circ F = F \circ \partial$.

$$\begin{array}{ccc} S_{i-1} & \xleftarrow{\partial} & S_i \\ \downarrow F & & \downarrow F \\ T_{i-1} & \xleftarrow{\partial} & T_i \end{array}$$

Remark 1.14. The homology of chain complexes is functorial: If $F : S_* \to T_*$ is a morphism of chain complexes, then we define $H_i(F) : H_i(S_*) \to H_i(T_*)$ as follows: Let $[a] \in H_i(S_*)$ be represented by $a \in Ker(\partial|_{S_i})$. Then $F(a) \in Ker(\partial|_{T_i})$ and we put $H_i(F)([a]) = [F(a)]$. The relation $\partial \circ F = F \circ \partial$ implies that this is well-defined. We also see $H_i(id_{S_*}) = id_{H_i(S_*)}$ and $H_i(G \circ F) = H_i(G) \circ H_i(F)$.

Remark 1.15. We call $S_*(X) = S_0(X) \xleftarrow{\partial} S_1(X) \xleftarrow{\partial} \dots$ the singular chain complex. We see that it is a chain complex and that its homology groups are the $H_i(X)$. We see that the map $X \mapsto S_*(X)$ is functorial: If $f : X \to Y$ is continuous, we put

$$f_* : S_i(X) \to S_i(Y);\ \sum a_\sigma \sigma \mapsto \sum a_\sigma f \circ \sigma.$$

We see that $f_* \circ \partial = \partial \circ f_*$. Thus f_* is a morphism of complexes and $(id_X)_* = id_{S_*(X)}$, $(g \circ f)_* = g_* \circ f_*$.

Definition 1.16. For $f : X \to Y$ continuous, we put

$$f_* : H_i(X) \to H_i(Y);\ \left[\sum a_\sigma \sigma\right] \mapsto \left[\sum a_\sigma f \circ \sigma\right].$$

Thus $(f_* : H_i(X) \to H_i(Y)) = H_i(f_* : S_*(X) \to S_*(Y))$. So it is functorial.

Remark 1.17. As the $H_i(X)$ are functorial, they are in particular topological invariants, i.e. if $f : X \to Y$ is a homeomorphism, then $H_i(f) : H_i(X) \to H_i(Y)$ is an isomorphism.

Remark 1.18. Let $\delta_n := id_{\Delta_n}$ A singular simplex $\sigma : \Delta_n \to X$ gives

$$\sigma_* : S_n(\Delta_n) \to S_n(X);\ \tau \mapsto \sigma \circ \tau.$$

Note that by definition $\sigma_*(\delta_n) = \sigma$. This observation is often used to use functoriality to reduce the proof of properties of singular simplices to the special case of δ_n. We will see an instance of this in the next section.

2. The homotopy invariance theorem

We want to show that the homology groups are homotopy invariants, not just topological invariants.

Theorem 2.1. *Let $f, g : X \to Y$ be homotopic maps, then $f_* = g_* : H_i(X) \to H_i(Y)$ for all $i \geq 0$.*

Corollary 2.2. *If X and Y are homotopy equivalent, then $H_i(X) = H_i(Y)$ for all $i \geq 0$.*

Corollary 2.3. *If X is contractible, then* $H_i(X) = \begin{cases} 0 & i > 0 \\ \mathbb{Z} & i = 0 \end{cases}$

In particular this is true for $X = \mathbb{R}^n$.

We shall show that the homotopy induces a chain homotopy between $f_*, g_* : S_*(X) \to S_*(Y)$.

Definition 2.4. Let $F, G : S_* \to T_*$ be morphisms of complexes. A *chain homotopy* from F to G is a sequence of maps $s : S_i \to T_{i+1}$, such that

$$\partial s - s\partial = G - F.$$

i.e.

$$\begin{array}{ccc} S_i & \xleftarrow{\partial} & S_{i+1} \\ \Big\downarrow\Big\downarrow{\scriptstyle F\,G} & \overset{s}{\swarrow} & \Big\downarrow\Big\downarrow{\scriptstyle F\,G} \\ S_i & \xleftarrow{\partial} & S_{i+1} \end{array}$$

If a chain homotopy from F to G exists, then F and G are called homotopic.

Proposition 2.5. *If F and G are homotopic, then $H_i(F) = H_i(G)$ for all i.*

Proof. If $a \in Ker(\partial|_{S_i})$, then $s(\partial a) = 0$, thus $G(a) - F(a) = \partial s(z)$, so $G(a)$ and $F(a)$ have the same homology class. □

Proof. (of the Theorem). First we reduce to the special case of maps $\lambda_t : X \to X \times I$; $x \mapsto (x, t)$. Then we have a homotopy $\lambda_i : \lambda_0 \simeq \lambda_1$.

Assume the theorem for λ_0 and λ_1. Let $F : X \times I \to Y$ be a homotopy from f to g. Then $F \circ \lambda_0 = f$, $F \circ \lambda_1 = g$ and thus

$$f_* = F_* \circ (\lambda_0)_* = F_* \circ (\lambda_1)_* = g_*.$$

Thus we only need to prove $(\lambda_0)_* = (\lambda_1)_*$.

For this we will construct a chain homotopy $P : S_n(X) \to S_{n+1}(X \times I)$ from $(\lambda_0)_*$ to $(\lambda_1)_*$ i.e. we want

$$(\partial P - P\partial)(\sigma) = \lambda_1 \circ \sigma - \lambda_0 \circ \sigma,$$

for all singular n-simplices σ. Now we want to use functoriality to reduce to the case $X = \Delta_n$, $\sigma = \delta_n$.

We want to construct P in a functorial way, i.e. we want that for any $f : X \to Y$

$$\begin{array}{ccc} S_n(X) & \xrightarrow{P} & S_{n+1}(X \times I) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle (f \times 1_I)_*} \\ S_n(Y) & \xrightarrow{P} & S_{n+1}(Y \times I) \end{array}$$

commutes. This requirement gives us as only choice $P(\sigma) = (\sigma \times 1_I)_*(P(\delta_n))$. On the other hand this definition is obviously functorial. Write $\overline{\lambda}_i$ for λ_i in case $X = \Delta_n$. Assume that we have define $P(\delta_n)$ in such a way that

$$(\partial P - P\partial)(\delta_n) = \overline{\lambda}_1 \circ \delta_n - \overline{\lambda}_0 \circ \delta_n.$$

Let $\sigma : \delta_n \to X$ be a singular n-simplex. Then, using that also $\partial f_* = f_* \partial$ by definition,

$$\begin{aligned}(\partial P - P\partial)(\sigma) &= (\partial P - P\partial)(\sigma_* \delta_n) = (\sigma \times 1_I)_*(\partial P - P\partial)(\delta_n)) \\ &= (\sigma \times 1_I)_*(\overline{\lambda}_1 - \overline{\lambda}_0)_*(\delta_n) = (\lambda_1 - \lambda_0)_*(\sigma).\end{aligned}$$

Now we have to define P on δ_n. The polyhedron $\Delta_n \times I \subset \mathbb{R}^{n+1}$ has $2(n+1)$ vertices $A_0, \dots, A_n$, $B_0, \dots, B_n$ given by $A_i := (e_i, 0)$, $B_i := (e_i, 1)$. We define

$$P(\delta_n) := \sum_{i=0}^{n} (-1)^i (A_0, \dots, A_i, B_i, \dots, B_n).$$

For instance

$$P(\delta_0) = (A_0, B_0),$$

$$P(\delta_1) = (A_0, B_0, B_1) - (A_0, A_1, B_1).$$

In general the effect of P is to make a $(n+1)$-chain made of prisms whose union is $\Delta_n \times I$.

Claim: $(\partial \circ P + P \circ \partial)(\delta_n) = \lambda_1 \circ \delta_n - \lambda_0 \circ \delta_n$.

The right hand side gives

$$\lambda_1 \circ \delta_n - \lambda_0 \circ \delta_n = (B_0, \dots, B_n) - (A_0, \dots, A_n).$$

Now we compute the left hand side.

$$\begin{aligned}\partial P(\delta_n) &= \sum_{i=0}^{n}(-1)^i \partial(A_0, \dots, A_i, B_i, \dots, B_n) \\ &= \sum_{j\le i}(-1)^{i+j}(A_0, \dots, \widehat{A_j}, \dots A_i, B_i, \dots, B_n) \\ &\qquad - \sum_{i\le j}(-1)^{i+j}(A_0, \dots, A_i, B_i, \dots \widehat{B_j}, \dots, B_n)\end{aligned}$$

All terms with $i = j$ cancel with the exception of $(B_0, \dots, B_n) - (A_0, \dots, A_n)$. Thus

$$\begin{aligned}\partial P(\delta_n) &= (B_0, \dots, B_n) - (A_0, \dots, A_n) \\ &\qquad + \sum_{j<i}(-1)^{i+j}(A_0, \dots, \widehat{A_j}, \dots A_i, B_i, \dots, B_n) \\ &\qquad - \sum_{i<j}(-1)^{i+j}(A_0, \dots, A_i, B_i, \dots \widehat{B_j}, \dots, B_n)\end{aligned}$$

On the other hand

$$\partial \delta_n = \sum_{j=0}^{k}(-1)^j(e_0, \dots, \widehat{e_j}, \dots, e_n).$$

Since

$$\begin{aligned}P((e_0, \dots, \widehat{e_j}, \dots, e_n)) &= -\sum_{i<j}(-1)^i(A_0, \dots, A_i, B_i, \dots \widehat{B_j}, \dots, B_n) \\ &\qquad - \sum_{i>j}(-1)^i(A_0, \dots, \widehat{A_j}, \dots A_i, B_i, \dots, B_n),\end{aligned}$$

the result follows. $\square$

3. Relation between fundamental group and homology

We now want to show that the fundamental group and the first homology group are closely related. We know that the fundamental group is in general not commutative, while the homology groups are all commutative. We will show that the first homology group is just the fundamental group made commutative (i.e. divided by its commutator subgroup).

