**Algebraic geometry for the masses, I**

This is the first part of a loose introduction to algebraic geometry, with incorrect definitions and no proofs. It is meant to enable anyone with previous knowledge of complex differential geometry to read papers in algebraic geometry, and understand some of the statements.[1]

We will define sheaves (sorry, not coherent sheaves yet), spec's and schemes.

## 1. Ringed spaces, easy version.

**Definition.** An **easy ringed space** $(X, \mathcal{O}_X)$ is a topological space $X$ together with, for every open subset $U$ of $X$, a subset $\mathcal{O}_X(U)$ of all complex valued functions on $U$, called **regular functions** on $U$. These data are supposed to satisfy the following properties:
(i) for every $f, g \in \mathcal{O}_X(U)$, $f+g$ and $f \cdot g$ are in $U$; that is $\mathcal{O}_X(U)$ is a ring (that's why ringed space). Usually we also require constant functions to be regular.
(ii) (sheaf property, see below) Let $U$ be an open subset of $X$, and $\{V_i\}$ an open cover of $U$. Let $f : U \to \mathbf{C}$ be a function. Then $f \in \mathcal{O}_X(U)$ if and only if $f|_{V_i} \in \mathcal{O}_X(V_i)$ for all $i$. That is, regularity is a local property.

**Remark.** Note that if $(X, \mathcal{O}_X)$ is a ringed space, for every two open subsets $V \subset U$ of $X$ we have a restriction map from $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. Moreover, given an open cover $V_i$ of $U$ and functions $f_i \in \mathcal{O}_X(V_i)$ such that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$ for all $i, j$, there is a unique glued function $f \in \mathcal{O}_X(U)$ such that $f|_{V_i} = f_i$.

As you may have suspected, I invented "easy" ringed spaces for this lecture. We will define general ringed spaces later.

**Example 1.** If $X$ is $\mathbf{R}^n$ or $\mathbf{C}^n$, we can define regular functions to mean continuous, or $C^k$ (for some $1 \le k \le \infty$) or $C^\omega$ or complex analytic. In each case we get an easy ringed space. Every open subset of an easy ringed space is naturally an easy ringed space.

**Definition.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be easy ringed spaces. A **morphism** (or regular map) from $X$ to $Y$ is a continuous map sending regular function to regular functions. That is $\phi : X \to Y$ such that for every $V \subset Y$ open and $f \in \mathcal{O}_Y(V)$ we have $f \circ \phi \in \mathcal{O}_X(\phi^{-1}(V))$.

An **isomorphism** is a morphism with a two-sided inverse which is also a morphism.

**Remark 1.** A $C^k$ $n$-manifold is an easy ringed space locally isomorphic to $(B^n, C^k)$; that is every point is contained in an open subset isomorphic, as e.r.s., to the unit ball in $\mathbf{R}^n$ with $C^k$ functions being regular. Similarly for (holomorphic) complex manifolds.

[1] versione 10.3.

**2.** A $C^k$ differentiable map of $C^k$ differentiable manifolds is a morphism of e.r.s. (i.e., it's $C^k$ differentiable iff the pullback of $C^k$ functions is $C^k$). Same with holomorphic.

Note that in all the examples above the local structure is the same at every point. This will be very untrue in the next examples.

## 2. Reduced schemes.

We will write $\mathbf{C}[x_1, \ldots, x_n]$ or just $\mathbf{C}[x]$ for the ring of polynomials in $n$ variables; each of them is a holomorphic function from $\mathbf{C}^n$ to $\mathbf{C}$. However, polynomials are "not enough" functions to get geometry going. So we introduce some more.

Let $F, G \in \mathbf{C}[x]$, and assume $G \neq 0$. We say that $F/G$ is a **rational function** on $\mathbf{C}^n$; of course in fact it is only defined as a function on the subset $U_G = \{x \in \mathbf{C}^n | G(x) \neq 0\}$. We say that two rational functions $F/G$ and $F_1/G_1$ are equal if they are equal as functions on $U_{GG_1}$, or equivalently if $FG_1 = GF_1$ as polynomials. The field of rational functions on $\mathbf{C}^n$ is usually denoted by $\mathbf{C}(x_1, \ldots, x_n)$; it comes from $\mathbf{C}[x_1, \ldots, x_n]$ the same way $\mathbf{Q}$ comes from $\mathbf{Z}$.

A subset $X \subset \mathbf{C}^n$ will be called **Zariski closed** if it is the common zero locus of some polynomials. Its complement will be called Zariski open; this defines the **Zariski topology** on $\mathbf{C}^n$ and on all its subsets. The usual topology on $\mathbf{C}^n$ will be called euclidean topology, whenever confusion may arise.

Note that euclidean topology can be defined analogously to Zariski topology, using $C^\infty$ (or $C^k$) functions instead of polynomials.

**Side remark.** A Zariski closed subset of $\mathbf{C}^n$ will be called **irreducible** if it is not the union of two proper closed subsets. E.g. in $\mathbf{C}^2$ $\{x^2 - y^2 - 1 = 0\}$ is irreducible but $\{x^2 - y^2 = 0\}$ is not. Every Zariski closed subset of $\mathbf{C}^n$ is a union of finitely many irreducible ones (this reflects the algebraic fact that the polynomial ring is noetherian, a consequence of Hilbert's basis theorem).

Let $X$ be a Zariski closed subset of $\mathbf{C}^n$, $U$ a Zariski open subset of $X$. We say that a function $f : U \to \mathbf{C}$ is **regular** at $x \in U$ if there is a rational function $F/G \in \mathbf{C}(x)$ such that $G(x) \neq 0$ and the functions $f$ and $F/G$ agree on $X \cap U_G$. We define the ring of regular functions $\mathcal{O}_X(U)$ to be the set of functions from $U$ to $\mathbf{C}$ which are regular at every point of $U$.

It turns out that $(X, \mathcal{O}_X)$ so defined is an easy ringed space, which we will call an **affine reduced scheme**; when we want to stress the fact that it is naturally contained in

$\mathbf{C}^n$, we say that it's a **reduced closed subscheme** of $\mathbf{C}^n$.

Recall that an **ideal** in a ring $A$ (commutative with 1) is an additive subgroup which contains all multiples of its elements, that is it's also an $A$-vector space (**module** is the official term).

There is a way to associate to ideals in $\mathbf{C}[x]$ closed subsets of $\mathbf{C}^n$ and conversely, as follows.

Given an ideal $I$, we can define $Z(I)$ its zero locus by

$$Z(I) = \{x \in \mathbf{C}^n \mid F(x) = 0 \ \forall F \in I\}.$$

Given a closed subset $X$, we can define the associated ideal to be

$$I(X) = \{F \in \mathbf{C}[x] \mid F(x) = 0 \ \forall x \in X\}.$$

Note the obvious simmetry in the definition.

**Big theorem**. *For every Zariski closed subset $X$ of $\mathbf{C}^n$, we have $X = Z(I(X))$. For every ideal $I$ in $\mathbf{C}^n$, we have $I(Z(I)) = \sqrt{I}$.*

Here $\sqrt{I} = \{F \mid \exists k \ F^k \in I\}$. It's easy to convince oneself that this is the best one can hope for; e.g., $I(Z(x^2, y^3)) = (x, y)$ in $\mathbf{C}[x, y]$. The big theorem (usually it's called Hilbert's Nullstellensatz) is true with any algebraically closed field instead of $\mathbf{C}$, but it's false otherwise. That's why algebraic geometers don't like to work over $\mathbf{R}$: because it's difficult.

In particular there's a one-to-one correspondece between Zariski closed and radical ideals, that is ideals $I$ with $\sqrt{I} = I$.

Let $A$ be a finitely generated reduced (= without nilpotents) complex algebra, and choose generators $x_1, \ldots, x_n$; we get a surjection $\mathbf{C}[x_1, \ldots, x_n] \to A$, and let $I$ be the kernel. $A$ is reduced iff $I$ is radical, by the way. We define $\mathit{Spec}\, A$ to be the easy ringed space $(X, \mathcal{O}_X)$ defined above, where $X = Z(I)$. Note that we can recover $A$ from $(X, \mathcal{O}_X)$ because it turns out that the natural map $A \to \vee_X(X)$ is an isomorphism (this is also a theorem).

The big deal is that the e.r.s. we obtain does not depend on the choice of generators, that is on the particular description of $A$ as $\mathbf{C}[x_1, \ldots, x_n]/I$ we choose. In fact, more is true: a morphism (of e.r.s) from $X = \mathit{Spec}\, A$ to $Y = \mathit{Spec}\, B$ is the same as a ring homomorphism $B \to A$, i.e. the opposite direction. One way is clear: if we have a morphism $\phi : X \to Y$, every element $f \in \vee_Y(Y)$ is a regular function, hence $f \circ \phi \in \mathcal{O}_X(X)$, and we can define $\phi^* : B \to A$ by $f \mapsto f \circ \phi$.