If $\gamma : I \to X$ is a loop in X, we may regard it as a singular 1-simplex (because by definition $\Delta_1 = I$). Fix a point $x_0 \in X$ and let $L(x_0)$ be the set of loops at x_0. Then we have a map $\chi : L(x_0) \to S_1(X)$, sending a loop σ to itself viewed as a singular simplex. The following result implies that this induces a group homomorphism $\chi : \pi_1(X, x_0) \to H_1(X)$.

PROPOSITION 3.1. *If two loops γ, τ are homotopic, then theys are homologuous as singular 1-simlices. Moreover, given two loops γ_1, γ_2 at x_0, we have*

$$\chi(\gamma_1\gamma_2) = \chi(\gamma_1) + \chi(\gamma_2).$$

PROOF. Choose a homotopy $F : I \times I \to X$ with

$$\begin{aligned} F(t,0) &= \gamma, \\ F(t,1) &= \tau, \\ F(1,s) &= F(0,s) = x_0, \ \forall_s. \end{aligned}$$

Consider the points $e_0 = (0,0)$, $e_1 = (1,0)$, $e_2 = (0,1)$, $Q = (1,1) \in \mathbb{R}^2$.

Note that

$$\begin{aligned} \gamma &= F \circ (e_0, e_1) \\ \tau &= F \circ (e_2, Q) \end{aligned}$$

and

$$F \circ (e_1, Q) = F \circ (e_0, e_2) = x_0.$$

Define the singular 2-chain

$$\sigma := F \circ (e_0, e_1, Q) - F \circ (e_0, e_2, Q).$$

Then

$$\begin{aligned}\partial\sigma &= F \circ (e_1, Q) - F \circ (e_0, Q) + F \circ (e_0, e_1) \\ &\quad - F \circ (e_2, Q) + F \circ (e_0, Q) + f \circ (e_0, e_2) \\ &= x_0 + \gamma - \tau - x_0 = \gamma - \tau\end{aligned}$$

This proves that γ and τ are homologuous.

To prove the second claim, we need to define a singular 2-simplex σ with

$$\partial\sigma = \gamma + \tau - \tau\gamma.$$

Consider the triangle

where $T = (1/2, 0)$. If $Q \in \Delta_2$ lies on or below the line Σ joining e_1 with T consider the line joining e_0 with Q parametrized linearly, such that $t = 0$ at e_0 and $t = 1$ at the intersection point with Σ, and set $\sigma(Q) := \gamma_1(t)$.

Analoguously, if Q lies on or above Σ, consider the line joining e_2 with Q and parametrize it in such a way that $t = 1$ at e_2 and $t = 0$ at the intersection with Σ and set $\sigma(Q) = \tau(t)$.

Then

$$\begin{aligned}\partial(\sigma) &= \sigma \circ (e_1, e_2) - \sigma \circ (e_0, e_2) + \sigma \circ (e_0, e_1) \\ &= \tau - \gamma\tau + \gamma.\end{aligned}$$

Thus $\chi([\gamma][\tau]) = \chi([\gamma]) + \chi([\tau])$. □

Now we want to see that χ is surjective and the kernel of χ is the commutator subgroup of $\pi_1(X)$.

Definition 3.2. The commutator subgroup $C(G)$ of a group G is the subgroup generated by all $ghg^{-1}h^{-1}$ with $g, h \in G$. One easily shows that $C(G)$ is a normal subgroup of G and $G/C(G)$ is abelian, called the abelianization of G. By definition it is clear that, if $\pi : G \to H$ is a homomorphism to an abelian group, then $ker(\pi) \supset C(G)$.

It turns out that $H_1(X)$ is the abelianization of $\pi_1(X)$.

Theorem 3.3. *Let X be path-connected Then $\chi : \pi_1(X, x_0) \to H_1(X)$ is surjective with kernel the commutator subgroup of $\pi_1(X, x_0)$.*

Proof. Let $z = \sum n_i \sigma_i$ be a 1-cycle, i.e.

$$0 = \partial z = \sum n_i(\sigma_i(1) - \sigma_i(0)).$$

In this sum, some of the points can coincide, the sum of the coefficients corresponding to every point must vanish.

Choose paths α_i from x_0 to $\sigma_i(0)$ and β_i from x_0 to $\sigma_i(1)$, where the paths only depend of the points (i.e. if some points concide, then also the paths are the same).

Then

$$0 = \sum n_i(\beta_i - \alpha_i),$$

because the paths correspond precisely to the points. Now put

$$\overline{\sigma}_i := \alpha_i + \sigma_i - \beta_i.$$

Then

$$z = \sum n_i \sigma_i = \sum n_i \overline{\sigma}_i.$$

Let γ_i be the loop $\alpha_i \sigma_i \beta_i^{-1}$ Then

$$\chi\big([\prod n_i \gamma_i]\big) = [\sum n_i \overline{\sigma}_i] = [z].$$

Thus χ is surjective.

Finally we have to determine the kernel of χ. First, since $H_1(X)$ is abelian, $ker(\chi)$ contains the commutator subgroup. We need to show that every element of $ker(\chi)$ lies in the commutator subgroup. The proof is not difficult, but a little bit confusing.

Let γ be a loop, which is a 1-boundary, i.e.

$$\gamma = \partial\Big(\sum n_i \sigma_i\Big)$$

Write $\partial\sigma_i = (\gamma_{0,i} - \gamma_{1,i} + \gamma_{2,i})$ for paths $\gamma_{0,i}, \gamma_{1,i}, \gamma_{2,i}$. It means that one of the $\gamma_{i,j}$, say of the γ_{i_0,j_0} is equal to γ and occurs in the sum with coefficient 1. After collecting terms, all other paths occur with coefficient 0.

Choose paths

$$\begin{aligned}
&\alpha_{0,i} \text{ from } x_0 \text{ to } \gamma_{2,i}(0) = \gamma_{1,i}(0)\\
&\alpha_{1,i} \text{ from } x_0 \text{ to } \gamma_{0,i}(0) = \gamma_{2,i}(1)\\
&\alpha_{2,i} \text{ from } x_0 \text{ to } \gamma_{1,i}(1) = \gamma_{0,i}(1)
\end{aligned}$$

again the paths should only depend on the points, i.e. if some points coincide we take the same paths, and if the point is x_0 we choose the constant path x_0. Consider the loops

$$\begin{aligned}
\beta_{0,j} &:= \alpha_{1,j}\gamma_{0,j}\alpha_{2,j}^{-1}\\
\beta_{1,j} &:= \alpha_{0,j}\gamma_{1,j}\alpha_{2,j}^{-1}\\
\beta_{2,j} &:= \alpha_{0,j}\gamma_{2,j}\alpha_{1,j}^{-1}
\end{aligned}$$

Note that $\beta_{i_0,j_0} \simeq \gamma_{i_0,j_0} = \gamma$. Note that the loops

$$\beta_j := \beta_{0,j}\beta_{1,j}^{-1}\beta_{2,j} = \alpha_{1,j}\gamma_{0,j}\gamma_{1,j}^{-1}\gamma_{i,2}\alpha_{1,j}^{-1}$$

are homotopic to the constant loop x_0 since the image of a 1-simplex is contractable.