The big deal is that we can also reverse the argument, i.e., given a ring (well, complex algebra with 1) homomorphism $\phi^* : B \to A$ there is a unique $\phi : Spec\, A \to Spec\, B$ morphism of e.r.s. such that $\phi^*(f) = f \circ \phi$.

This is in fact very easy to prove.

A **reduced scheme** (locally of finite type over the complex numbers) is an easy ringed space such that every point has an open neighborhood which, as an e.r.s., is isomorphic to a reduced affine scheme.

## 3. First properties of reduced affine schemes

First of all, $\mathbf{C}^n$ itself is naturally a reduced affine scheme, being a Zariski closed subset of itself – defined by no equations at all! That is $\mathbf{C}^n = Spec\, \mathbf{C}[x_1, \ldots, x_n]$. When we want to stress that we look at $\mathbf{C}^n$ with its structure of affine scheme, we usually call it $\mathbf{A}^n$ (pronounce affine $n$-space), or even $\mathbf{A}^n(\mathbf{C})$ if we want to specify the base field.

Next, a (reduced irreducible) affine hypersurface in $\mathbf{C}^n$ is the zero locus of a non-constant polynomial $F \in \mathbf{C}[x_i]$ with non nonconstant factors; we can write it as $X = Spec\,(\mathbf{C}[x_i]/(F))$.

Note that $X$ is a complex submanifold of $\mathbf{C}^n$ (in the usual topology) in the open subset

$$X^{sm} = \{x \in X | \partial F/\partial x_i(x) \neq 0 \text{ for some } i = 1, \ldots, n\}.$$

If $\bar{x} \in X_{sm}$, we have that $T_{\bar{x}} X_{sm} \subset T_{\bar{x}} \mathbf{C}^n = \mathbf{C}^n$ is defined by the equation

$$\sum F_i(\bar{x}) x_i = 0, \text{ where } F_i = \partial F/\partial x_i.$$

More generally, for $X = Spec\,(\mathbf{C}[x]/I)$, with $I = \sqrt{I}$, we define $T_{\bar{x}} X \subset \mathbf{C}^n$ with the equations

$$\{\sum F_i(\bar{x}) x_i = 0 | \text{for every } F \in I\}.$$

If $I$ is generated by $F^1, \ldots F^k$, it is also true that

$$T_{\bar{x}} = \{\sum F_i^j(\bar{x}) x_i = 0 | \text{for every } j = 1, \ldots, k\}.$$

In the nonempty dense subset $X_{sm}$ (called the **smooth locus**) of $X$ which is a submanifold of $\mathbf{C}^n$, this coincides with the usual tangent space.

If $X$ is an irreducible reduced affine scheme, we define its dimension to be the dimension of its smooth locus.

**Examples. 1.** $\{x^2 + y^2 = z^2\}$ is an irreducible reduced subscheme of $\mathbf{C}^3$ of dimension 2, smooth outside the origin.

**2.** $\{xy = xz = 0\}$ is a reduced subscheme of $\mathbf{C}^3$; it is smooth outside the origin, and is the union of two irreducible smooth subschemes, one of dimension 2 and one of dimension 1. That's why we don't define dimension of reducible affine schemes.

## 4. Ringed spaces.

Let $X$ be a toplogical space. A **sheaf of abelian groups** $\mathcal{F}$ on $X$ is the datum, for each open subset $U$ of $X$, of an abelian group $\mathcal{F}(U)$, and for every inclusion of open subsets $V \subset U$ of a restriction map $f \mapsto f|_V$ from $\mathcal{F}(U) \to \mathcal{F}(V)$, such that if $W \subset V$, then restricting first to $V$ and then to $W$ is the same as restricting to $W$ directly. So far, it's a presheaf (actually we assume that the restriction is a group homomorphism).

For every open subsets $U$ of $X$ and every open cover $V_i$ of $U$, we require the presheaf to satisfy the **sheaf axioms**:

(i) given $f, g \in \mathcal{F}(U)$ such that $f|V_i = g|_{V_i}$ for all $i$, then $f = g$;

(ii) given for every $i$ an element $f_i \in \mathcal{F}(V_i)$ such that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$, there is a $f \in \mathcal{F}(U)$ such that $f|_{V_i} = f_i$; such an $f$ is unique by (i).