Hence $\prod_i [\beta_i]^{n_i} = 1$ in $\pi_1(X, x_0)$. Let $\pi_1' := \pi_1(X, x_0)/C(\pi_1(X, x_0))$, and let $\widetilde{\beta}_i$ be the image of $[\beta_i]$ in π_1'. We use additive notation for π_1'. Then $\sum_i n_i\widetilde{\beta}_i = 1$. Since π_1 is commutative, we can collect terms and get

$$\begin{aligned}\sum_i n_i\widetilde{\beta}_i^{n_i} &= \sum_i n_i(\widetilde{\beta}_{0,i} - \widetilde{\beta}_{1,i} + \widetilde{\beta}_{2,i}) \\ &= \widetilde{\gamma}\end{aligned}$$

Note that the fact that the $\alpha_{i,j}$ except $\alpha_{i_0,j_0} = \gamma$ cancel out in the first sum is equivalent to the $\beta_{i,j}$ except $\beta_{i_0,j_0} \simeq \alpha_{i_0,j_0} = \gamma$ canceling out. Thus $\gamma \in C(\pi_1(X, x_0))$. □

Corollary 3.4. *If X is simply connected, then $H_1(X) = 0$.*

Corollary 3.5.

$$H_1(S^n) = \begin{cases} \mathbb{Z} & n = 1, \\ 0 & n > 1 \end{cases}$$

Corollary 3.6. $H_1(X \times Y) = H_1(X) \oplus H_1(Y)$.

4. Relative Homology

We have to extend the notion of homology to pairs $A \subset X$ of topological spaces.

Definition 4.1. A *pair* of topological spaces is a topological space together with a subspace $A \subset X$ (subspace means that $A \subset X$ and that A carries the induced topology). A *morphism* $f : (X, A) \to (Y, B)$ of pairs is a continuous map $f : X \to Y$ with $f(A) \subset B$.

Now we want to define relativ homology groups $H_i(X,A)$ for pairs (X,A).

DEFINITION 4.2. The relative i-chains are

$$S_n(X,A) := S_n(X)/S_n(A),$$

and we define a boundary map

$$\bar{\partial} : S_n(X,A) \to S_{n-1}(X,A), [a] \mapsto [\partial a].$$

Thus the diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{\pi} & S_n(X,A) \\ \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \bar{\partial}} \\ S_{n-1}(X) & \xrightarrow{\pi} & S_{n-1}(X,A) \end{array}$$

where the horizontal maps are the quotient maps, commutes. As $\partial(S_n(A)) \subset S_{n-1}(A)$, $\bar{\partial}$ is well-defined. As $\partial \circ \partial = 0$ it follows that $\bar{\partial} \circ \bar{\partial} = 0$. Thus

$$S_*(X,A) := (S_0(X,A) \xleftarrow{\bar{\partial}} S_1(X,A) \xleftarrow{\bar{\partial}} \dots)$$

is a chain complex. The *relative homology groups*

$$H_n(X,A) := Ker(\bar{\partial}|_{S_n(X,A)})/Im(\bar{\partial}|_{S_{n-1}(X,A)})$$

are the homology groups of this complex.

We can describe the relative homology also directly in terms of $S_n(X)$: The group of relative n-cycles modulo A is

$$Z_n(X,A) := \{c \in S_n(X) \mid \partial c \in S_{n-1}(A)\}.$$

The group of relative n-boundaries modulo A is

$$B_n(X,A) := \partial(S_{n+1}(X)) + S_n(A) \subset S_n(X).$$

Then by the isomorphism theorem

$$(M/P)/(N/P) = M/N$$

we get

$$H_n(X,A) = Z_n(X,A)/B_n(X,A).$$

EXAMPLE 4.3. A path $\sigma : I \to X$ is a relative 1-cycle if modulo A if its end points lie in A.

REMARK 4.4. If A is empty, then $S_n(A) = 0$ for all n, thus $H_n(X, \emptyset) = H_n(X)$. Thus the usual homology is a special case of the relative homology.

REMARK 4.5. Relative homology is functorial in the pair (X, A), i.e. if $f : (X, A) \to (Y, B)$ is a morphism of pairs, then we have a homomorphism

$$f_* : H_n(X, A) \to H_n(Y, B)$$

such that

$$id_{(X,A)*} = id_{H_n(X,A)}, \qquad (g \circ f)_* = g_* \circ f_*.$$

The proof is the same as before. For $\sum a_i \sigma_i \in Z_n(X, A)$ we put

$$f_*\big(\big[\sum a_i \sigma_i\big]\big) := \big[\sum a_i f \circ \sigma_i\big].$$

EXAMPLE 4.6. In particular we have the inclusion map $j : (X, \emptyset) \to (X, A)$ which is the identity on X. giving

$$j_* : H_n(X) \to H_n(X, A).$$

PROPOSITION 4.7. *Let $(X_k)_k$ be the family of path-components of X, and put $A_k = X_k \cap A$. Then for all $n \geq 0$,*

$$H_n(X, A) \simeq \sum_k H_n(X_k, A_k).$$

PROOF. The proof is the same as for $H_n(X)$. $\square$

PROPOSITION 4.8. *If A is nonempty and X is path connected, then $H_0(X, A) = \emptyset$*

PROOF. Choose $x_0 \in A$. For each $x \in X$ let α_x be a path from x_0 to x. For $c = \sum \nu_x x \in S_0(X)$, we get

$$\partial\Big(\sum \nu_x \alpha_x\Big) = c - \sum \nu_x x_0.$$

Thus $c \in B_0(X, A)$. $\square$

It follows that in general $H_0(X,A)$ is the free abelian group on the connected components of X which do not meet A.

Remark 4.9. Two maps $f,g:(X,A)\to(Y,B)$ are called homotopic if they are homotopic as maps $X\to Y$ by a homotopy F with $F(A\times I)\subset B$. It is easy to modify the above proof of homotopy invariance of homology to show that if $f,g:(X,A)\to(Y,B)$ are homotopic, then

$$f_*=g_*:H_n(X,A)\to H_n(Y,B).$$

5. Long exact homology sequence

We are not really interested very much in the relative homology groups in their own right, we want to use them as a tool to compute ordinary homology groups. Thus the most important property of the relative homology groups $H_n(X,A)$ is the long exact homology sequence, which relates them to the usual homology groups $H_n(X)$ and $H_n(A)$.

First we make a short interlude in homological algebra.

Definition 5.1. A sequence $A_0\xrightarrow{f_0}A_1\xrightarrow{f_1}\dots\xrightarrow{f_{n-1}}A_n$ of homomorphisms of abelian groups is called exact at A_i if $Ker(f_i)=Im(f_{i-1})$. It is called exact if it is exact at $A_1,\dots,A_{n-1}$.

In particular

$$0\longrightarrow A\xrightarrow{f}B\xrightarrow{g}C\longrightarrow 0$$

is exact (called a short exact sequence) if f is injective g is surjective and $Ker(g)=Im(f)$.

For instance the sequence

$$0\to\mathbb{Z}\xrightarrow{\cdot n}\mathbb{Z}\longrightarrow\mathbb{Z}/(n)\longrightarrow 0$$

is exact.

Note that by definition a chain complex (with $S_{-1}=0$) (S_*,∂) is exact if and only if all homology groups vanish. Thus the homology measures the failure of exactness.

Definition 5.2. Let $S_*\xrightarrow{F}T_*\xrightarrow{G}U_*$ be morphisms of chain complexes. The sequence

$$0\longrightarrow S_*\xrightarrow{F}T_*\xrightarrow{G}U_*\longrightarrow 0$$

is called a short exact sequence of chain complexes if for all i

$$0 \longrightarrow S_i \xrightarrow{F} T_i \xrightarrow{G} U_i \longrightarrow 0$$

is a short exact sequence of abelian groups.

We have the following fundamental fact which lies on the basis of many results about homology and cohomology:

THEOREM 5.3. *Let*

$$0 \longrightarrow S_* \xrightarrow{F} T_* \xrightarrow{G} U_* \longrightarrow 0$$

be a short exact sequence of chain complexes. Then there exists a long exact homology sequence

$$\ldots \xrightarrow{\delta} H_n(S_*) \xrightarrow{F_*} H_n(T_*) \xrightarrow{G} H_n(U_*) \xrightarrow{\delta} H_{n-1}(S_*) \xrightarrow{F_*} H_{n-1}(T_*) \xrightarrow{G} H_{n-1}(U_*) \xrightarrow{\delta}$$
$$\ldots \xrightarrow{\delta} H_0(S_*) \xrightarrow{F_*} H_0(T_*) \xrightarrow{G} H_0(U_*) \longrightarrow 0$$

PROOF. This is proven by a technique called diagram chasing, i.e. one takes elements in one of the groups and maps them either forward by one of the maps one has, or takes a preimage under one of the maps, until they do what you want. At any given step one only has a small number of choices, thus given enough patience one can usually find the proof for a given true statement.

First we want to construct $\delta : H_n(U_*) \to H_{n-1}(S_*)$. I write $Z_n(U)$ for $Ker(\partial|_{U_n})$, $B_{n-1}(U) := Im(\partial|_{U_n})$ and similarly for S_*, T_*.

(0) Let $[a] \in H_n(U_*)$ for $a \in Z_n(U)$. We need to construct $\delta([a]) := [c]$ for $c \in Z_{n-1}(S)$. We do this according to the diagram, where the horizontal maps are F, G and the vertical maps are ∂ and the numbers refer to the order in which the elements are constructed.

$$\begin{array}{cccccc}
 & S & & T & & U \\
 & & & & & \\
n & & & b_{(1)} & \xrightarrow{G} & a_{(0)} \\
 & & & \downarrow \partial & & \downarrow \partial \\
n-1 & c_{(3)} & \xrightarrow{F} & \partial b_{(2)} & \xrightarrow{G} & 0 \\
 & \downarrow \partial & & & & \\
 & 0 & & & &
\end{array}$$

(1) There exists $b \in T_n$ with $G(b) = a$ (because G is surjective).

(2) We have $\partial b \in T_{n-1}$. Note that

$$G(\partial b) = \partial G(b) = \partial a = 0.$$

(3) Thus by the exactness of the sequence, there exists a unique $c \in S_{n-1}$ with $F(c) = \partial b$. Furthermore

$$F(\partial c) = \partial F(c) = \partial\partial b = 0.$$

As F is injective we get $\partial c = 0$, thus $c \in Z_{n-1}(S)$, and we put $\delta([a]) = [c] \in H_{n-1}(S_*)$.

EXERCISE 5.4. Check that this is independent of the choice of b.

We now need to check exactness of the sequence. We check it at U_n and leave the rest as an exercise. First we check $\delta \circ G_* = 0$. Let $[b] \in H_n(T_*)$ with $b \in Z_n(T)$. Then for $[a] = G_*([b])$ we can take b as the b in the above definition of c. Thus $\partial b = 0$ and therefore $c = 0$, i.e. $\partial G_*([b]) = 0$.

Now let $[a] \in H_n(U)$ with $a \in Z_n(U)$ and $\delta([a]) = 0$. We need to find $b' \in Z_n(T)$ with $G_*([b']) = [a]$. That $\delta([a]) = 0$ means that in the above diagram

$$\begin{array}{ccccc} d & & b & \xrightarrow{G} & a \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\ c & \longrightarrow & \partial b & & \end{array}$$

we have $c \in B_{n-1}(S)$, i.e. there exists $d \in S_n$ with $c = \partial d$. Put $b' = b - F(d)$. Then $G(b') = G(b) = a$ (because $G \circ F = 0$) and

$$\partial b' = \partial b - \partial F(d) = \partial b - \partial b = 0$$

Therefore $b' \in Z_n(T)$ and $[a] = G_*([b'])$. □

COROLLARY 5.5. *(long exact cohomology sequence) Let $A \subset X$ be a subspace. There is a long exact homology sequence*

$$\ldots \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{j_*} H_{n-1}(X,A) \xrightarrow{\delta}$$
$$\ldots \xrightarrow{\delta} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X,A) \longrightarrow 0$$

where i_* is the pushforward by the inclusion $A \to X$ and j_* is the pushforward by the inclusion $X \to (X,A)$.

Proof. By definition we have that the maps $i_* : S_*(A) \to S_*(X)$ and $j_* : S_*(X) \to S_*(X, A)$ commute with ∂, i.e. they are maps of complexes. By definition i_* is injective and $S_n(X,A) = S_n(X)/S_n(A)$, i.e

$$0 \to S_*(A) \xrightarrow{i_*} S_*(X) \xrightarrow{j_*} S_*(X,A) \to 0$$

is an exact sequence of complexes. □

Exercise 5.6. Show that the exact homology sequence is functorial in (X, A), i.e. if $f : (X, A) \to (Y, B)$ is a map of pairs, then the diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow \ldots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\ldots \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow \ldots
\end{array}$$

is commutative.

Example 5.7. Let

$$E^n := \left\{ x \in \mathbb{R}^n \;\middle|\; |x| \le 1 \right\}$$

be the n-ball and

$$S^{n-1} := \left\{ x \in \mathbb{R}^n \;\middle|\; |x| = 1 \right\}$$

the $(n-1)$-sphere. E^n is contractable, therefore $H_k(E^n) = 0$ for $k > 0$ and by the exact homology sequence

$$\partial : H_k(E^n, S^{n-1}) \to H_k(S^{n-1})$$

is an isomorphism for all $k > 1$. We have an exact sequence

$$0 \to H_1(E^n, S^{n-1}) \to H_0(S^{n-1}) \to H_0(E^n) \to 0$$

If $n > 1$ the last two groups are $\mathbb{Z}$ and thus $H_1(E^n, S^{n-1}) = 0$. One also sees $H_1(E^1, S^0) = \mathbb{Z}$.

The exact homology sequence will be useful for the computation of homology groups, once we have a method for computing the relative homology. This we will introduce in the next section.

6. The Excision Theorem

Let X be a topological space and $A \subset X$ a subspace. The relative homology $H_n(X, A)$ has mostly been introduced as a tool to compute the usual homology $H_n(X)$. This obviously only can work if we have a method for computing relative homology. The most important result for doing this is the excision theorem. It says that certain subspaces $U \subset A$ can be cut out (excised) without effecting the relative homology.

Definition 6.1. The inclusion map $j : (X \setminus U, A \setminus U) \to (X, A)$ is called an excision if it induces an isomorphism $j_* : H_k(X \setminus U, A \setminus U) \to H_k(X, A)$ for all $k \geq 0$. In this case we say that U can be excised.

Theorem 6.2. *(Excision) Assume that the closure $\overline{U}$ of U in X is a subset of the interior A^o of A. Then U can be excised.*

Definition 6.3. Let $(Y, B) \subset (X, A)$. (Y, B) is called a deformation retract of (X, A), if there is a retraction $r : (X, A) \to (Y, B)$, such that for the inclusion $i : (Y, B) \to (X, A)$, $i \circ r$ is homotopic to the identity on (X, A).

Theorem 6.4. *Let $V \subset U \subset A$ and assume*

(1) *V can be excised,*
(2) *$(X \setminus U, A \setminus U)$ is a deformation retract of $(X \setminus V, A \setminus V)$.*

Then U can be excised.

Proof. Let $i : (X \setminus U, A \setminus U) \to (X \setminus V, A \setminus V)$ be the inclusion. Then $i_* : H_k(X \setminus U, A \setminus U) \to H_k(X \setminus V, A \setminus V)$ is an isomorphism for all k. Since homology is a functor and V can be excised, also U can. □

Before proving the excision theorem, we give some applications.

Proposition 6.5. *Let E_n^+, E_n^- be the closed northern and southern hemisphere of S^n, $n \geq 1$ (thus $E_n^+ \cap E_n^-$ is the equator S^{n-1}. Then*

$$(E_n^+, S^{n-1}) \to (S_n, E_n^-)$$

is an excision.

PROOF. Let

$$V := \{x \in S^n \mid x_{n+1} < -1/2\}.$$

Then by the excision theorem V can be excised. We see that (E_n^+, S^{n-1}) is a deformation retract of $(S^n \setminus V, E_n^- \setminus V)$ (move up along great circles). Therefore $U := \{x \in S^n \mid x_{n+1} < 0\}$ can be excised.

$\square$

Now we can compute the homology of S^n.

COROLLARY 6.6. *For all $n \geq 1$ we have*

$$H_k(S^n) = \begin{cases} \mathbb{Z} & q = n \text{ or } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$H_k(E_n, S^{n-1}) = \begin{cases} \mathbb{Z} & q = n \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Projecting on the first n coordinates gives a homeomorphism $(E_n^+, S^{n-1}) \to (E^n, S^{n-1})$. We have seen that the connecting homomorphism $H_k(E^n, S^{n-1}) \to H_{k-1}(S^{n-1})$ is an isomorphism for $k > 1$. As E_n^- is contractible, the long exact homology sequence gives that $H_k(S^n) \to H_k(S^n, E_n^-)$ is an isomorphism. Combining this with the isomorphism $H_k(E^n, S^{n-1}) \to H_k(S^n, E^n)$ of the proposition we get

$H_k(S^n) = H_k(S^n, E_n^-) = H_k(E^n, S^{n-1}) = H_{k-1}(S^{n-1})$ for all $k > 1$, $n > 0$. So we have for $k > 1$,$n > 0$ that $H_k(S^n) = H_{k-l}(S^{n-l})$ and $H_k(E_n, S^{n-1}) = H_{k-1}(S^{n-1})$. As $H_k(S^0) = 0$ for $k > 0$, it follows that $H_k(S^n) = 0$ for $k > n$. If $0 < k \leq n$, then $H_k(S^n) = H_k(E^n, S^{n-1}) = H_1(S^{n-k+1})$. Therefore it suffices to compute $H_1(E_n, S^{n-1})$ and $H_1(S^n)$.

We have shown

$$H_1(E^n, S^{n-1}) = \begin{cases} 0 & n > 1 \\ \mathbb{Z} & n = 1 \end{cases}$$

We have an exact sequence

$$0 \to H_1(S^n) \xrightarrow{a} H_1(S^n, E_n^-) \xrightarrow{b} H_0(E_n^-) \xrightarrow{c} H_0(S^n) \to 0.$$

As c is an isomorphism, b must be 0 and therefore a is an isomorphism. So

$$H_1(S^n) = H_1(S^n, E_n^-) = H_1(E^n, S^{n-1}) = \begin{cases} 0 & n > 1, \\ \mathbb{Z}. & n = 1 \end{cases}$$

□

Now we can prove the general version of Brouwers fixed point theorem. The proof is the same as in the $n = 2$ case.