Analogously we define sheaves of complex vector spaces, sheaves of rings, etc. Even sheaves of sets.

The sheaf axioms mean that the elements of $\mathcal{F}(U)$ (so-called **sections** of $\mathcal{F}$ over $U$) are local in nature.

**Examples.** Say $M$ is a real $C^\infty$ manifold. Then $C^\infty$ differential $p$-form are naturally a sheaf of real vector spaces on $M$; same the $C^\infty$ vector fields, and all kinds of tensors. Just so that you don't start believing that everything is a sheaf, note that closed $p$-forms are a sheaf but exact ones are not, just a presheaf!

**Definition.** A **ringed space** $(X, \mathcal{O}_X)$ is a topological space $X$ together with a sheaf of rings (here by rings we mean commutative complex algebras) on $X$.

Mathematicians are also fond of locally ringed spaces. Every ringed space we will have occasion to encounter is also locally ringed, so don't worry about that.

Of course every "easy" ringed space is also a ringed space, and in general we will want our ringed spaces so that for evry $U$, the ring $\mathcal{O}_X(U)$ is "almost" a ring of functions on

$U$. The almost comes from the fact that now nilpotents are allowed.

**Example.** Let $X$ be a topological space with only one point, $A$ any ring. Then setting $\mathcal{O}_X(X) = A$ and $\mathcal{O}_X(\emptyset) = \{0\}$ makes $(X, \mathcal{O}_X)$ into a ringed space. We will not consider such wild generalizations, only the tamer(!) objects introduces in next section.

## 5. Affine schemes.

Let now $I$ be any ideal in $\mathbf{C}[x_1, \ldots, x_n]$, and let $A = \mathbf{C}[x_i]/I$. We define the **spectrum** of $A$, a ringed space $Spec\, A = (X, \mathcal{O}_X)$ as follows.

The topological space $X$ is just $Z(I)$ with the Zariski topology. For any $F \in \mathbf{C}[x_i]$, let $X_F = \{x \in X | F(x) \neq 0\}$; this is a (possibly empty) Zariski open subset of $X$.

We define $\mathcal{O}_X(X_F) = A_F$, the localization of $A$ at $F$. By definition, this means the following:

$$A_F = \{G/F^K \mid G \in \mathbf{C}[x],\, k \geq 0\}/\text{equivalence relation}$$

where $G/F^k \equiv H/F^m$ iff $GF^m - F^k$ is "zero" on $X_F$; by this we mean the, for some $a \geq 0$, $F^a(GF^m - HF^k) \in I$.

The idea is that multiplying by $F$ does not affect a function being zero in $X_F$, where $F$ is nonzero anyway; in general, to define to functions being equal on $X$, we want their difference to be in the ideal defining $X$.

Note that $X_F \cap X_{F_1} = X_{FF_1}$, and we have a restriction homorphism $\mathcal{O}_X(X_F) \to \mathcal{O}_X(X_{FF_1})$ given by $G/F^k \mapsto (GF_1^k)/(FF_1)^k$.

Then we define a sheaf of rings $\mathcal{O}_X$ on $X$ by using the sheaf axioms; note that every Zariski open subset of $X$ has a finite open cover by subset of the type $X_F$. The miracle is that no contradictions arise, and we get a ringed space.

As in the reduced case, $Spec\, A$ depends only on the algebra $A$ and not on its expression as $\mathbf{C}[x]/I$; again, a morphism of ringed spaces $\phi : Spec\, A \to Spec\, B$ is the same as a complex algebra homomorphism "the other way round" $\phi^* : B \to A$.

**Definition.** A **scheme** (locally of finite type over the complex numbers) is a ringed space locally isomorphic to the spectrum of a finitely generated complex algebra.

## 6. An easy example of scheme.

My favorite non-reduced scheme is $T = Spec\, \mathbf{C}[\varepsilon]/(\varepsilon^2)$ (sorry, the choice of $T$ is my own – I don't think it has any standard letter, or name).

The choice of the letter $\varepsilon$ instead is traditional, and is supposed to remind you that $\varepsilon$ is small, so small that $\varepsilon^2$ is negligible; in fact, it's so negligible that it's zero.