Corollary 6.7. *S^{n-1} is not a retract of S^n.*

Proof. Suppose $f : E^n \to S^{n-1}$ is a continuous map whose restriction to S^{n-1} is the identity, then we get

$$H_{n-1}(S^{n-1}) \to H_{n-1}(E^n) \to H_{n-1}(S^{n-1}),$$

such that the composition is the identity. Thus for $n > 1$ this is

$$\mathbb{Z} \to 0 \to \mathbb{Z},$$

which is impossible. (Exercise: do the case $n = 1$.) □

Theorem 6.8. *(Brouwers Fixed point theorem) Any continuous map $E^n \to E^n$ has a fixed point.*

Proof. The proof is the same as before. □

In order to prove the theorem we will have to show that we can compute $H_*(X, A)$ with simplices which lie either entirely in $X \setminus U$ or in A^o.

Definition 6.9. Let $U := (U_i)_i$ be an open cover of a topological space X. For each $k \geq 0$ let

$$S_k^{\mathcal{U}}(X) := \left\{ \sum a_\sigma \sigma \in S_k(X) \;\middle|\; \forall_k \exists_i : \sigma(\Delta_k) \subset U_i \right\}.$$

These are the linear combinations of simplices whose image is contained in one of the open sets of the cover. Then

$$S_*^{\mathcal{U}}(X) = (S_0^{\mathcal{U}}(X) \xleftarrow{\partial} S_1^{\mathcal{U}}(X) \xleftarrow{\partial} \dots)$$

is a sub-chain complex of $S_*(X)$. We define $S_k^{\mathcal{U}}(X, A) = S_k^{\mathcal{U}}(X)/S_k^{\mathcal{U}}(A)$. The inclusions $S_k^{\mathcal{U}}(X) \to S_k(X)$ and $S_k^{\mathcal{U}}(A) \to S_k(A)$ induce maps $i : S_k^{\mathcal{U}}(X, A) \to S_k(X, A)$. Thus we get the chain complex $S_*^{\mathcal{U}}(X, A)$ with a chain map $i : S_*^{\mathcal{U}}(X, A) \to S_*(X, A)$.

Theorem 6.10. *(of cover) $S_*^{\mathcal{U}}(X) \to S_*(X)$, $S_*^{\mathcal{U}}(A) \to S_*(A)$, $S_*^{\mathcal{U}}(X, A) \to S_*(X, A)$ induce isomorphisms on homology for all $k \geq 0$.*

Proof. (of Excision Theorem from Theorem of cover) Consider the cover $\mathcal{U} := \{X \setminus \overline{U}, A^o\}$ of X. Then $S_*^{\mathcal{U}}(X, A) = S_*^{\mathcal{U}}(X)/S_*^{\mathcal{U}}(A)$. We have $S_k^{\mathcal{U}}(X) = S_k(X \setminus U) + S_k(A^o)$. We have

$$S_k^{\mathcal{U}}(X)/S_k^{\mathcal{U}}(A) = \frac{S_k(X \setminus U) + S_k(A^o)}{S_k(A \setminus U) + S_k(A^o)} = S_k(X \setminus U)/S_k(A \setminus U).$$

Thus we obtain by the theorem of cover

$$H_k(X, A) = H_k(S_*^{\mathcal{U}}(X, A)) = H_k(S_*(X \setminus U, A \setminus U)) = H_k(X \setminus U, A \setminus U).$$

$\square$

In order to prove the theorem of the cover we need to subdivide simplices into smaller simplices. We do this by baricentric subdivision.

Definition 6.11. Let $p_0, \dots, p_k \in \mathbb{R}^n$. Let $\sigma := (p_0, \dots, p_k)$ be the corresponding singular k-simplex. Let $B \in \mathbb{R}^n$. The *join* $B * \sigma$ is defined as

$$B * \sigma := (B, p_0, \dots, p_k) : \Delta_{k+1} \to \mathbb{R}^n.$$

We extend this by linearity to a map

$$B* : S_k(\mathbb{R}^n) \to S_{k+1}(\mathbb{R}^n);\; B * \Big(\sum_\sigma a_\sigma \sigma \Big) = \sum_\sigma a_\sigma B * \sigma.$$

Note that for $c = \sum a_i \sigma_i \in S_k(\mathbb{R}^n)$ we have

$$\partial(B * c) = \begin{cases} c - B * (\partial c) & k > 0, \\ c - \sum_i a_i B & k = 0. \end{cases}$$

We want to define subdivision operators

$$Sd : S_n(X) \to S_n(X)$$

which are functorial, i.e. if $f : X \to Y$ is a continuous map, then the diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{Sd} & S_n(X) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ S_n(Y) & \xrightarrow{Sd} & S_n(Y) \end{array}$$

commutes. Thus it suffices to define $Sd(\delta_n)$, because for $\sigma : \Delta_n \to X$ a singular simplex, we can put

$$Sd(\sigma) := S_n(\sigma_*(\delta_n)) = \sigma_*(Sd(\delta_n)).$$

We inductively define

$$Sd(\delta_0) = \delta_0$$
$$Sd(\delta_n) = B_n * Sd(\partial \delta_n)$$

Here B_n is the baricenter of Δ_n:

$$B_n := \sum_{i=0}^{n} \frac{1}{n+1} e_i.$$

LEMMA 6.12. *$\partial Sd = Sd \partial$.*

PROOF. It is enough to show $\partial Sd(\delta_n) = Sd\partial(\delta_n)$. We prove this by induction on n. The case $n = 0$ is obvious. Now

$$\partial Sd(\delta_n) = \partial(B_n * Sd(\partial\delta_n)) = Sd(\partial\delta_n) - B_n * \partial(Sd(\partial\delta_n)).$$

By induction

$$B_n * \partial Sd\partial\delta_n = B_n * Sd(\partial^2\delta_n) = 0.$$

□

PROPOSITION 6.13. *Sd and $id_{S_*(X)}$ are chain homotopic.*

PROOF. Define $T : S_n(X) \to S_{n+1}(X)$ again to be functorial, i.e. so that for all $f : X \to Y$ the diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{T} & S_{n+1}(X) \\ \downarrow f_* & & \downarrow f_* \\ S_n(Y) & \xrightarrow{T} & S_{n+1}(Y) \end{array}$$

commutes. Thus it is again enough to define $T(\delta_n)$. We put $T(\delta_0) = 0$ and inductively

$$T(\delta_n) := B_n * (\delta_n - Sd\delta_n - T\partial\delta_n).$$

Claim:

$$\partial T + T\partial = Id - Sd.$$

Again it is enough to show this for δ_n. We do this by induction on n. By induction

$$\partial T\partial\delta_n = (Id - Sd - T\partial)(\partial\delta_n) = \partial\delta_n - \partial Sd\delta_n.$$

Thus we get

$$\begin{aligned} \partial T\delta_n &= \partial B(\delta_n - Sd\partial\delta_n - T\partial\delta_n) \\ &= \delta - Sd\delta - T\partial\delta - B(\partial\delta_n - \partial Sd\delta_n - \partial T\partial\delta_n) \\ &= \delta_n - Sd\delta_n - T\partial\delta_n. \end{aligned}$$

□

REMARK 6.14. (1) Note that the operators Sd, T send $S_*^{\mathcal{U}}(X)$ into itself.

(2) By restriction we also have the corresponding operators $Sd, T : S_*(A) \to S_*(A)$. Thus by passing to the quotient we get $Sd, T : S_*(X, A) \to S_*(X, A)$ which again send $S_*^{\mathcal{U}}(X, A)$ to itself.

(3) It also follows that on $S_*(X, A)$ we have

$$\partial Sd = Sd\partial$$
$$\partial T + T\partial = Id - Sd.$$

Let $\sigma = (p_0, \dots, p_k)$ be a singular simplex in $\mathbb{R}^n$. Then $\sigma(\delta_k)$ is a compact set. Let $d(\sigma)$ be its diameter.

(Exercise: $d(\sigma)$ is the maximum of the length of the edges of σ.)

LEMMA 6.15. *Each singular simplex appearing in the k-chain $Sd(\sigma)$ has diameter at most $\frac{kd(\sigma)}{k+1}$.*

PROOF. Exercise. □

PROPOSITION 6.16. *Let σ be a singular n-simplex in X, let $\mathcal{U} := (U_i)_i$ be an open cover of X. Then there is an $r > 0$ such that $Sd^r\sigma \in S_n^{\mathcal{U}}(X)$.*

PROOF. Since Δ_n is compact, there is an $\epsilon > 0$ such that σ maps an ϵ-neighbourhood of any point in Δ_n into one of the U_i (exercise). Since

$$\lim_{r\to\infty} \frac{n^r}{(n+1)^r} = 0,$$

there is an $r > 0$ such that $Sd^r\delta_n$ is a linear combination

$$Sd^r\delta_n = \sum_i a_i\tau_i$$

of simplices τ_i of diameter smaller then ϵ. But then

$$Sd^r\sigma = \sum_i a_i\sigma\circ\tau_i \in S_n^{\mathcal{U}}(X).$$

□

PROOF. (of the theorem of cover). Let $Y = X$ or $Y = A$ or $Y = (X, A)$. We have to show that the canonical map $j : H_n(S_n^{\mathcal{U}}(Y)) \to H_n(Y)$ is an isomorphism.

Surjectivity: Let $[c] \in H_n(Y)$ represented by $c \in S_i(Y)$ with $\partial c = 0$. For r sufficiently large we have $Sd^r(c) \in S_n^{\mathcal{U}}(Y)$ and $\partial Sd^r(c) = Sd^r(\partial c) = 0$. Putting $b := \sum_{i=0}^{r-1} Sd^ic$, we get $\partial b = 0$ and

$$c - Sd^r(c) = (Id - Sd)b = (\partial T + T\partial)b = \partial Tb.$$

Thus $[c] = [Sd^r(c)]$ and $[Sd^r(c)]$ is in the image of j.

Injectivity: Let $a \in S_n^{\mathcal{U}}(Y)$, with $j([a]) = 0$. This means $a = \partial b$ with $b \in S_{n+1}(Y)$. Then $Sd^r(a) = \partial Sd^r(b)$ and for r sufficiently large $Sd^r(b) \in S_{n+1}^{\mathcal{U}}(Y)$. Finally in $S_n^{\mathcal{U}}(Y)$ we have that $a - Sd^r(a)$ is a boundary (as in the proof of the surjectivity). Thus a is a boundary in $S_n^{\mathcal{U}}(Y)$. Thus j is injective. □

7. Mayer-Vietoris sequence

The most useful tool for computing the homology of a space is the Mayer-Vietoris sequence. Under suitable assumptions it allows to compute the homology of $X \cup Y$ in terms of that of X, that of Y and that of $X \cap Y$.

DEFINITION 7.1. A triple (X, X_1, X_2) of a topological space and subspaces $X_1, X_2 \subset X$ with $X = X_1 \cup X_2$ is called an *exact triad* if the inclusion map $(X_1, X_1 \cap X_2) \to (X, X_2)$ is an excision (i.e. induces an isomorphism on homology).

Main example: If $X = X_1 \cup X_2$ and X_1 and X_2 are open in X, then (X, X_1, X_2) is an exact triad. Putting $A = X_1$, $U = X_1 \setminus X_1 \cap X_2$, in the statement of the excision theorem, we see $X \setminus U = X_2$, thus U is a closed subset of X contained in the open A. So the excision theorem applies.

THEOREM 7.2. *(Mayer-Vietoris sequence): Let (X, X_1, X_2) be an exact triad. Denote $i_k : X_k \to X$ and $j_k : X_1 \cap X_2 \to X_k$ the inclusions. Then there is a long exact sequence*

$$\longrightarrow H_n(X_1 \cap X_2) \xrightarrow{j_{1*} \oplus j_{2*}} H_n(X_1) \oplus H_n(X_2) \xrightarrow{(-i_{1*}, i_{2*})} H_n(X) \longrightarrow H_{n-1}(X_1 \cap X_2) \dots$$
$$\longrightarrow H_0(X_1 \cap X_2) \xrightarrow{j_{1*} \oplus j_{2*}} H_0(X_1) \oplus H_0(X_2) \xrightarrow{(-i_{1*}, i_{2*})} H_0(X) \longrightarrow 0$$

Furthermore the sequence is functorial in exact triads.

PROOF. (in the case $X_1, X_2 \subset X$ are open). Let $\mathcal{U}$ be the open cover $\{X_1, X_2\}$ of X. We have an exact sequence of chain complexes

$$0 \longrightarrow S_*(X_1 \cap X_2) \xrightarrow{j_{1*} \oplus j_{2*}} S_*(X_1) \oplus S_*(X_2) \xrightarrow{(-i_{1*}, i_{2*})} S_*^{\mathcal{U}}(X) \longrightarrow 0.$$

That $j_{1*} \oplus j_{2*}$ is injective is obvious, also that $(-i_{1*}, i_{2*})$ is surjective is obvious. Furthermore $(-i_{1*}, i_{2*}) \circ (j_{1*} \oplus j_{2*}) = -j_* + j_* = 0$, where $j : X_1 \cap X_2 \to X$ is the inclusion. Finally if $-i_{1*}(a_1) + i_{2*}(a_2) = 0$ for $a_1 \in S_n(X_1)$, $a_2 \in S_n(X_2)$, it follows that $a_1 = a_2 \in S_n(X_1 \cap X_2)$.

As the homology of $S_*^{\mathcal{U}}(X)$ is equal to $H_n(X)$ the result follows from the long homology sequence associated to an exact sequence of chain complexes. □

For the general case we need a lemma in homological algebra.

LEMMA 7.3. *(Barrat-Whitehead Lemma). Given a commutative diagram of abelian groups and homomorphisms with exact rows*

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & C_{n+1} & \xrightarrow{h_{n+1}} & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & \dots \\
 & & \downarrow{\scriptstyle \gamma_{n+1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \gamma_n} & & \\
 & \longrightarrow & \overline{C}_{n+1} & \xrightarrow{\overline{h}_{n+1}} & \overline{A}_n & \xrightarrow{\overline{f}_n} \dots & \overline{B}_n & \xrightarrow{\overline{g}_n} & \overline{C}_n & \xrightarrow{\overline{h}_n} & \dots
\end{array}$$

in which the γ_i are isomorphisms. There is a long exact sequence

$$\dots \longrightarrow A_n \xrightarrow{\alpha_n \oplus f_n} \overline{A}_n \oplus B_n \xrightarrow{(-\overline{f}_n, \beta_n)} \overline{B}_n \xrightarrow{h_n \gamma_n^{-1} \overline{g}_n} A_{n-1} \quad \dots$$

which is functorial in such diagrams.

PROOF. The proof is by diagram chasing. We show exactness at $\overline{B}_n$.

$$h_n\gamma_n^{-1}\overline{g}_n(-\overline{f}_n, \beta_n)(a,b) = h_n\gamma_n^{-1}\overline{g}_n(-\overline{f}_n(a), \beta_n(b)) = h_n\gamma_n^{-1}\overline{g}_n(\beta_n(b))$$

as $\overline{g}_n\overline{f}_n = 0$. But $\gamma_n^{-1}\overline{g}_n\beta_n = g_n$ and $h_n g_n = 0$. Thus $h_n\gamma_n^{-1}\overline{g}_n(-\overline{f}_n, \beta_n) = 0$.

Now let $b \in \overline{B}_n$ with $h_n\gamma_n^{-1}\overline{g}_n(b) = 0$. There is a $b_1 \in B_n$ with $g_n(b_1) = \gamma_n^{-1}\overline{g}_n(b)$. Then $\overline{g}_n(b) = \gamma_n g_n(b_1) = \overline{g}_n\beta_n(b_1)$. Thus $\overline{g}_n(b - \beta_n(b_1)) = 0$ and there is an $a \in \overline{A}_n$ such that $\overline{f}_n(a) = b - \beta_n(b_1)$. Then $(-\overline{f}_n, \beta_n)(-a, b_1) = b - \beta_n(b_1) + \beta_n(b_1) = b$. □

PROOF. (of the Mayer Vietoris sequence). Apply the Barrat Whitehead lemma to the diagram

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H_n(X_1 \cap X_2) & \xrightarrow{j_{1*}} & H_n(X_1) & \longrightarrow & H_n(X_1, X_1 \cap X_2) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle j_{2*}} & & \downarrow{\scriptstyle i_{1*}} & & \downarrow & & \\
 & \longrightarrow & H_n(X_2) & \xrightarrow{i_{2*}} & H_n(X) & \longrightarrow & H_n(X, X_2) & \longrightarrow & \dots
\end{array}$$

□

Reduced homology. Sometimes in order to avoid case distinctions it is useful to have a slightly modified definition of the 0^{th} homology group.

DEFINITION 7.4. On $S_0(X)$ replace the boundary map ∂ by

$$\partial^{\#}\Big(\sum_x \nu_x x\Big) = \sum \nu_x.$$

Leave the other boundary operators the same as before. It is easy to check that $\partial^{\#}\partial = 0$. The homology of this chain complex is called the *reduced homology* of X, and denoted by $H_n^{\#}(X)$. By definition $H_n^{\#}(X) = H_n(X)$ if $n \neq 0$, and if X is connected, then $H_0^{\#}(X) = 0$. For $A \neq \emptyset$ we put $H_n^{\#}(X, A) := H_n(X, A)$.

Exercise 7.5. If $A \neq \emptyset$, the long exact homology sequence of (X, A) is also exact with reduced homology.

Remark 7.6.

(1) If $U \neq A$ and $(X \setminus U, A \setminus U) \to (X, A)$ is an excision, then $H_n^{\#}(X \setminus U, A \setminus U) = H_n^{\#}(X, A)$ for all n.

(2) If $X_1 \cap X_2$ is nonempty this implies that we also have the Mayer-Vietoris sequence for reduced homology.

Example 7.7. A graph is a space which is a union of finitely many closed arcs (called edges) (i.e. spaces homemorphic to closed intervals), which have at most the endpoints in common.

We want to compute the homology of a graph G. G is homotopically equivalent to an r-leaved rose G_r

(just contract successively all edges which are not loops). So we want to compute the homology of G_r. $G_1 = S^1$, and for $r > 1$ $G_r = S^1 \cup G_{r-1}$ with $S^1 \cap G_{r-1} = \{p\}$. (G_r, G_{r-1}, S^1) is an exact triad, because $(G_{r-1}, p) \to (G_r, S^1)$ is an excision. This is by excising $S^1 \setminus p$. If we excise a slightly smaller open the excision theorem applies, but then (G_{r-1}, p) is a deformation retract.

Thus for $n > 0$ we get an exact sequence

$$0 = H_n(p) \to H_n(S^1) \oplus H_n(G_{r-1}) \to H_n(G_r) \to H^{\#}_{n-1}(p) = 0.$$

Thus $H_n(S^1) \oplus H_n(G_{r-1}) = H_n(G_r)$. So by induction we get

$$H_n(G_r) = \begin{cases} 0 & n > 1, \\ \mathbb{Z}^r & n = 1. \end{cases}$$

EXAMPLE 7.8. The torus $T = S^1 \times S^1$ can be obtained as two annuli both called A identified along their boundary which is the union of two circles C_1, C_2 (i.e. each C_i is identified with C_i in the other copy of A).

The inclusions of the circles into A are homotopy equivalences. Thus the excision theorem implies that $(A, C_1 \sqcup C_2) \to (T, A)$ is an excision. So we get the exact sequence

$$0 \to H_2(T) \to H_1(C_1 \sqcup C_2) \xrightarrow{(j_{1*}, j_{2*})} H_1(A) \oplus H_1(A) \to H_1(T) \to H^{\#}_0(C_1 \sqcup C_2) \to 0.$$

We see that $H_1(C_1 \sqcup C_2) = \mathbb{Z}^2 = H_1(A) \oplus H_1(A)$ and the matrix of (j_{1*}, j_{2*}) is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Thus the kernel has dimension 1 and we get $H_0(T) = H_2(T) = \mathbb{Z}$, $H_1(T) = \mathbb{Z}^2$.

8. Spherical complexes

We introduce a technique for constructing topological spaces and develop a technique for computing the homology groups of spaces constructed this way. Essentially all reasonable spaces can be constructed in this way.

DEFINITION 8.1. Let $A \subset X$ be a subspace of a topological space. Let $f : A \to Y$ be a continuous map. We put $X \cup_f Y := (X \coprod Y)/\sim$ where the equivalence relation is generated by x being equivalent to $f(x)$ for all $x \in A$. We denote by $g : X \coprod Y \to X \cup_f Y$ the quotient map, and by $\overline{f}$ the restriction of g to X. This space is given the quotient topology, i.e. $U \subset X \cup_f Y$ is open if and only if $g^{-1}(U)$ is open.

g maps Y homeopmorphically to a subspace of $X \cup_f Y$, which we identify with Y. Then with our identification $\overline{f}|_A = f$.

We want to study this under some additional assumptions.

DEFINITION 8.2. A pair (X, A) is called a *collared pair* if

(1) X is Hausdorff,
(2) $A \subset X$ is closed,
(3) points in $X \setminus A$ can be separated from A, i.e. for any $x \in X \setminus A$, there are disjoint opens U, V such that $x \in U$ and $A \subset V$,
(4) A has a collaring in X, i.e. there is an open neighbourhood B of A in X, such that A is a strong deformation retract of B (i.e. there exists a homotopy $F : B \times I \to B$ with $F(\bullet, 0) = id_B$, $F|_{A\times I} = id_{A\times I}$, $F(\bullet, 1) : B \to A$.)

EXAMPLE 8.3. (E^n, S^{n-1}) is a collared pair.

If $f : S^{n-1} \to Y$ is a continuous map, we say $E^n \cup_f Y$ is obtained from Y by adjoining an n-cell via f.

PROPOSITION 8.4. *Let Y be a Hausdorff space and let $Z := E_n \cup_f Y$ be obtained by adjoining an n-cell. Then (Z, Y) is a collared pair. In fact if B is a collaring for*

S^{n-1}*, then* $Y \cup \overline{f}(B)$ *is a collaring for* Y*. Moreover* $\overline{f}|_{E^n \setminus S^{n-1}} : E^n \setminus S^{n-1} \to Z \setminus Y$ *is a homeomorphism.*

PROOF. $g^{-1}(Z \setminus Y) = E^n \setminus S^{n-1}$, which is open in Z. Thus Y is closed and $\overline{f} : E^n \setminus S^{n-1} \to Z \setminus Y$ is a homeomorphism.

Let B be a collaring of S^{n-1} in E^n. Since $B \sqcup Y$ is open in $E^n \sqcup Y$ and saturated under the equivalence relation, we get that $Y \cup \overline{f}(B)$ is open in Z. Let $D : B \times I \to B$ be a continuous map with

$$D(a,t) = a \text{ for all } a \in A,$$
$$D(b,0) = b \text{ for all } b \in B,$$
$$D(b,1) \in A \text{ for all } b \in B.$$

Let $\overline{D} : (Y \cup \overline{f}(B)) \times I \to Y \cup \overline{f}(B)$ be given by

$$\overline{D}(z,t) := \begin{cases} z & z \in Y, \\ \overline{f}(D(b,t)) & z = \overline{f}(b), \text{ for } b \in B. \end{cases}$$

Then $\overline{D}$ is continuous because it comes from a continuous map on $(B \sqcup Y) \times I$, and it makes Y into a strong deformation retract of $Y \cup \overline{f}(B)$.

Let $x \in E^n \setminus S^{n-1}$. Let U, V be disjoint opens in E^n such $x \in U$ and $S^{n-1} \subset V$. Then the disjoint opens $Y \cup \overline{f}(V)$ and $\overline{f}(U)$ separate Y from $\overline{f}(x)$.

Let $x, y \in Z$. We have to see they can be separated by open sets. Then there are three possibilities.

(1) $x, y \in Z \setminus Y$. As $Z \setminus Y$ is open and homeomorphic to $E^n \setminus S^{n-1}$, they can be separated by open sets.

(2) $x \in Y$ $y \notin Y$. Then this follows as Y and be separated from y.

(3) $x, y \in Y$. Let Y_1, Y_2 be disjoint opens in Y separating x, y. Let $r : B \to S^{n-1}$ be a retraction and $B_i := r^{-1}(f^{-1}(Y_i))$. Then B_i is open in E^n and $Y_1 \cup \overline{f}(B_1)$, $Y_2 \cup \overline{f}(B_2)$ are disjoint opens separating x and y. □

PROPOSITION 8.5. *Let* Z *be Hausdorff, let* $Y \subset Z$ *be a closed subspace and let* $\overline{f} : E^n \to Z$ *be a map sending* S^{n-1} *to* Y*. Assume that* $\overline{f} : E^n \setminus S^{n-1} \to Z \setminus Y$ *is a homeomorphism. Let* $f := \overline{f}|_{S^{n-1}}$*.*

Then $Z = E^n \cup_f Y$*.*

Proof. $\overline{f} : E^n \to Z$ and the inclusion $Y \to Z$ define a continuous map $\overline{g} : E^n \sqcup Y \to Z$. The induced map $g : E^n \cup_f Y \to Z$ is bijective. Furthermore $\overline{g}|_{E^n}$ is closed because E^n is compact, and $\overline{g}|_Y$ is closed. Thus $\overline{g}$ is closed. Therefore also g is closed and so it is a homeomorphism. $\square$

Definition 8.6. A spherical complex is a topological space which is obtained from a finite set of points by successively attaching finitely many cells (of varying dimensions).

Example 8.7. (Examples of spherical complexes)

(1) Let Y be a point p, then $E^n \cup_f p \simeq S^n$.
(2) Let $Y = S^{n-1}$, $f = id_{S^{n-1}}$, then $E^n \cup_{id} S^{n-1} = E^n$.
(3) Let $Y = S^1$, $f : S^0 \to S^1$ the constant map to a point. Then Z is a two-leaved rose G_2.

Repeating this process we get an r-leaved rose G_r.

(4) Let $Y = G_2$ denote the cycles α, β.

define $f : S^1 \to G_2$ according to the diagram

Then $E^2 \cup_f G_2 \simeq T = S^1 \times S^1$.

Now we want to see that real and complex projective spaces are also spherical complexes.

$\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^n_{\mathbb{R}}$ are the spaces of complex (respectively real) one-dimensional subspaces in $\mathbb{C}^{n+1}$ (respectively $\mathbb{R}^{n+1}$. Points have homogeneous coordinates $(a_0 : \dots : a_n)$, with $a_i \in \mathbb{R}$ (respectively $\mathbb{C}$).

PROPOSITION 8.8. *(1) $\mathbb{P}^n_{\mathbb{C}}$ is obtained from $\mathbb{P}^{n-1}_{\mathbb{C}}$ by attaching a $2n$-cell.*
(2) $\mathbb{P}^n_{\mathbb{R}}$ is obtained from $\mathbb{P}^{n-1}_{\mathbb{R}}$ by attaching an n-cell.

PROOF. In both cases $\mathbb{P}^{n-1}$ can be identified with the set of $(a_0 : \dots : a_{n-1} : 0)$ in $\mathbb{P}^n$, and the complement is homeomorphic to $\mathbb{C}^n$ (respectively $\mathbb{R}^n$). Define a map $\overline{f} : E^{2n} \to \mathbb{P}^n_{\mathbb{C}}$ (respectively $\overline{f} : E^n \to \mathbb{P}^n_{\mathbb{R}}$ by

$$(a_0, \dots, a_{n-1}) \mapsto (a_0, \dots, a_{n-1}, 1 - \|(a_0, \dots, a_{n-1})\|)$$

Here the a_i are complex (respectively real) numbers. Then $\overline{f} : E^n \setminus S^{n-1} \to \mathbb{P}^n \setminus \mathbb{P}^{n-1}$ is a homeomorphism and the result follows by the proposition. $\square$

Now we want to compute the homology of spherical complexes. We will use a very easy but often useful result from homological algebra.

LEMMA 8.9. *(five lemma). Let*

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

be a commutative diagram of abelian groups and homomorphisms in which the rows are exact and the four outer vertical maps are isomorphism. Then α is an isomorphism.

PROOF. The proof is by easy diagram chasing. $\square$

THEOREM 8.10. *Let $Z := E^n \cup_f Y$ be obtained by attaching an n-cell. Then $\overline{f} : (E^n, S^{n-1}) \to (Z, Y)$ induces an isomorphism $\overline{f}_* : H_k(E^n, S^{n-1}) \to H_k(Z, Y)$ for all k.*

PROOF. Let B be a collaring of S^{n-1} in E^n, and let $\overline{B}$ be the corresponding collaring of Y in Z. We have a commutative diagramm

$$\begin{array}{ccc} H_k(E^n, S^{n-1}) & \xrightarrow{i} & H_k(E^n, B) \\ \downarrow f_1 & & \downarrow f_2 \\ H_k(Z, Y) & \xrightarrow{j} & H_k(Z, \overline{B}) \end{array}$$

The horizontal maps are induced by inclusions and the vertical ones by $\overline{f}$. We will show that i, j, f_2 are isomorphisms. This then shows that f_1 is an isomorphism. The commutative diagram

$$\begin{array}{ccc} H_k(E^n \setminus S^{n-1}, B \setminus S^{n-1}) & \longrightarrow & H_k(E^n, B) \\ \downarrow f_3 & & \downarrow f_2 \\ H_k(Z \setminus Y, \overline{B} \setminus Y) & \longrightarrow & H_k(Z, \overline{B}) \end{array}$$

in which the horizontal maps are excisions and f_3 is induced by a homeomorphism, shows that f_2 is an isomorphism. To show that j (resp. i) is an isomorphism, we use that Y (respectively S^{n-1}) is a deformation retract of $\overline{B}$ (respectively B). Thus we get a commutative diagramm with exact rows

$$\begin{array}{ccccccccc} H_k(Y) & \longrightarrow & H_k(Z) & \longrightarrow & H_k(Z, Y) & \longrightarrow & H_{k-1}(Y) & \longrightarrow & H_{k-1}(Z) \\ \downarrow & & \downarrow & & \downarrow j & & \downarrow & & \downarrow \\ H_k(\overline{B}) & \longrightarrow & H_k(Z) & \longrightarrow & H_k(Z, \overline{B}) & \longrightarrow & H_{k-1}(\overline{B}) & \longrightarrow & H_{k-1}(Z) \end{array}$$

in which the outer arrows are isomorphism. Thus j is an isomorphism by the five lemma. $\square$

Finally we can compute how the homology changes under attaching an n-cell.

THEOREM 8.11. *Let $Z = E^n \cup_f Y$ be obtained from Y by attaching an n-cell.*

(1) $H_k(Z) = H_k(Y)$ *for* $k \neq n, n-1$,
(2) $H^{\#}_{n-1}(Z) = H^{\#}_{n-1}(Y)/f_*(H_{n-1}(S^{n-1}))$,
(3) $H^{\#}_n(Z) = H^{\#}_n(Y) \oplus ker(f_* : H_{n-1}(S^{n-1}) \to H_{n-1}(Y))$.

PROOF. We know that $\partial : H_k(E^n, S^{n-1}) \to H^{\#}_{k-1}(S^{n-1})$ is an isomorphism. Using the commutative diagram

$$\begin{array}{ccc} H_k(E^n, S^{n-1}) & \longrightarrow & H^{\#}_{k-1}(S^{n-1}) \\ \downarrow & & \downarrow{\scriptstyle f_*} \\ H_k(Z, Y) & \longrightarrow & H^{\#}_{k-1}(Y) \end{array}$$

we can replace $H_k(Z, Y)$ by $H^{\#}_{k-1}(S^{n-1})$ in the long exact cohomology sequence for (Z, Y). So we get the exact sequence

$$H_k(S^n) \xrightarrow{f_*} H_k(Y) \to H_k(Z) \to H^{\#}_{k-1}(S^{n-1}) \xrightarrow{f_*} H^{\#}_{k-1}(Y) \to H^{\#}_{k-1}(Z).$$

(1) follows because $H^{\#}_{k-1}(S^{n-1}) = 0$ unless $k = n$. The sequence

$$H^{\#}_{n-1}(S^{n-1}) \xrightarrow{f_*} H^{\#}_{n-1}(Y) \to H^{\#}_{k-1}(Z) \to 0$$

implies (2). Finally we have a sequence

$$0 \to H^{\#}_n(Y) \to H^{\#}_n(Z) \to ker(f_*) \to 0.$$

We know that $ker(f_*)$ is isomorphic to 0 or to $\mathbb{Z}$. Therefore the sequence splits and $H^{\#}_n(Z) = H^{\#}_n(Y) \oplus ker(f_*)$. □

COROLLARY 8.12. *If Z is a spherical complex, then $H_k(Z)$ is finitely generated for all k, and if n is the highest dimension of a cell, then $H_k(Z) = 0$ for $k > n$.*

Now we can compute the homology groups of projective spaces.

COROLLARY 8.13.

$$H_k(\mathbb{P}^n_{\mathbb{C}}) = \begin{cases} 0 & k > 2n \text{ or } k \text{ odd,} \\ \mathbb{Z} & k \text{ even, } 0 \le k \le 2n \end{cases}$$

PROOF. The proof is by induction on n, $\mathbb{P}^0_{\mathbb{C}}$ is a point. $\mathbb{P}^n_{\mathbb{C}}$ is obtained from $\mathbb{P}^{n-1}_{\mathbb{C}}$ by attaching a $2n$-cell. Therefore $H_k(\mathbb{P}^n_{\mathbb{C}}) = H_k(\mathbb{P}^{n-1}_{\mathbb{C}})$ for $k \neq n-1, n$. Note that $H_{2n-1}(\mathbb{P}^{n-1}_{\mathbb{C}}) = 0$. Therefore f_* is the zero map. Thus $H_{2n-1}(\mathbb{P}^n_{\mathbb{C}}) = H_{2n-1}(\mathbb{P}^{n-1}_{\mathbb{C}}) = 0$ and $H_{2n}(\mathbb{P}^n_{\mathbb{C}}) = H_{2n}(\mathbb{P}^{n-1}_{\mathbb{C}}) \oplus \mathbb{Z} = \mathbb{Z}$. □

The argument shows that if a sperical complex Z has only cells of even dimensions, then $H_{2k-1}(Z) = 0$ for all k and $H_{2k}(Z) = \mathbb{Z}^r$ where r is the number of $2k$-cells in Z. There is a number of cases from algebraic geometry, like Grassmanians and Flag varieties, where this happens.

The case of the real projective space is more complicated, because now we have cells of all dimensions, therefore the attaching maps will not automatically induce the zero map in homology. So one has to study these induced maps, which is a bit complicated. The result is

THEOREM 8.14.

$$H_k(\mathbb{P}^n_{\mathbb{R}}) = \begin{cases} 0 & k > n, \\ \mathbb{Z} & k = n \text{ or } k \text{ even and } 0 \neq k \leq n-1, \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd and } k \leq n-1. \end{cases}$$